Measuring quantum complexity in quantum spacetime

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Lessons from Quantum Computation Theory

1 – difference in quantum vs. classical physics comes in two layers:

- quantum correlations are stronger than classical correlations and violate Bell's inequalities
- assuming $P \neq NP$, quantum physics is exponentially harder to simulate than classical physics [Gottesman 98]

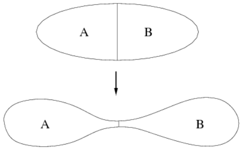
2 – there is a hierarchy in quantumness:

- highly entangled states created by quantum circuits made of Clifford gates (CNOT,S,H), stabilizer operations, can be efficiently simulated on a classical computer in polynomial time (Gottesman-Knill Theorem) [Gottesman'98] $\sim$ classical!
- set of gates beyond the Clifford group, e.g. (CNOT,H, & T), are necessary to prepare a generic state that is complex = hard to simulate classically and unlock quantum advantage [Campbell et a.'17]

> entanglement & complexity <=> deeper layer of quantumness

Entanglement resource in quantum gravity

- entanglement/geometry correspondence: from $(S \propto A)_{BH}$ [Bekenstein&Hawking] to $(S^{\text{CFT}}_{\text{ent}} \propto A_{AdS})$ bulk area scaling of holographic entanglement entropy of boundary states in AdS/CFT [Ryu&Takayanagi '06]



$$S_A(|\psi\rangle) = -\text{tr}(\rho_A \log \rho_A), \quad \text{for } |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \;\; S_A = 0$$

> entanglement as the spacetime fabric: classical geometry emergent from the hierarchy of correlations of the quantum theory on the boundary

Entanglement resource in quantum gravity

huge efforts in the last two decades

- in AdS/CFT
  - which types of entanglement have smooth geometric representations? [van Ramsdonk '10, Bianchi & Myers '12, Preskill '15, ∞]
- in LQG & non-perturbative, background independent approaches
  - featuring short-range entanglement in quantum spin networks (area law, thermal behaviour) to select quantum geometry states with "good" semiclassical behaviour

    [Livine & Terno 05-08, Girelli & Livine 06, Donnelly 08-14, Bianchi & Myers 12, GC, Rovelli 14, Livine & Charles 16-18, Bianchi &Yokomizo 15-18, Dittrich et al.14-18, GC, D.Oriti, Zhang 17-18, Bianchi, Donà 19, Colafranceschi et al. 20-22, Bianchi, Livine 23]

Entanglement is not enough

we recently realize that entanglement is not enough – open issues

- BHs barrier: a lot of the geometry remains uninterpreted in terms of quantum information
  [Susskind 14, Brown, Roberts, Susskind, Swingle and Zhao 16, Myers et al.,...]
- tension between the geometric growth of the Einstein–Rosen bridge and the early saturation of entanglement entropy > BHs interior conjectured to dually evolve as quantum chaotic system [Hartman et al14]

> quantum complexity necessary to extend the entanglement/geometry correspondence and to fully describe black hole interior dynamics [Harlow and P. Hayden13, Susskind16, Aaronson16, Stanford & Susskind14, Caputa et al 17, Jefferson & Myers 17, Chapman et al 18, Chagnet et al. 22, Policastro 22, Cao et al. 24,Leone et al 21].

Entanglement is not enough

similar situation on discrete quantum gravity models.

- flat entanglement spectrum in toy models of holographic duality based on quantum error correction codes (beside AdS/CFT): constant Rényi entropy $>$ absence of bulk fluctuations $=$ missing component in holography

$>$ need quantum complexity in the CFT to have backreaction in the bulk: non-local nonstabilizer complexity controls the level of geometric response [Cao 23, Cao, Hamma et al 24]

$>$ harnessing complexity of key importance for efficient simulation of toy models of quantum geometry on quantum computers [M.Han et al.'19, van der Meer et al. '22, Mielczarek '18-'19, Czelusta '20].

How do we measure quantum complexity?

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How do we measure quantum complexity?

