Università di Genova

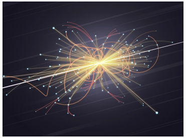
GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

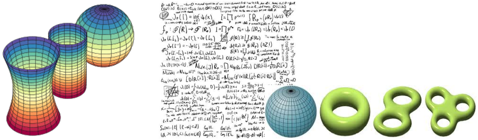
The BV Construction in NCG: Towards the Infinite-Dimensional Case

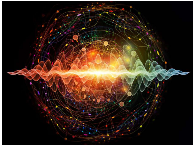
Roberta A. Iseppi

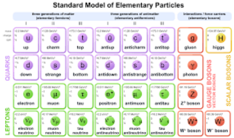
Cost Action CaLISTA General Meeting 2025

15 September 2025



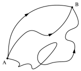






The BV construction: where it was discovered

Context: quantization of a gauge theory (X_0, S_0) via a path integral approach $\rightsquigarrow Z := \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]$



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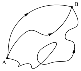

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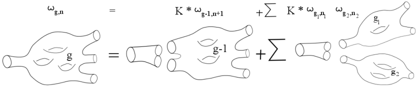
➡ Functorial Approach: TQFT = Functor of symmetric monoidal categories

$$\mathcal{C}ob_n \longrightarrow Vect_{\mathbb{C}}$$



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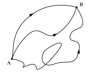

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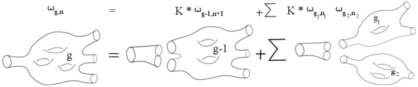

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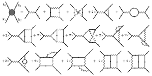
- Perturbative Approach: implement the principle of stationary phase

$$\int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu] \underset{\hbar\to 0}{\backsim} \sum_{x_0\in\{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar}S_0(x_0)} \,|\det S_0''(x_0)|^{-\frac{1}{2}}\, e^{\frac{\pi i}{4} sign(S_0''(x_0))}(2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|}\Phi_{\Gamma}.$$

Γ= Feynman diagram



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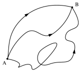

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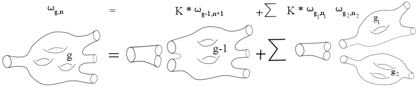

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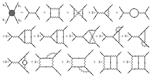

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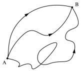

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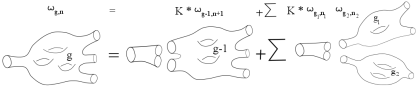

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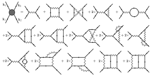

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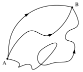

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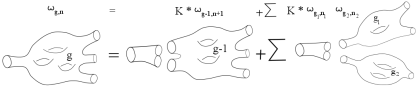

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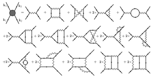

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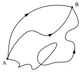

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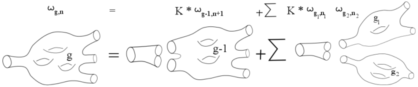

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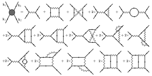

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How to eliminate these redundant symmetries?

- take the quotient w.r.t. the action of the group $\rightsquigarrow$ orbifolds
- add extra auxiliary variables $\rightsquigarrow$ ghost fields ✓

$$\int_{-\infty}^{+\infty} e^{-x^2}dx \rightsquigarrow \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy$$

The introduction of ghost fields

Def. A ghost field φ is characterized by its ghost degree $\deg(\varphi) \in \mathbb{Z}$ & its parity: $\epsilon(\varphi) \in \{0, 1\}$ where $\epsilon(\varphi) = 0$ is bosonic/real and $\epsilon(\varphi) = 1$ is fermionic/Grassm. s.t. $\deg(\varphi) \equiv \epsilon(\varphi)$ *mod* $\mathbb{Z}/\mathbb{Z}2$. For a ghost φ, its antighost φ^* has $\deg(\varphi^*) = -\deg(\varphi) - 1$ & $\epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1$ *mod* $\mathbb{Z}/2\mathbb{Z}$.

