Corfu Workshop on Quantum Gravity and Strings

The Improved Null Energy Condition (INEC)

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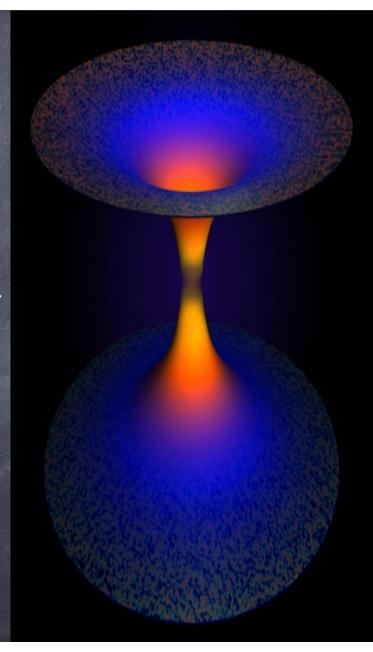
Outline

- General Relativity and semiclassical gravity.

- The Quantum Focusing Conjecture

- The Improved Quantum Null Energy Condition (INEC)

- WiP- Attempt to prove the INEC Null Plane and Sphere.

Geometry=Energy

- The metric fixes the LHS. What is the RHS?

- The RHS is the energy-momentum (stress-energy) tensor (EMT) $ds^2 = g_{ab} dx^a dx^b$ it describes the energy sources of spacetime – particles, fluids, BHs, Cosmological Constant...

- Many times – fluid approximation

$$T_{\mu\nu} = Diag\{\rho, -p, -p, -,p\}$$

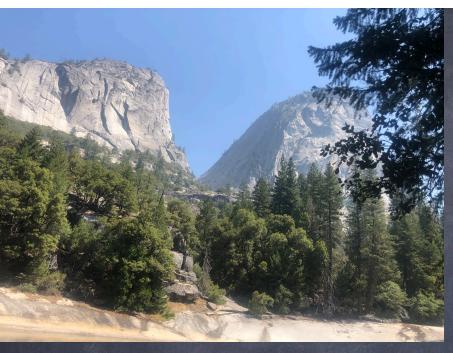$$\boxed{G_{\mu\nu}(g_{\mu\nu}) = 8\pi G T_{\mu\nu}}$$

Energy Sources

- We do not have a theory telling us what are the "allowed" EMT.

- Actually, the Synge method allows us to construct any EMT that we want – pick a metric g_{ab}, and solve for the EMT using Einstein's equations.
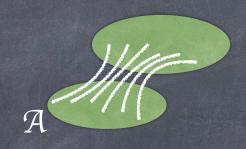
- So let us ponder about what are plausible EMTs...

$$G_{\mu\nu}(g_{\mu\nu}) = 8\pi G T_{\mu\nu}$$

Energy Conditions

- In the fluid approximation, the equation of state relates the pressure and energy density $p = w\rho$ $T_{\mu\nu} = Diag\{\rho, -p, -p, -,p\}$

- In the 60's, when singularity theorems where proven we knew about matter (w=0) and radiation (w=1/3)

- By now, spatial curvature (w=-1/3), CC, Inflation, DE (w=-1) more...already violates most energy conditions.

- So based on the known forms of matter, energy conditions were suggested as an additional assumption for solving the EFE.

- Null Energy Condition: For every future pointing null vector field kᵃ: $$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$$
  - Fluid approx. w>=-1

Classical Expansion

- Consider a congruence of geodesics through some cross-section infinitesimal area $\mathcal{A}$ as we move forward with the affine parameter λ, the classical scalar expansion is defined as $\theta \equiv \dfrac{1}{\mathcal{A}}\dfrac{d\mathcal{A}}{d\lambda}$

The Raychaudhuri eq.

- θ is the expansion scalar which measures the expansion of a congruence of geodesics, k^μ a null vector, λ the affine parameter.

- For simplicity zero shear, torsion, etc.