- most work focused so far on quantum computational complexity – given a reference state how hard it is to construct a quantum target state within a given precision – in discrete systems and FT:
  Jefferson & Myers 17, Hackl 18, Chapman et al 18, Chagnet et al. 22]
  e.g. Nielsen's geometric approach

$$\mathcal{C}(U) = \min_{\gamma: I \to U} \int ds\, F(\dot{\gamma}(s))$$

  (geodesic length in unitary space)

our focus on qualitative approach based on quantum resource theory:

- since stabilizer quantum channels $\sim$ classical!, quantum channels that do not belong to this class are called resource operations or dynamical resources
- free states $\mathcal{F}(\mathcal{H})$ are created by the set of stabilizer quantum channels, hence elements of $S(\mathcal{H})/\mathcal{F}(\mathcal{H})$ are resource states

$>$ measure quantum complexity via **nonstabilizerness (MAGIC)**

Stabilizer resource theory for n-qubit systems

let $\mathcal{H} \simeq \mathbb{C}^{2\otimes n}$ n-qubit system; $\mathbb{P}_n$ the group of all n-qubit Pauli strings:

- operations that leave Pauli strings invariant define the normalizer of the Pauli group

$$\mathcal{C}(n) := \{C \in \mathcal{U}(n)\, , \text{ s.t. } \forall P \in \mathbb{P}_n\, , CPC^\dagger = P' \in \mathbb{P}_n\}$$

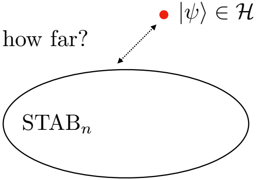
  that is a subset of the unitary group known as the Clifford group
- the Clifford group consists of unitaries generated by the circuits using Hadamard, Phase, and CNOT gates
- given a computational basis $\{|i\rangle\}$ of $\mathcal{H}$, free states are defined as the set of pure stabilizer states of $\mathcal{H}$ corresponding to the full Clifford orbit of $\{|i\rangle\}$

$$\text{STAB} = \{C\,|i\rangle\ , C \in \mathcal{C}(n)\}$$

[Veitch14]

Measuring non-stabilizerness

- measuring MAGIC amounts to quantify which resources allow me to leave the orbit



$>$ to we quantify non-stabilizerness (magic), we need a monotone function M which is

(i) faithful: $M(|\psi\rangle) = 0$ iff $(|\psi\rangle) \in \mathrm{STAB}$, otherwise $M(|\psi\rangle) > 0$;

(ii) Clifford invariant: for $C \in \mathcal{C}(n), M(C\,|\psi\rangle) = M_\alpha(|\psi\rangle)$;

(iii) additive: $M(|\psi\rangle \otimes |\phi\rangle) = M(|\psi\rangle) + M(|\phi\rangle)$

how focus on Pauli spectrum: nonstabilizerness reflects in the spread of the Pauli spectrum decomposition $>$ entropic measure do the job

Entropic measure of non-stabilizerness

- For any $|\psi\rangle \in \mathcal{H}\,(d = \dim(\mathcal{H}) = 2^n)$, take the decomposition of $\psi = |\psi\rangle\langle\psi|$, in the Pauli operator basis

$$\psi = \frac{1}{d} \sum_{P \in \mathbb{P}_n} \mathrm{Tr}(\psi P) P$$

  – the Pauli spectrum is $\mathrm{spec}(|\psi\rangle) = \{\mathrm{Tr}(P\psi) = \langle\psi| P |\psi\rangle \mid P \in \mathbb{P}_n\}$

- the empirical distribution function in operator space

$$\Pi(x) = \frac{1}{d^2} \sum_{x_P \in \mathrm{spec}(|\psi\rangle)} \delta(x - x_P)$$

  captures statistically how aligned the state is with each Pauli observable

- IPR: define the α-moment of $\Pi(x)$ as

$$\Xi^\alpha(|\psi\rangle) := d \int dx\, \Pi(x)\, x^{2\alpha} = d^{-1} \sum_{P \in \mathbb{P}_n} \mathrm{tr}^{2\alpha}(P\psi)$$