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- X_0: vector sp $\cong \mathbb{A}_{\mathbb{R}}^{n^2}$
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BV extended theory

- $\widetilde{X} = \bigoplus_{i \in \mathbb{Z}} [\widetilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\widetilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\widetilde{X}]^0 = X_0$

  $\downarrow$ graded locally free $\mathcal{O}_{X_0}$-mod.
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- $\widetilde{S} \in [\mathcal{O}_{\widetilde{X}}]^0$, s.t. $\widetilde{S}|_{X_0} = S_0$ & $\{\widetilde{S}, \widetilde{S}\} = 0$ sol. classical master eq.

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Note:

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[3] Each BV-extended theory naturally induces a cohomology complex: the BV-complex.

The BV construction: the key idea

BV cohom. complex: ▸ Cochain sp.: $\mathcal{C}^i(\widetilde{X}, d_{\tilde{S}}) = [\mathcal{O}_{\widetilde{X}}]^i$ ▸ Coboundary op.: $d_{\tilde{S}} := \{\widetilde{S}, -\}$, $d_{\tilde{S}}^2 = 0$

These cohomology groups capture relevant physical information about (X_0, S_0):

$$H^0_{BV}(\widetilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}$$

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The classical/quantum BV construction

$$\begin{tikzcd}
& \underset{\text{BV complex}}{\mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}/S_q})} \arrow[r, phantom, "\cong"] & \underset{\text{total complex}}{\mathcal{C}^\bullet_{BV}(X_t, d_{S_t/S_{q,t}})} & \underset{\text{BRST complex}}{\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t/S_{q,t}})|_{\mathcal{L}}} \\
\underset{\text{initial gauge th.}}{(X_0, S_0)} \arrow[r, "+\text{ gh./anti-gh.}"] & \underset{\text{BV-extended th.}}{(\widetilde{X}, \widetilde{S}/S_q)} \arrow[u] \arrow[r, "+\text{ aux. flds}"] & \underset{\text{total th}}{(X_t, S_t/S_{q,t})} \arrow[u] \arrow[r, "\text{gauge-}", "\text{fixing}"'] & \underset{\text{gauge-fixed th.}}{(X_t, S_t/S_{q,t})|_{\mathcal{L}}} \arrow[u]
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Key idea: [B.V.] The integral $(*)$ is invariant under the change of Lagrangian submanifold $\mathcal{L}$ in the homotopy class of $[X_0] \subset X_t$ and of action S_q in the quantum BV cohomology class of S_0

$$(*) \quad \int_{X_0} e^{\frac{i}{\hbar} S_0} \, [d\mu] \underset{BV}{\cong} \int_{[\mathcal{L}] \subset X_t} e^{\frac{i}{\hbar} S_q} \, d\mu_{BV}$$

The BV construction: the key idea

BV cohom. complex: ▸ Cochain sp.: $\mathcal{C}^i(\widetilde{X}, d_{\tilde{S}}) = [\mathcal{O}_{\tilde{X}}]^i$ ▸ Coboundary op.: $d_{\tilde{S}} := \{\widetilde{S}, -\}$, $d^2_{\tilde{S}} = 0$

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- The BV construction ⇝ cohomological approach to the study of gauge theories.

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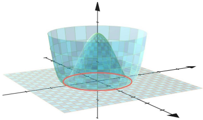

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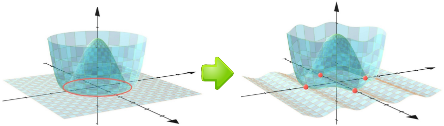
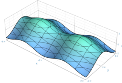


How noncommutative geometry?

Key idea: extend the classical notion of *manifold* by translating the geometrical concept in algebraic terms



Differential geometry $\hookrightarrow$ Noncommutative geometry

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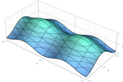

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Topology: locally compact, Hausdorff topologial spaces $\leftrightsquigarrow$ commutative C^*-algebras *Gelfand-Naimark Th [1943]*

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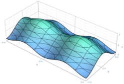

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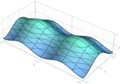

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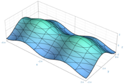

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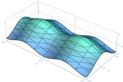

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Metric: closed, connected Riemannian spin manifold $\leftrightsquigarrow$ even, real spectral triples $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ s.t. $\mathcal{A} =$ commutative & 8 axioms