- Raychaudhuri eq. $\quad \dfrac{d\theta}{d\lambda} = -\dfrac{1}{2}\theta^2 - R_{\mu\nu}k^\mu k^\nu = -\dfrac{1}{2}\theta^2 - 8\pi G T_{\mu\nu}k^\mu k^\nu =$

- If the NEC holds we have $\quad \theta' \leq -\dfrac{1}{2}\theta^2$

- Solving gives $\dfrac{1}{\theta} \geq \dfrac{1}{\theta_0} + \dfrac{\lambda}{2}$ => if $\theta_0 < 0$ a caustic is formed at finite λ

- Basis of singularity theorems

Classical Results

- Black hole singularities

- Big Bang singularity

- Area theorem of black holes (The area can only grow)

- …..

- Useless in the real world!

Quantum Fields violate the NEC!

Semiclassical Gravity

- Consider classical background space-time and on top of that quantum fields.

- Quantum fields may perturb the space-time in a self-consistent manner.

- Quantum corrected EMT, (in principle) all orders in hbar, 1st order in G $8\pi G\langle T_{\mu\nu} k^\mu k^\nu\rangle$

- BHs – Bekenstein GSL=Generalized Second Law, Hawking radiation,..

- Cosmology – inflation, theory of large scale structure,..
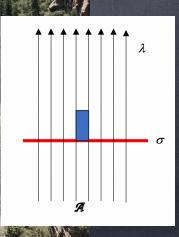

Classical
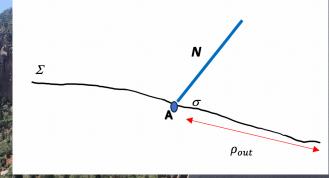
Bekenstein Hawking Entropy

- Bekenstein 72' $\quad S_{gen} = \dfrac{k_B c^3}{4G\hbar} A_{EH} + S; \quad dS_{gen} \geq 0$

- <u>All fundamental constant of nature appear in the formula!!</u>

- S is the von Neumann (entanglement) entropy of quantum fields on one side of the cross-section area (non-local) $\quad S = -\,Tr\rho \log \rho$

- A_{EH} – the area of the BH at the event horizon.

- BHs are thermodynamical objects. They have entropy, temperature. They radiate away and disappear – BH information paradox.

- Generalized entropy is finite, unlike the entanglement entropy.

Area-> Generalized Entropy
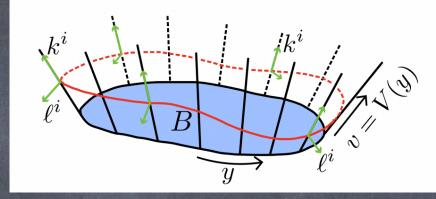
- Nowadays, for infinitesimal areas $\mathcal{A}$ as well

  ($k_B = c = 1$): $S_{gen} = \dfrac{\mathcal{A}}{4G\hbar} + S$ Semiclassical result

- Consider classical works in GR and replace $\mathcal{A} \to 4G\hbar S_{gen}$, and try to re-derive results

- Many successful generalizations - GSL (also for cosmology), modified singularity theorems, advances in the BH Information paradox...HOW?

Quantum Expansion

- If we replace A->S_{gen}, then the expansion can change to quantum expansion. By analyzing the quantum expansion we can derive semiclassical results.

$$\Theta = \frac{4G\hbar}{\sqrt{h(y)}}\frac{\delta S_{gen}}{\delta V(y)} \qquad \Longleftrightarrow \qquad \Theta = \theta + \frac{4G\hbar}{\mathcal{A}}S'$$
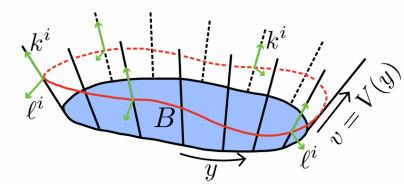
$\mathcal{A}$
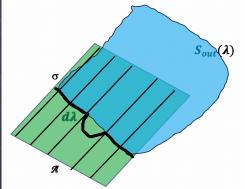
- Examples: Islands, "Quantum Extremal Surfaces" $\Theta = 0$, were crucial for showing that BH evaporation is unitary – major advance in resolving the BH information paradox. Almheiri, Engelhardt, Marolf and Maxfield 2019, Pennington 2020 ...IBD, Merav Hadad, Elizabeth Wildenhaim, 2023
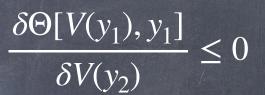
Quantum Focusing Conjecture