Entropic measure of non-stabilizerness

- for $\alpha = 1$, $\text{tr}^{2\alpha}(P\psi)$ gives the probability of finding P in the representation of $|\psi\rangle$

– Ξ^α large = spectrum localized in operator space

– Ξ^α small: state's Pauli expectations are broadly distributed (delocalized)

> quantify the spread by computing the Rényi entropy of $\Xi^\alpha(|\psi\rangle)$: [Leone, Oliviero, Hamma'22]

$$\mathsf{M}_\alpha(|\psi\rangle) := (1-\alpha)^{-1} \log \Xi^\alpha(|\psi\rangle)$$

– $\psi \in$ STAB it has support on a subspace of the operator space spanned by the stabilizers with exactly d elements (e.g. one qubit $d(\mathcal{H}) = 2$, $\mathcal{C}(1) = \{1, Z\}$) : $\Xi^\alpha = 1 \rightarrow M_\alpha(|\psi\rangle) = 0$

– $M_\alpha(|\psi\rangle) > 0$ otherwise (good monotone) [Leone & Bittel 24]

Stabilizer Rényi Entropy (SE)

> $M_\alpha(|\psi\rangle)$ quantifies the effective number of Pauli operators with non-negligible expectation in ψ: how spread-out the state is in the operator space

– low $M_\alpha(|\psi\rangle)$ localization in operator basis > low complexity

– high $M_\alpha(|\psi\rangle)$ delocalization > high complexity

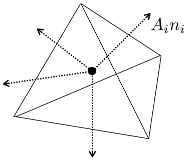
? how do we try this notion in quantum gravity? proper definition of complexity outside the traditional spin-chain formulation, in particular for quantum field theory (QFT) states in progress [Cao, White, Swingle 20,23,24]

> natural framework given by spin-network states, an orthonormal basis of the Hilbert space of loop quantum gravity [Rovelli & Smolin 95, Baez 96]

- minimal example: quantum tetrahedron – is it MAGIC free?

Minimal example: quantum tetrahedron

- consider F vectors $\vec{J_i} \in \mathbb{R}^3$ with norms $|\vec{J_i}| = j_i$ s.t. they sum up to zero. Non-coplanar normals identify a unique polyhedron [Minkowski 1897]



(e.g. $F = 4$)

- each vector live on a sphere: $\vec{J_i} \in S^2_{j_i} \subset \mathbb{R}^3$. The space of such vectors modulo rotations has the structure of a symplectic manifold [Kapovich & Millson 98]

$$\mathcal{S}_F = \left\{ n_i \in (S^2)^{\times F} | \sum_{i=1}^{N} j_i n_i = 0 \right\} / SO(3)$$

- $\mathcal{S}_F$ space of shapes of polyhedra at fixed areas (moduli space of closed F-gons in $\mathbb{R}^3$ with edge lengths j_i, modulo rotations)

Quantum F-gon = intertwiner space

quantization:

- each $S^2_{j_i}$ is quantized as the SU(2) irreducible representation $V(j_i)$, of spin j_i, dimension $2j_i + 1$.
- up to a dimensionful constant, the generators $\hat{J}_i$ of $SU(2)$ give the quantization of the vectors $\vec{J}_i$

> then $\mathcal{H}_{\text{unconstrained}} = \bigotimes_{i=1}^{F} V(j_i)$

- quantization of the closure constraint = zero total angular momentum (Gauss constraint of a SU(2) Yang-Mills)

> the physical Hilbert space is the SU(2)-invariant intertwiner space

$$\mathcal{H}_F = \text{Inv}_{\text{SU(2)}}\left[\bigotimes_{i=1}^{F} V(j_i)\right]$$

! intertwiners states in H_F are the building blocks (nodes) of $F - \textit{valent}$ quantum spin-networks in LQG (similarly in lattice gauge theory) [Baez 96]

Mapping quantum geometry to qubits system

focus on 4-valent intertwiner state $|I\rangle \in \mathcal{H}_{F=4} \equiv \mathcal{H}_I$ with all $j = 1/2$,