Reconstruction th. Connes [2008]

M ➡ $(\mathcal{C}^{\infty}(M), L^2(M, S), \partial\!\!\!/_M, J, \gamma)$ canonical spectral triple

- $L^2(M, S)$ = square-integrable sections of the spinor bundle S
- $\partial\!\!\!/_M := -i(\hat{c} \circ \nabla^s)$ Dirac operator
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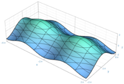

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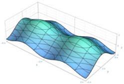

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The notion of spectral triple & more

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- an involutive unital algebra $\mathcal{A}$, faithfully represented as operators on a Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$;
- a separable Hilbert space $\mathcal{H}$;
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Def. A real structure on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:

$$[a, Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A} \qquad \text{and} \qquad [[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}$$

commutation relation first-order condition

Moreover:

KO-dim.	0	1	2	3	4	5	6	7
$J^2 = \pm Id$	1	1	-1	-1	-1	-1	1	1
$JD = \pm DJ$	1	-1	1	1	1	-1	1	1
$J\gamma = \pm \gamma J$	1		-1		1		-1	

➡ it reflects the properties of the periodicity of KO-homology and real K-theory

From spectral triples to gauge theories

Classically:

$$\begin{tikzcd} G \circlearrowright B \arrow[d] \\ M \end{tikzcd}$$

a gauge theory is understood as a principal bundle over a manifold M describing the spacetime, while the physics is modeled in terms of *connections*, *sections* of the bundle and *automorphisms of the structure group.*

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$$X_0 = \text{field configuration space} \qquad S_0 : X_0 \to \mathbb{R}, \text{ action functional}$$

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Def. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, its spectral action is defined as:

$$S[\varphi] := Tr\big(f(\tfrac{D+\varphi}{\Lambda})\big)$$

where:

- $\varphi \in \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi, a_i, b_i \in \mathcal{A}\}$ inner fluctuations of the operator D
- Λ fixes the energy scale
- f is a test function, plays a role only via its momenta $\quad f_0 := f(0), \quad f_k := \int_0^{+\infty} f(\nu)\nu^{k-1} d\nu$

The Standard model as an almost-commutative spectral triple

Chamseddine, Connes, Marcolli [2007]

spectral triple $(\mathcal{A}, \mathcal{H}, D)$ ➡ gauge theory $(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$
- $\mathcal{H}$ = Hilbert space
- $D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator

- $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\} \rightsquigarrow$ conf. sp = inner fluctuations
- $S_0[D+\varphi] = Tr(f(D+\varphi)) \rightsquigarrow$ action func. = spectral action
- $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group = unitary elements in $\mathcal{A}$

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M = compact Riem. spin manifold
with canonical spectral triple

$(\mathcal{C}^\infty(M), L^2(M,S), D_M, J_M, \gamma_M)$

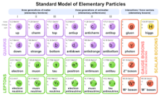
✖

F= finite noncomm. space
with finite real spectral triple

$(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_F, J_F, \gamma_F)$

Gauge group:
$U(1) \times SU(2) \times SU(3)$

96 particles



Standard Model of particles, with neutrino mixing and minimally coupled to gravity,

Product: $(\mathcal{C}^\infty(M) \otimes [\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})], L^2(M,S) \otimes \mathbb{C}^{96}, D_M \otimes Id + \gamma_M \otimes D_F, J_M \otimes J_F, \gamma_M \otimes \gamma_F)$

The Standard Model as an almost-commutative spectral triple [2]

Two notions of action: Spectral action: $S[D+\varphi] = Tr(f(D+\varphi));$

- for f a regular function (good decay, cut off...);
- for φ a self-adjoint element, with $\varphi = \sum_j a_j[D, b_j]$, $a_j, b_j \in \mathcal{A}$

Fermionic action: $S[\psi] = \frac{1}{2}\langle (J)\psi, D\psi\rangle,$

- for $\langle\ ,\ \rangle$ the inner product structure on $\mathcal{H}$;
- for $\psi \in \mathcal{H}_f \subseteq \mathcal{H}$ ➡ we can impose a Grassmannian nature to the elements in $\mathcal{H}_f$

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$$S_0(A,\psi) = \underbrace{Tr(f(D_{SM}/\Lambda))}_{\text{spectral action}} + \underbrace{\langle J_{SM}(\psi), D_{SM}\psi\rangle}_{\text{fermionic action}}$$

➡ The full Lagrangian (in Euclidean signature) of the Standard Model can be obtained as asymptotic expansion of the action determined by applying the spectral action principle, plus the fermioninc action.