(Bousso, Fisher, Leichenauer and Wall 2015)

$$\Theta[V(y), y_1] \equiv \frac{4G\hbar}{\sqrt{h(y_1)}} \frac{\delta S_{gen}}{\delta V(y_1)}$$

- Based on the Raychaudhuri eq. (assuming NEC) that $\theta' \leq 0$.

- Claim – in semiclassical gravity, no NEC assumption is needed.

- The off-diagonal part $y_2 \neq y_1$ is proven by entropy subadditivity. The diagonal part is a conjecture.

$$\frac{\delta\Theta[V(y_1), y_1]}{\delta V(y_2)} \leq 0$$

- Basically, an energy condition in the presence of gravity.

$$\Theta' \leq 0$$

- Results: 1) Entropy bounds

- 2) GSL – (certain examples)

- 3) Quantum Null Energy Condition!

QNEC-Quantum Null Energy Condition

- Consider the QFC and use the Raychaudhuri eq.

$$\Theta' = \theta' + \frac{4G\hbar}{\mathcal{A}}(S'' - \theta S') = -\frac{1}{2}\theta^2 - 8\pi G\langle T_{\mu\nu}k^\mu k^\nu\rangle + \frac{4G\hbar}{\mathcal{A}}(S'' - \theta S') \leq 0$$
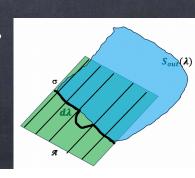
$$\langle T_{\mu\nu}k^\mu k^\nu\rangle \geq \frac{\hbar}{2\pi\mathcal{A}}(S'' - \theta S') - \frac{1}{16\pi G}\theta^2$$

Energy cond. w. gravity

- hbar->0 reproduces the classical NEC.

$$\langle T_{\mu\nu}k^\mu k^\nu\rangle \geq 0$$
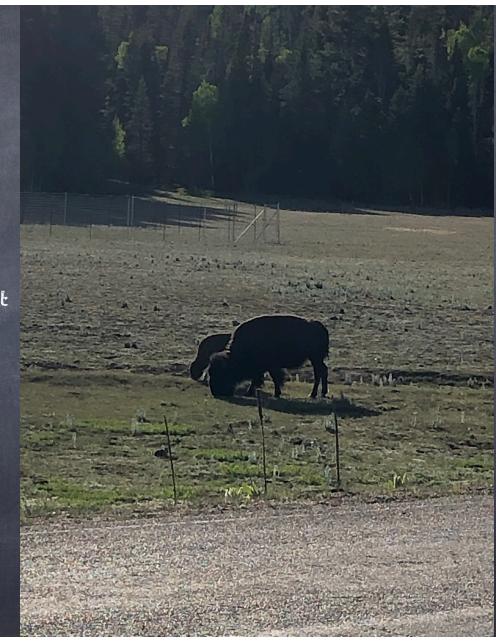
- Consider instead the limit $\theta \to 0$, congruences with vanishing classical expansion:

$$-8\pi G\langle T_{\mu\nu}k^\mu k^\nu\rangle + \frac{4G\hbar}{\mathcal{A}}S'' \leq 0 \quad \Rightarrow \quad \langle T_{\mu\nu}k^\mu k^\nu\rangle \geq \frac{\hbar}{2\pi\mathcal{A}}S''$$

QNEC

$$\langle T_{\mu\nu} k^\mu k^\nu \rangle \geq \frac{\hbar}{2\pi\mathcal{A}} S''$$

- S'' can be negative, weaker than the classical NEC.