– as $V_{1/2} \to \mathbb{C}^2$ $|j = 1/2, \vec{m}\rangle$ spins map to qubits

! (Shur) $V_{1/2}^{\otimes 4} = \sum_J D_{1/2}^J \otimes V_J = 2V_0 \oplus 3V_1 \oplus V_2$

- the intertwiner space $\mathcal{H}_I$ corresponds to the 2dim degeneracy space $D_{1/2}^0$, which again maps to $\mathbb{C}^2$

> think of $|I\rangle \in \mathcal{H}_I := \mathrm{Inv}_{\mathrm{SU}(2)}[\mathcal{H}_{1/2}^{\otimes 4}]$ as a logical intertwiner qubit

- given the basis $\{|0\rangle_s, |1\rangle_s\} \in \mathcal{H}_I$, the generic LIQ state reads

$$|I\rangle = \cos\frac{\theta}{2}|0_s\rangle + \sin\frac{\theta}{2}e^{i\phi}|1_s\rangle$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ angles on the Bloch sphere

Quantum tetrahedron as a logical qubit

- can represent $|I\rangle \in \mathcal{H}_I$ both in the logical basis $\{|0_s\rangle, |1_s\rangle\} \in \mathcal{H}_I$ and in the computational basis $\{|0\rangle, |1\rangle\}^{\otimes 4}$ of the 4-qubits space $V_{1/2}^{\otimes 4}$
- use the definition of intertwiner state as recoupling of four spin-j

$$|I\rangle = N \sum_{K=0}^{2j} \sum_{M=-K}^{K} \sum_{\{\vec{m}\}} C^{K,M}_{jm_1 jm_2} C^{K,-M}_{jm_3 jm_4} |j, \vec{m}\rangle$$

to express the LIQ basis in terms of the computational basis

$$|0_s\rangle = \frac{1}{2}\left(|0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle\right)$$

$$|1_s\rangle = \frac{1}{\sqrt{3}}\left[|0011\rangle + |1100\rangle - \frac{1}{2}\left(|0101\rangle + |1010\rangle + |0110\rangle + |1001\rangle\right)\right]$$

Q: making $|I\rangle$ from $\mathcal{H}_{1/2}^{\otimes 4}$ requires non-stabilizer resources?

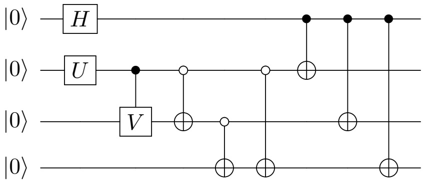
Quantum tetrahedron as a logical qubit

goal: investigate non-stabilizerness of the logic basis states

- start from the reference state $|0\rangle^{\otimes 4}$ in the 4-qubit Hilbert space and look for the unitary transformations such that

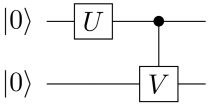
$$|0_s\rangle = \mathcal{U}_{0_s}|0\rangle^{\otimes 4} \qquad |1_s\rangle = \mathcal{U}_{1_s}|0\rangle^{\otimes 4}$$

- express the unitary operators $\mathcal{U}_{0_s}$ and $\mathcal{U}_{1_s}$ in terms of a set of unitary gates acting on the reference state
- for the generic intertwiner state, the associated circuit reads [Han'19, Czelusta et al.'20]



Quantum intertwiner circuit

- reduced 2-qubit system: $U(\theta,\phi), V(\theta,\phi) \in SU(2)$ are the only non-Clifford gates



$|0_s\rangle$ for $(U_0, V_0, \theta = 0, \phi = 0)$ the reduced 2-qubit system given by the action of $CV_0\,(U_0 \otimes 1)$ on $|0\rangle^{\otimes 2}$

$$|0_s\rangle_2 = \frac{1}{\sqrt{2}}\left(|10\rangle - |11\rangle\right)$$

- the state $\psi_{0_s} = |0_s\rangle_2 \langle 0_s|_2$ expressed in the Pauli basis reads

$$\psi_{0_s} = \frac{1}{d}\sum_{P\in P_2} \mathrm{tr}(\psi_{0_s}P)P = \frac{1}{4}\left(1\otimes 1 - 1\otimes \mathrm{X} - \mathrm{Z}\otimes 1 + \mathrm{Z}\otimes \mathrm{X}\right)$$