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But: ➡ *Compactness*: does not work well with the notion of *causality* $\rightsquigarrow$ local version of spectral triples

➡ *Riemannian*: to describe gravity we need Lorentian signature $\rightsquigarrow$ at the moment still missing

Towards the quantization of the spectral action

But... all of this is still classical. ❓ How can we quantize our theory?

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$$\boxed{Z = \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]}$$

$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$, inner-fluct. of D

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IF one considers the theory induced by the canonical spectral triple:

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- X_0 = space of all metrics on M

➡ Possible approach to quantum gravity (Barrett, Glaser, Khalkhali, ...)

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Problem 1: the measure $[d\mu]$ is not well defined

Idea: to see the manifold M as the limit of a finite object $\rightsquigarrow M_n(\mathbb{C})$, $n \to \infty$, matrix models

$$X_0 = \{A \in M_n(\mathbb{C}) : A^* = A\} \quad \& \quad D = \text{fixed self-adjoint matrix}$$

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Recall: the fully noncommutative case has:

$X_0 =$ infinite-dim. vector space & $D =$ unbounded self-adjoint op., with compact resolvant

The BV construction for finite spectral triples

- We want to study the BV construction for gauge theories induced by finite spectral triples

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Questions and goals:

- Can the BV construction be described in terms of spectral triples?
- Can the BRST cohomology be related to other (better understood) cohomological theories?

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Noncommutative geometry

$$\begin{tikzcd}
\begin{array}{c}(M_n(\mathbb{C}), \mathbb{C}^n, D, f)\\ \text{initial spectral triple}\end{array} \arrow[r] \arrow[dd, leftrightarrow] & \begin{array}{c}?\\ \text{BV spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}?\\ \text{total spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}?\\ \text{gauge-fixing fermion}\end{array} \arrow[d] \\
 & \begin{array}{c}\mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\widetilde{S}}) \cong ?\\ \text{BV complex} \cong ??\end{array} & \begin{array}{c}\mathcal{C}^\bullet_{BV}(X_t, d_{S_t}) \cong ?\\ \text{BV tot complex} \cong ??\end{array} & \begin{array}{c}\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi \cong ?\\ \text{BRST complex} \cong ??\end{array} \\
\begin{array}{c}(X_0, S_0)\\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\widetilde{X}, \widetilde{S})\\ \text{extended theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)\\ \text{total theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)|_\Psi\\ \text{gauge-fixed theory}\end{array} \arrow[u]
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BV construction & BRST cohomology

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arXiv:2410.11823

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The BV construction for finite spectral triples [2]

Step 1: $\boxed{(\mathcal{A}_0, \mathcal{H}_0, D_0) \ \& \ f \xrightarrow{\text{BV construction}} (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})}$ J_{BV} : to go from $\mathbb{C}$ to $\mathbb{R}$.

How to extract the information from the initial spectral triple $(\mathcal{A}_0, \mathcal{H}_0, D_0)$?

ghost fields: Which role are the ghost fields going to play in the BV-spectral triple?

extended action: How can we determine S_{BV} starting from (D_0, f)?

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initial spectral tr.

⇓

induced gauge theory

$$(\mathcal{A}_0, \mathcal{H}_0, \underbrace{D_0)\ \&\ f}$$

$$X_0 = \Omega^1(\mathcal{A}_0) \qquad S_0 = Tr(f(D_0 + \varphi))$$

BV spectral tr.