- G dropped out of the calculation - independent of gravity.

- Field theory statement, works in Minkowski space.

- Field theory proof - 70 pages with long calculations (Bousso, Fisher, Koeller, Leichenauer and Wall 2015, Balakrishnan, Faulkner, Khandker and Wang 2019, Kudler-Flam, Leutheusser, Rahman, Satishchandran and Speranza 2023).

Other limits?

- QFC is a conjecture, perhaps its wrong?

- Restricted QFC (proven in "braneworld scenario")
  Shahbazi-Moghaddam 2023 claims that at $\Theta = 0 \Rightarrow \Theta' \leq 0$

- Still gives all the desired results of GSL, singularity theorems, BH information paradox etc.

- Let's consider the $\Theta \to 0$ limit.

Improved Null Energy Condition (INEC) IBD 2023

- $\Theta = \theta + \dfrac{4G\hbar}{\mathscr{A}}S' = 0 \quad \Rightarrow \quad \theta = -\dfrac{4G\hbar}{\mathscr{A}}S'$

- Substitute into the QFC:

$$\langle T_{\mu\nu}k^\mu k^\nu \rangle \geq \frac{\hbar}{2\pi\mathscr{A}}\left(S'' - \frac{1}{2}\theta S'\right) = \frac{\hbar}{2\pi\sqrt{\mathscr{A}}}\left(\frac{S'}{\sqrt{\mathscr{A}}}\right)'.$$

- Stronger condition $-\theta S' \geq 0$

- Independent of G

Other dimensions

- In D=2,3 the term drops out and we reproduce the QNEC.

- $$\langle T_{\mu\nu} k^\mu k^\nu \rangle \geq \frac{\hbar}{2\pi\mathscr{A}} \left(S'' - \frac{D-3}{D-2} \theta S' \right)$$

- In D→infinity we get the pointwise QHANEC.
$$\langle T_{\mu\nu} k^\mu k^\nu \rangle \geq \frac{\hbar}{2\pi\mathscr{A}} (S'' - \theta S') = \frac{\hbar}{2\pi} \left(\frac{S'}{\mathscr{A}} \right)'$$

Interpretations of the INEC

- 1. The INEC is true and will be proven in field theory in Minkowski spacetime.

- 2. The QFC and the restricted version work only perturbatively with $G\hbar/\mathscr{A} \ll 1$ s.t. the $-\theta S' \geq 0$ is always subdominant, so the QNEC holds. Corrections from curvature or EE need to be added.

- 3. $\Theta = 0$ exist only in the redundant case of $\theta = S' = 0$, and these are the only points where the QNEC is saturated.

- 4. $8\pi G\langle T_{\mu\nu}k^\mu k^\nu\rangle$ is only 1st order in hbar => perturbative point wise version of the QHANEC

$$R_{ab}k^a k^b = 8\pi G T_{ab}^{(0)}k^a k^b \qquad\Longrightarrow\qquad \langle T_{ab}^{(1)}k^a k^b\rangle \geq \frac{1}{2\pi\mathscr{A}}(S'' - \theta S')$$

WiP-Sketch of proof

- Consider curved surfaces, and expand S in powers of the surface $\sigma(\lambda)$, and differentiate w.r.t the affine parameter λ.

- We need to connect EE/modular hamiltonian to the EMT, usually known only for flat surfaces with $\theta = 0$.