Stabilizer 2-Rényi Entropy of the logical basis

- computing the Stabilizer 2-Rényi Entropy of $|0_s\rangle$ we get

$$M_2(|0_s\rangle) = -\log d^{-1} \sum_{P\in\mathbb{P}_2} \mathrm{tr}^4(\psi_{0_s} P) = 0 \quad => \quad |0_s\rangle \text{ is STAB}$$

$|1_s\rangle$ for $(U_1, V_1, \theta = 0, \phi = \pi)$ the reduced 2-qubit system returns

$$|1_s\rangle_2 = \sqrt{\frac{2}{3}}\,|00\rangle - \frac{1}{\sqrt{6}}\,|10\rangle - \frac{1}{\sqrt{6}}\,|11\rangle$$

- the state ψ_{1_s} in the Pauli basis (10 non-vanishing terms)

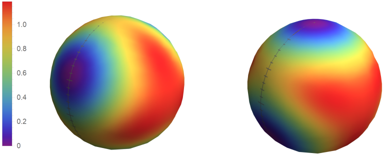
$$\psi_{1_s} = \frac{1}{4}\Big(1\otimes 1 + \frac{1}{3}1\otimes \mathrm{X} + \frac{2}{3}1\otimes \mathrm{Z} - \frac{2}{3}\mathrm{X}\otimes 1 - \frac{2}{3}\mathrm{X}\otimes \mathrm{X} - \frac{2}{3}\mathrm{X}\otimes \mathrm{Z} + \frac{2}{3}\mathrm{Y}\otimes \mathrm{Y} + \frac{1}{3}\mathrm{Z}\otimes 1 - \frac{1}{3}\mathrm{Z}\otimes \mathrm{X} + \frac{2}{3}\mathrm{Z}\otimes \mathrm{Z}\Big)$$

- we find $M_2(\psi_{1_s}) = 0,847997$: $|1_s\rangle$ needs NSTAB resources

> $|I\rangle$ not a tensor product state: both entanglement and magic are necessary for the gauge structure

Maximal SE at volume eigenstates

- by varying (θ, ϕ), compute the SE of the generic intertwiner



!! SE has two maxima $\mathrm{M}_2(|V_\pm\rangle) = 1.16993$ corresponding to

$$|V_+\rangle = \frac{1}{\sqrt{2}}(|0_s\rangle - i\,|1_s\rangle), \quad |V_-\rangle = \frac{1}{\sqrt{2}}(|0_s\rangle + i\,|1_s\rangle$$

- the eigenstates of the LQG volume operator on the Bloch sphere placed at the equator with angular coordinates $(\theta = \frac{\pi}{2}; \phi = \frac{\pi}{2}, \phi = \frac{3\pi}{2})$.

Averaged NSTAB of the intertwiner subspace

Q what is the NSTAB gap of the intertwiner subspace $\mathcal{H}_G$?

- a meaningful answer requires averaging (linearized) SE_2 in $\mathcal{H}_G = \Pi_G \mathcal{H}_{\text{tot}}$

$$M^i := \mathbb{E}_{U_G} M^i_{\text{lin}}(\psi_{U_G}) = 1 - d_i^{-1} \sum_{P \in \mathbb{P}_n} \operatorname{tr}\left(P^{\otimes 4}\, \mathbb{E}_{U_G} \psi_{U_G}^{\otimes 4}\right)$$

  with $i = \{0, G\}$ labelling register and G-inv space, $\mathbb{E}_{U_G}$ the unitary group average with respect to the Haar measure over $\mathcal{H}_G$ and $\psi_{U_G} := U_G \psi U_G^\dagger$