⇓

BV-extended theory

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, \underbrace{D_{BV}, J_{BV}})$$

$$\tilde{X} = X_0 + X_0^*[1] + \mathcal{H}_{BV,f} \qquad S_{BV} = \frac{1}{2} S_{ferm}$$

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BV-extended theory $\tilde{X} = X_0 + X_0^*[1] + \mathcal{H}_{BV,f}$ $S_{BV} = \frac{1}{2} S_{ferm}$

The Hilbert space: describes the ghost sector of the BV-extended theory.

$$\mathcal{H}_0 = \mathbb{C}^n \xrightarrow{+\text{ ghost/anti-ghost fields}} \boxed{\mathcal{H}_{BV} = [M_n(\mathbb{C})]_{-2} \oplus [M_n(\mathbb{C})]_{-1}} \oplus \boxed{[M_n(\mathbb{C})]_0 \oplus [M_n(\mathbb{C})]_1}$$

where

$$\mathcal{H}_{BV,f} = [i\mathfrak{su}(\mathfrak{n})]_{-2} \oplus [i\mathfrak{su}(\mathfrak{n})]_{-1} \oplus [i\mathfrak{su}(\mathfrak{n})]_1 \oplus [i\mathfrak{su}(\mathfrak{n})]_2$$
$$= \mathcal{Q} \oplus \mathcal{Q}^*[1]$$

➡ fully determined by
$\mathfrak{su}(n) = \mathfrak{u}(\mathcal{A}_0)/\mathcal{Z}(\mathfrak{u}(\mathcal{A}_0))$

The BV construction for finite spectral triples [3]

The operator D_{BV} determines the BV-action $S_{BV} := \widetilde{S} - S_0$ as induced fermionic action.

$$D_{BV} = \begin{pmatrix} 0 & R \\ R^* & S \end{pmatrix} \qquad \begin{array}{l} R : \mathcal{Q} \to \mathcal{Q}^*[1] \\ S : \mathcal{Q} \to \mathcal{Q} \end{array}$$

The linear operators R and S are represented, as block matrices, by

$$R := \frac{1}{2}\begin{pmatrix} 0 & -ad(C) \\ ad(C) & -ad(x) \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & ad(x^*) \\ ad(x^*) & ad(C^*) \end{pmatrix}$$

where
$$ad(z) : M_n(\mathbb{C}) \to M_n(\mathbb{C}); \quad \varphi \mapsto [\alpha(z), \varphi]_-.$$

Explicitly, the matrix representation of these linear operators has in position (p, r) the term: $-\sum_q i \cdot f_{pqr}\; z_q$ → Structure constants of $\mathfrak{su}(n)$

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The self-adjoint operator D_{BV} is completely obtained by:

- linearity in the antifields, which enforces the zero-block matrix in position $(1, 1)$ in D_{BV};
- degree condition, that is, the induced fermionic action has to have total ghost degree 0, which determines the variables to insert in each block;

Conditions of the BV construction

- structure constants of $\mathfrak{su}(\mathfrak{n}) = \mathfrak{u}(\mathcal{A}_0)/\mathcal{Z}(\mathfrak{u}(\mathcal{A}_0))$, which dictate the entries in each block matrix.

by the gauge symmetries

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How extend the construction to the general case $(\mathcal{A}_0, \mathcal{H}_0, D_0)$, with $\mathcal{A}_0$ an infinite dim, noncomm. $*$-algebra?

The variation of the spectral action under inner fluctuations

Chanseddine, Connes 2006

The canonical spectral triple: $(\mathcal{C}^\infty(M), L^2(M,S), D_M, J_M, \gamma_M)$, M = compact Riem. spin mfld, $dim(M) \leqslant 4$

Theorem: Let's supposed the vanishing of the tadpole. Then, for M a spin manifold of dim. 4, the inner fluctuation of the scale-independent part of the spectral action is given by

$$Tr(|D+A|^0) - Tr(|D|^0) = \underbrace{\tfrac{1}{4}\int_{\tau_0}(dA + A^2)^2}_{[1]} - \tfrac{1}{2}\int_{\psi}(AdA + \tfrac{2}{3}A^3)$$

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[1] Yang-Mills functional with

τ_0= positive Hochschild 4-cycle where the positivity in Hochschild cohomology is the condition:

$$\int_{\tau_0} \omega\omega^* \geqslant 0, \quad \forall \omega \in \Omega^2$$

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[2] Chern-Simons functional with $\psi =$ cyclic 3 cocycle