- If there exists a bifurcating Killing horizon then we know the modular H:

$$H_\gamma = 2\pi \int d^{d-2}y \int_{\gamma(y)}^{\infty} d\lambda (\lambda - \gamma(y)) T_{\lambda\lambda}(\lambda, y)$$

- We will take shape derivatives after deforming the surface and use the relation between the EE, modular H and Relative entropy, and check whether the INEC holds

$$\Delta H_\gamma = \Delta S_{rel} + \Delta S$$

WiP-Sketch of proof
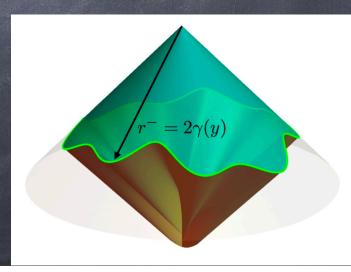
- The QNEC reduces to $\quad H_\gamma'' = 2\pi T_{\lambda\lambda}(\lambda, y) = S'' + S_{rel}'' \geq S''$

- Relative Entropy S_{rel} measures the "distance" between two states and is non-negative and monotonic

- Since $S''_{rel} >= 0$ this completes the proof. $\quad S_{rel} = Tr\rho(\log\rho - \log\sigma)$

- The INEC similarly reduces to $\quad S_{rel}'' - \dfrac{1}{2}\theta S_{rel}' \geq 0$

- We also know that $S'_{rel} =< 0$, but we don't know the size of expansion or derivatives, stay tuned...

WiP- The sphere case

- Consider the modular H on the Lightcone/sphere

- The entangling surface is on the past light cone $r^+=0$, $r^- = 2\gamma(\Omega)$, which is a boundary of somre region $\mathcal{R}(\lambda)$ with reduced density matrix $\rho_{\mathcal{R}}$

$$H_\gamma = 2\pi \int d\Omega \int_0^{\gamma(\Omega)} d\lambda \, \lambda^{d-1} \frac{\gamma(\Omega) - \lambda}{\gamma(\Omega)} T_{\lambda\lambda}$$

- λ is the affine param. Of null rays, and vanishes at the tip of the cone.

$r^- = 2\gamma(y)$

WiP- The sphere case

$$H_\gamma = 2\pi \int d\Omega \int_0^{\gamma(\Omega)} d\lambda\, \lambda^{d-1} \frac{\gamma(\Omega) - \lambda}{\gamma(\Omega)} T_{\lambda\lambda}$$

- Consider a 1-param. Family of cuts
$\gamma(\Omega; \lambda) \equiv \gamma(\Omega; 0) + \lambda\dot{\gamma}(\Omega)$

- Calculate the derivatives of the modular H

$r^- = 2\gamma(y)$
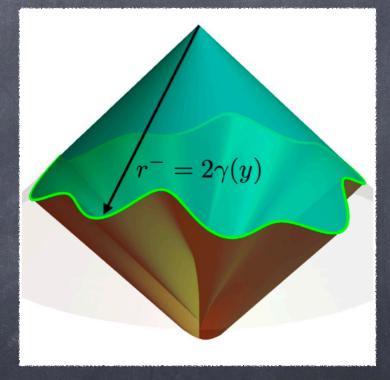
- Write the INEC in functional form:

$$\int d\Omega \left\{ \frac{\delta^2 S_{\text{out}}}{\delta\gamma(\Omega)^2}\dot{\gamma}(\Omega)^2 - \left(\frac{d-3}{d-2}\right)\theta\frac{\delta S_{\text{out}}}{\delta\gamma(\Omega)}\dot{\gamma}(\Omega) - 2\pi\dot{\gamma}(\Omega)^2\gamma(\Omega)^{d-2}\, T_{\lambda\lambda}[\gamma(\Omega),\Omega] \right\} \le 0$$

$$\int d\Omega \left\{ \frac{\delta^2 S_{\text{out}}}{\delta\gamma(\Omega)^2}\dot{\gamma}(\Omega)^2 - \left(\frac{d-3}{\gamma(\Omega)}\right)\frac{\delta S_{\text{out}}}{\delta\gamma(y)}\dot{\gamma}^2(\Omega) - 2\pi\dot{\gamma}(\Omega)^2\gamma(\Omega)^{d-2}\, T_{\lambda\lambda}[\gamma(\Omega),\Omega] \right\} \le 0$$