- we find (lemma) $\mathbb{E}_{U_G} \psi_{U_G}^{\otimes k} = c_G(d, d_G)\, \Pi_G{}^{\otimes k} \mathbb{E}_U \psi_U^{\otimes k}$, then

$$M^i = 1 - c_i(d, d_i)\, d_i^{-1} \sum_{P \in \mathbb{P}_n} \operatorname{tr}\left[P^{\otimes 4} \Pi_G{}^{\otimes 4} \mathbb{E}_U \psi_U^{\otimes 4}\right]$$

- define *SE-gap* $\Delta M(\mathcal{H}_G) := M^{(0)} - M^{(G)}$ find $\Delta M(\mathcal{H}_I) = 8/45$

Averaged NSTAB of the intertwiner subspace

> $\Delta M(\mathcal{H}_I) > 0$: projecting a generic 4-qubit state onto the gauge invariant subspace has a cost in terms of non-stabilizer resources

!! $\Delta M(\mathcal{H}_I) > 0$: non-stabilizer resource is an intrinsic feature of the quantum geometry state – and gauge reduction more generally

- coming back to quantum computing:

!! $M(\mathcal{H}_I) > 0$ also reflects in the computational complexity in simulating such states:

> in the experimental realization ($\tilde{\psi}$) of a state $|\psi\rangle$ (e.g.[Czelusta'21], IBM superconducting quantum computer)

$$N_{\tilde{\psi}} \geq \frac{2}{\epsilon^2} \ln\left(\frac{2}{\delta}\right) \exp[M_2(|\psi\rangle)]; \quad \mathcal{F}_{\max} \leq 1 - \epsilon$$

the minimum number of preparations needed to achieve a desired value of fidelity within a desired accuracy ϵ, is bounded from below by SE [Leone et al. '23]

Summary of results

– Interpretative impact

- constructing the quantum tetrahedron out of a collection of four qubits – general gauge invariant states in LGT – inherently requires non-stabilizer resources (MAGIC)
- SE quantifies MAGIC and is easy to compute

- eigenstates of the oriented volume have near-maximal amount of SE:

> possible magic/geometry correspondence in these states to be investigated gravity extends at a deeper layer of quantumness

– Experimental impact

- non-stabilizer resources reflect in the computational complexity of a simulation of quantum gravity states (amplitudes): M_2 sets bounds on fidelity

> harnessing complexity necessary to efficient simulations of quantum geometry states running on a quantum computer

Ongoing directions

- extend to qudits with Heisenberg-Weyl group for $d-$level systems (see e.g. [Wang '23]) to include generic intertwiners

> check whether for a generic intertwiner the volume eigenstates are states with maximum SE ($\sim>$ complexity=volume [Susskind'16])

- extend to collection of intertwiners (spin network states):

> expect extra non-stabilizer resources coming from graph connectivity and non-trivial holonomies dressing the links

> is NSTAB complexity extensive?

- measure SE of quantum geometry amplitudes: e.g. quantum 6j-symbol (path integral of topological field theory, low dim gravity,...)

$$\psi_\Gamma(\{g_e\}) = \langle 0|^{\otimes 4n} \mathcal{U}_{\mathcal{E}}^\dagger(\{g_e\}) \mathcal{U}_I |0\rangle^{\otimes 4n}$$

Thank You

faithful measure of magic, what for?

- essential resource to achieve universal quantum computation
- the non-stabilizing power - how much stabilizer entropy a unitary operator can achieve - of a quantum evolution can be cast in terms of out-of-time-order correlation functions (OTOCs) and that is thus a necessary ingredient of quantum chaos
  [Leone, Oliviero, Hamma'22, Goto et al '22]
- in AdS/CFT, classical spacetime emerges from the chaotic nature of the dual quantum system: magic of dual quantum sys strongly involved in the emergence of spacetime geometry
  [White, Cao, Swingle 21, Hamma, Cao '23]

> what is the role of this second layer of quantumness on other fields of physics (quantum space, gauge field theory, particle physics, etc...)?
[GC et al '24, Esposito et '24 Savage et al'24]