Note: under the gauge transformation

$$\gamma_u(A) = udu^* + uAu^*, \quad u\in U(\mathcal{A})$$

the CS function fulfills the following invariance rule

$$CS_{\psi}(\gamma_u(A)) = CS_{\psi}(A) + \frac{1}{3}\langle\psi, u\rangle$$

where $\langle\psi,u\rangle$ is the pairing between $HC^3(\mathcal{A})$ and $\mathcal{K}_1(\mathcal{A})$

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➡ Under the tadpole hypothesis the pairing of a 3-cyclic cocycle with an element in $\mathcal{K}_1(\mathcal{A})$ vanishes
$\rightsquigarrow$ gauge invariance of the CS functional

Chern-Simons and Yang-Mills as "building blocks"

Van Nuland, Van Suijlekom, 2022

Can we extend this result to noncommutative infinite-dimensional $*$-algebras?

Theorem: Given $(\mathcal{A}, \mathcal{H}, D)$ any spectral triple, the inner fluctuation of the spectral action is given by:

$$Tr(f(D+A)) - Tr(f(D)) = \sum_{k=1}^{\infty} \frac{1}{2k} \int_{\varphi_{2k}} YM_k(A) + \int_{\psi_{2k-1}} CS_{2k-1}(A)$$

In the above theorem:

- YM_k: higher Yang-Mills th., $\int_{\varphi_{2k}} F^k$, $F = dA + A^2$
- φ_{2k}: Hochschild cocycle
- CS_{2k-1}: generalised Chern-Simons theory
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Note:

- This theorem holds for any spectral triples, commutative or not, of any dimension, beyond the case of 4-dim. spin manifolds,
- This theorem holds at any order in the scale parameter Λ.
- The spectral action is globally invariant under the action of the gauge group.

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Chern-Simons and Yang-Mills as "building blocks"

Van Nuland, Van Suijlekom, 2022

Can we extend this result to noncommutative infinite-dimensional $*$-algebras?

Theorem: Given $(\mathcal{A}, \mathcal{H}, D)$ any spectral triple, the inner fluctuation of the spectral action is given by:

$$Tr(f(D+A)) - Tr(f(D)) = \sum_{k=1}^{\infty} \frac{1}{2k} \int_{\varphi_{2k}} YM_k(A) + \int_{\psi_{2k-1}} CS_{2k-1}(A)$$

In the above theorem:

- YM_k: higher Yang-Mills th., $\int_{\varphi_{2k}} F^k$, $F = dA + A^2$
- φ_{2k}: Hochschild cocycle
- CS_{2k-1}: generalised Chern-Simons theory
- ψ_{2k-1}: odd $(b, B)-$cocycle

Note:

- This theorem holds for any spectral triples, commutative or not, of any dimension, beyond the case of 4-dim. spin manifolds,
- This theorem holds at any order in the scale parameter Λ.
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Higher YM and generalised CS theories are "building blocks" in the inner fluctuation of the sp. action

Chern-Simons theory in the noncommutative setting

joint with T. Krajewski and C. Perez-Sanchez

Classically:

- M = compact oriented 3-dim. manifold
- G= simple, simply connected Lie group
- $\pi : P \longrightarrow M$, principal G-bundle
- $s : M \longrightarrow P$, section of the bundle P

Field content: $\mathcal{F}_{CS} \cong \Omega^1(M, \mathfrak{g})$

Action functional: $S_{CS}[A] := \int_M \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A, A]\rangle$

Gauge transformation: $A \mapsto A^g = gAg^{-1} + gdg^{-1}$

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Note: The action S_{CS} is not invariant under gauge transformation but this holds for $e^{\frac{i}{\hbar} \cdot S_{CS}}$, for $\hbar = \frac{1}{2k\pi}$, $k \in \mathbb{Z}$.