WiP- The sphere case

$$r^- = 2\gamma(y)$$

- Now we can differentiate twice the modular H and substitute into the INEC using $\Delta H_\gamma = \Delta S_{rel} + \Delta S$ and $S''_{rel} \geq 0$

- Resulting in
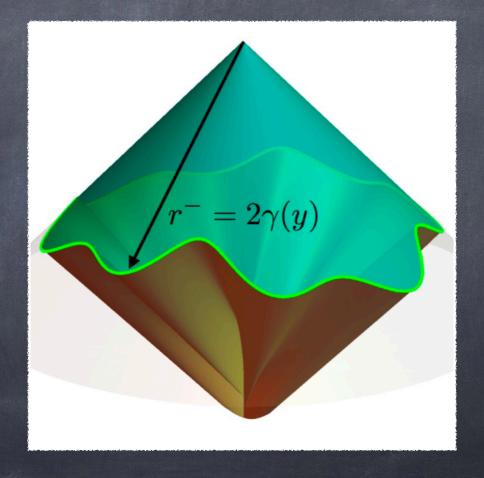
$$\frac{\delta^2 S_{\text{out}}}{\delta\gamma^2}\dot{\gamma}^2 - \left(\frac{d-3}{d-2}\right)\theta\frac{\delta S_{\text{out}}}{\delta\gamma}\dot{\gamma} - 2\pi\dot{\gamma}^2\gamma^{d-2}\,T_{\lambda\lambda}[\gamma,\Omega] \leq -\frac{2\pi(d-1)}{\gamma^3}\dot{\gamma}^2\int_0^\gamma d\lambda\,\lambda^d\,T_{\lambda\lambda}(\lambda,\Omega) + \left(\frac{d-3}{d-2}\right)\theta\frac{\delta S_{\text{rel}}}{\delta\gamma}\dot{\gamma}$$

WiP- The sphere case

- If the RHS is negative =>INEC. If not, unclear, i.e. a SUFFICENT CONDITION

- $$-\frac{2\pi(d-1)}{\gamma(\Omega)^2}\int_0^{\gamma(\Omega)} d\lambda\,\lambda^d\,T_{\lambda\lambda}(\lambda,\Omega) + (d-3)\frac{\delta S_{\text{rel}}}{\delta\gamma(y)} \leq 0$$

- When $\dot\gamma \leq 0 \Rightarrow \dfrac{\delta S_{\text{rel}}}{\delta\gamma} \leq 0$

- Hence the averaged NEC implies the INEC

$$\int_0^{\gamma(\Omega)} d\lambda\,\lambda^d\,T_{\lambda\lambda}(\lambda,\Omega) \geq 0 \Rightarrow \langle T_{\mu\nu}k^\mu k^\nu\rangle \geq \frac{\hbar}{2\pi\mathscr{A}}\left(S'' - \frac{D-3}{D-2}\theta S'\right)$$

$r^- = 2\gamma(y)$

WiP-Further Implications

- Can be written in terms of geometrical quantities:

- $T_{kk} \geq \dfrac{\hbar}{2\pi\mathcal{A}}S'' + \dfrac{1}{16\pi G}\theta^2 \Leftrightarrow R_{kk} - \dfrac{1}{2}\theta^2 \geq \dfrac{4G\hbar}{\mathcal{A}}S''$

- Using the classical Raychaudhuri eq. and the definition of θ results in $S''_{gen} \leq 0$

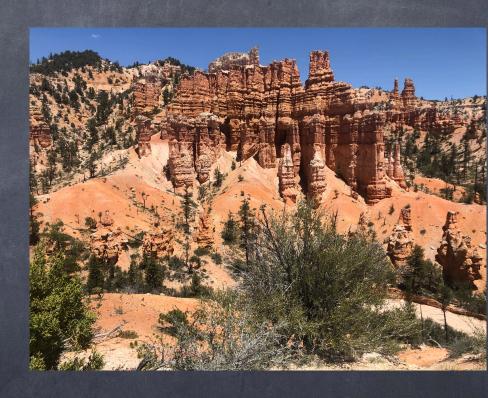- i.e. it is a maximal surface, not just extremal.