Stabilizer Resource Theory

- the Clifford group on n qubits is generated by $\{H, S, CNOT\}$:
  - Hadamard H – swaps $X \leftrightarrow Z$: $HXH^\dagger = Z, \quad HZH^\dagger = X$
  - Phase S – rotates $X \mapsto Y$, leaves Z invariant: $SXS^\dagger = Y$, $SZS^\dagger = Z$ where $S = \mathrm{diag}(1, i)$
  - for two qubits, the CNOT acts on qubits 1 (control) and 2 (target), transforming Pauli operators according to:

$$\mathrm{CNOT}(X \otimes I)\mathrm{CNOT}^\dagger = X \otimes X, \quad \mathrm{CNOT}(I \otimes X)\mathrm{CNOT}^\dagger = I \otimes X,$$

$$\text{CNOT } (\text{Z } \otimes I)\mathrm{CNOT}^\dagger = Z \otimes I, \quad \mathrm{CNOT}(I \otimes Z)\mathrm{CNOT}^\dagger = Z \otimes Z$$

- $\mathcal{C}(2^n) \cup$ Measurements defines the set of free operations of the stabilizer QRT
- the set of free states of $\mathcal{H}$ $\mathcal{F}(\mathcal{H})$ is the set of n-qubit stabilizer states $STAB(n)$ that is the full Clifford orbit of $\{|i\rangle\}$

$$\mathrm{STAB}(n) = \{C\,|i\rangle\ , C \in \mathcal{C}_n\} \qquad \text{[Veitch14]}$$

Stabilizer Formalism

- let $\mathcal{H} \simeq \mathbb{C}^{2 \otimes n}$ a n-qubit system and $\mathbb{P}_n$ the group of Pauli strings on $\mathcal{H}$.

Def the Pauli group on one qubit:

$$\mathbb{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\},$$

$$\text{where} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- the n-qubit Pauli group is the n-fold tensor product single-qubit Pauli operator, with phases in $\{\pm 1, \pm i\}$:

$$\mathbb{P}_n = \{\alpha P_1 \otimes \cdots \otimes P_n : P_k \in \{I, X, Y, Z\}, \alpha \in \{\pm 1, \pm i\}\}.$$

Def Suppose S is a subgroup of $\mathbb{P}_n$, and define V_S to be the set of n qubit states for which $\forall \left|\psi\right\rangle \in V_S, \forall P_S \in S,\ P_S \left|\psi\right\rangle = \left|\psi\right\rangle$

- V_S is defined as the vector space stabilized by S: elements of V_S are stabilizer states of S and S is said to be the stabilizer group of V_S.

Stabilizerness & Clifford Group

- apply a unitary operator U to a space V_S stabilized by S. For any $|\psi\rangle \in V_S$ and g,

$$U|\psi\rangle = Ug|\psi\rangle = (UgU^\dagger)U|\psi\rangle$$

the space UV_S is stabilized by the group $USU^\dagger := \{UgU^\dagger, g \in S\}$

! it is not guaranteed that $Ug_1U^\dagger, \ldots, Ug_lU^\dagger$ are still Pauli strings (generally any linear combination of Pauli strings)

Def (Normalizer of a subset of a group). Given a group G and a subset S of said group the normalizer $N_G(S)$ is defined as

$$N_G(S) := \{U \in G |\ UgU^{-1} \in G, \forall g \in G\}$$

- the normalizer of the Pauli group is a subset of the unitary group also referred to as the Clifford group $\mathcal{C}(n)$ on n-qubit systems

$$\mathcal{C}(n) := \{C \in \mathcal{U}(n)\,,\ \text{s.t.}\ \ \forall P \in \mathbb{P}_n\,,\, CPC^\dagger = P' \in \mathbb{P}_n\}$$

Logical Qubit & Intertwiners

Define the logical qubit encoded in the invariant (singlet) subspace of four spin-$\frac{1}{2}$ representations, i.e.