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In NCG:

Def. A cycle of dim. n is denoted by $(\Omega^\bullet(\mathcal{A}), d, \int)$, where

- $\mathcal{A}$ = unital $*$-algebra over $\mathbb{C}$
- $\Omega^\bullet(\mathcal{A}) = \oplus_{k \geqslant 0}\Omega^k(\mathcal{A})$, graded algebra s.t. there exists a representation $\rho : \mathcal{A} \to \Omega^0(\mathcal{A})$
- $d : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet+1}(\mathcal{A})$, derivation of deg 1, with graded Leibniz rule: $d(\omega\eta) = d(\omega)\eta + (-1)^{|\omega|}d(\eta)$
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Note: The notion of dimension is encoded by the integral, not by the algebra of forms as there is not a top-degree for forms here.

Chern-Simons theory in the noncommutative setting [2]

joint with T. Krajewski and C. Perez-Sanchez

Given a 3-cycle, one can define the induced Chern-Simons theory:

- Field content: $\mathcal{F}_{NC} \cong \Omega^1(\mathcal{A})$
- Action functional: $S_{CS,NC}[A] := \int AdA + \frac{2}{3}A^3$
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Note: the theory is gauge invariant under infinitesimal gauge transformations which are connected to the identity

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Step 1: The study of the critical locus

Classically: the critical points of the action functional S_{CS} are flat connections, that is, connections $A \in \Omega^1(M, \mathfrak{g})$ s.t. $F = 0$.

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The bilinear form $\langle \omega, \eta \rangle := \int \omega\eta$ is degenerate. To solve this problem we quotient w.r.t. the junk-forms

$$J(\mathcal{A}) = \{\omega \in \Omega^\bullet(\mathcal{A}), \omega \neq 0 \;\; \text{s. t.} \forall \eta \in \Omega^\bullet(\mathcal{A}) \text{ with } |\eta| = n - |\omega|, \eta \neq 0, \int \omega\eta = (-1)^{|\omega||\eta|} \int \eta\omega = 0\}$$

Towards a BV formalism for NC Chern-Simons theory

joint with T. Krajewski and C. Perez-Sanchez

Classically BV-extended field sp.

$$\mathcal{F}_{BV} = \underbrace{\Pi\Omega^0(M,\mathfrak{g})}_{\text{ghost fields, Grassmannian}} \oplus \underbrace{\Omega^1(M,\mathfrak{g})}_{\text{initial fields}} \oplus \underbrace{\Pi\Omega^2(M,\mathfrak{g})}_{\text{antifields, fermionic}} \oplus \underbrace{\Omega^3(M,\mathfrak{g})}_{\text{antighosts, bosonic}}$$

BV-action

$$S_{BV} = S_{CS} + \int_M \langle A^*, d_A C\rangle + \frac{1}{2}\int_M \langle C^*, [C,C]\rangle$$

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Note: A similar construction can be performed for the case of a (noncommutative) Yang-Mills theory.

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Note: A similar construction can be performed for the case of a (noncommutative) Yang-Mills theory.

- How to merge all the different BV-extensions, coming from the different contributions of CS and YM theories to the full spectral triple?
- How to perform all the other steps in the BV construction, including establishing the BV/BRST complexes and determining the gauge-fixing Lagrangian?

What is coming? Some interesting open problems

Project 1: The BV formalism for Chern-Simons theory in NCG

Idea: To extend the BV construction for the Chern-Simons theory from classical differential forms to universal forms induced by cyclic cocycles.

T. Krajewski

Project 2: the BV formalism for fuzzy geometries

Idea: To apply the previous result to a fuzzy geometry, which induces a Yang-Mills matrix model:

$S_{\text{YM}} = -\frac{1}{2} Tr_{N\otimes n}\big([D_\mu, D_\nu][D_\mu, D_\nu]\big)$ ➡ compute $\int_{M_N(\mathbb{C})^4_{\text{skew-adj}}} e^{-S_{YM}[D]}$, towards quantum gravity

C. Perez-Sanchez

Project 3: The BV formalism for noncommutative manifolds

Idea: To rethink the BV formalism in a purely noncommutative and infinite dimensional setting.

R. Nest

Project 4: Spectral triples and higher-groups

Idea: To extend the notion of spectral triple to have induced gauge theory with a higher-group as gauge group

A. Frabetti