WiP-Further Implications

- Consider the QNEC as the fundamental property that always holds. What is the weakest form of the QFC?

- Some Quantum Raychaudhuri (in)equality?

- $$\Theta' \leq -\frac{1}{2}\left(\Theta^2 - \left(\frac{4G\hbar}{\mathcal{A}}\right)^2 S'^2\right)$$

- Correctly reproduces the classical limit, QNEC always holds.

Summary

- From semi-classical gravity we can derive field theory consequences in a simple manner.

- We got an energy condition that is not based on familiarity, but first principles.

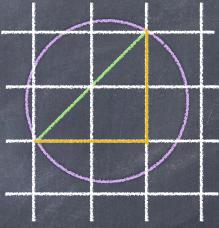- <u>TO DO:</u> Finish the Proof

- <u>TO DO:</u> Cosmological Implications

Thank You!

Simple 2D examples

- Euclidean metric $ds^2 = dx^2 + dy^2$

- Cab driver's metric $ds = |dx| + |dy|$

- Metric on a sphere – arc length $ds = Rd\theta$

$$G_{\mu\nu}(g_{\mu\nu}) = 8\pi G T_{\mu\nu}$$

General Relativity (GR)

- GR, devised by Einstein in 1915, is the accepted theory of Gravity - the weakest fundamental force known to us in Nature.

- It is a classical field theory that connects the geometry of space-time to the energy sources present.

- As a result, space-time, the Universe is dynamic. While obvious today, it was unheard of a century ago! (Einstein's CC etc.)

- The basic entity of the theory is the metric, g_{ab}, a 2nd-rank tensor that measures distances in space-time.

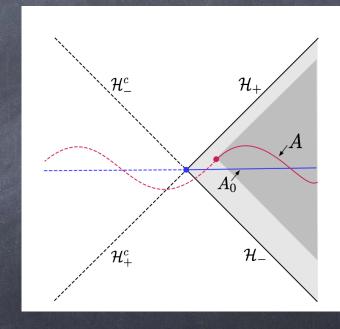- Throughout the talk I'll probably use natural units $\hbar = c = 1$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu; \; \mu = 0,1,2,3$$

Einstein's Field eq. (EFE): $\boxed{G_{\mu\nu}(g_{\mu\nu}) = 8\pi G T_{\mu\nu}}$

WiP-Sketch of proof

- In 4D Fits Casini,Teste Torroba a-theorem for vacuum 1704.01870

- $r\Delta S''(r) - \Delta S'(r) \leq 0$

- Perturbative approach for small deformations - Faulkner et al.
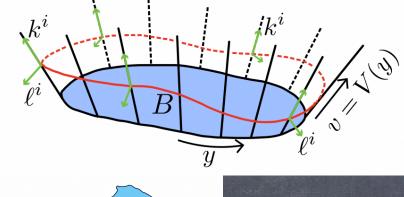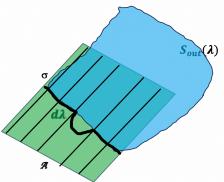
Quantum Focusing Conjecture

- Using functional derivatives

- $\Theta[V(y), y_1] \equiv \dfrac{4G\hbar}{\sqrt{V_g(y_1)}} \dfrac{\delta S_{gen}}{\delta V(y_1)}$

- $\dfrac{\delta}{\delta V(y_2)} \Theta[V(y_1), y_1] \leq 0$

- The quantum expansion Θ at one point of σ is the rate at which the generalized entropy S_{gen} changes under a small variation $d\lambda$ of σ, per cross-sectional area $\mathcal{A}$ of the variation

- The off-diagonal version $y_2 \neq y_1$ has been proven due to entropy subadditivity.

- The diagonal case of $y_2 = y_1$ is a conjecture, for simplicity, denote as $\Theta' \leq 0$, with affine parameter λ.