$$\mathsf{Inv}_{SU(2)}\left[\,(\mathcal{H}_{1/2})^{\otimes 4}\,\right].$$

- Each $\mathcal{H}_{1/2} \cong \mathbb{C}^2$ carries the spin-$\frac{1}{2}$ representation of $SU(2)$.
- The tensor product of 4 qubits decomposes into irreps of $SU(2)$.
- We want the invariant subspace:

$$\mathrm{Inv}_{SU(2)}\left[(\tfrac{1}{2})^{\otimes 4}\right] = \{\, |\psi\rangle \in (\mathbb{C}^2)^{\otimes 4} \mid U^{\otimes 4}|\psi\rangle = |\psi\rangle,\ \forall U \in SU(2)\,\}.$$

This is also called the intertwiner space, since it consists of $SU(2)$-intertwiners mapping the trivial rep into $(\frac{1}{2})^{\otimes 4}$.

Decomposition of $(\frac{1}{2})^{\otimes 4}$

- The Clebsch–Gordan series: $(\frac{1}{2}) \otimes (\frac{1}{2}) = 0 \oplus 1$.
- for four spins: $(\frac{1}{2})^{\otimes 4} = (0 \oplus 1) \otimes (0 \oplus 1)$
- Expanding $= (0 \otimes 0) \ \oplus \ (0 \otimes 1) \ \oplus \ (1 \otimes 0) \ \oplus \ (1 \otimes 1)$. with
  - $0 \otimes 0 = 0$.
  - $0 \otimes 1 = 1$.
  - $1 \otimes 0 = 1$.
  - $1 \otimes 1 = 0 \oplus 1 \oplus 2$.
- So total decomposition:$(\frac{1}{2})^{\otimes 4} = 2 \cdot 0 \ \oplus \ 3 \cdot 1 \ \oplus \ 2$.
- The trivial representation $j = 0$ appears twice. Then

$$\dim \mathrm{Inv}_{SU(2)} \left[(\tfrac{1}{2})^{\otimes 4} \right] = 2.$$

- The degeneracy space is a two-dimensional space, i.e. a logical qubit.

Basis for the intertwiner qubit

There are several natural bases. One convenient way is to pair spins:

(a) Pairing (12)(34):
Couple qubits 1-2 into a singlet or triplet.
Couple qubits 3-4 into a singlet or triplet.
Then fuse the two pairs.

- Two independent singlet states are:

1 Singlet–singlet state

$$|\mathcal{I}_0\rangle = |\psi^-\rangle_{12} \otimes |\psi^-\rangle_{34},$$

where $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.

2 Triplet-triplet singlet state

$$|\mathcal{I}_1\rangle = \frac{1}{\sqrt{3}}\big(|t_+\rangle_{12}\,|t_-\rangle_{34} + |t_-\rangle_{12}\,|t_+\rangle_{34} - |t_0\rangle_{12}\,|t_0\rangle_{34}\big),$$

where $|t_+\rangle = |00\rangle$, $|t_-\rangle = |11\rangle$, $|t_0\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$.

- These two are orthonormal and span the invariant subspace.

Logical Qubit Structure

- We can interpret this invariant subspace as encoding a logical qubit:
  - Define $|0_L\rangle = |\mathcal{I}_0\rangle$,
  - Define $|1_L\rangle = |\mathcal{I}_1\rangle$.
- Any linear combination $\alpha|0_L\rangle + \beta|1_L\rangle$ is invariant under global $SU(2)$ action.
- Thus this space is robust against global $SU(2)$ rotations – a natural **decoherence-free subspace** and also the intertwiner qubit used in spin networks and loop quantum gravity.
- The logical intertwiner qubit in $\mathrm{Inv}_{SU(2)}[(\mathcal{H}_{1/2})^{\otimes 4}]$ is the two-dimensional singlet (invariant) subspace of four spins-1/2. A natural basis is

$$|0_L\rangle = |\psi^-\rangle_{12} \otimes |\psi^-\rangle_{34},$$

$$|1_L\rangle = \frac{1}{\sqrt{3}}\big(|t_+\rangle_{12}|t_-\rangle_{34} + |t_-\rangle_{12}|t_+\rangle_{34} - |t_0\rangle_{12}|t_0\rangle_{34}\big),$$

which together span the intertwiner qubit