Generalized Palatini formalism

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Generalized geometry

A generalized metric on a CA is an involution $\tau \in \mathrm{Aut}(E)$ s.t. $\tau^2 = \hat{1}$ and

$$\mathbf{G}(\cdot\,,\,\cdot) := \langle\,\cdot\,,\,\tau(\cdot)\,\rangle_E \;\geq 0 \tag{1}$$

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A generalized connection is $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ s. t.:

$$\nabla_{f\phi}\psi = f\nabla_\phi\psi \qquad ; \qquad \nabla_\phi f\psi = f\nabla_\phi\psi + \rho(\phi)f\,\psi \tag{2}$$

$$\rho(\phi)\left\langle \psi, \psi' \right\rangle_E = \left\langle \nabla_\phi\psi, \psi' \right\rangle_E + \left\langle \psi, \nabla_\phi\psi' \right\rangle_E \tag{3}$$

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Divergence $\mathrm{div}_\nabla : \Gamma(E) \to C^\infty(M)$:

$$\mathrm{div}_\nabla\psi := \left\langle \nabla_{e_i}\psi \, , \, e^i_E \right\rangle_E \tag{4}$$

Generalized geometry

Torsion 3-form [Gualtieri, 2007]:

$$T_\nabla(\psi, \psi', \phi) = \left\langle \nabla_\psi \psi' - \nabla_{\psi'} \psi - [\psi, \psi']_E, \phi \right\rangle_E + \left\langle \nabla_\phi \psi', \psi \right\rangle_E \quad (5)$$

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A generalized Levi-Civita connection ∇ is s.t.:

$$\nabla \mathbf{G} = 0 \quad (6)$$

$$T_{\nabla} = 0 \quad (7)$$

Leading to:

$$\mathrm{LC}(E, \mathbf{G}) = \{ \text{ connections } \nabla \text{ on E } | \mathbf{G} = 0\ ;\ T_{\nabla} = 0 \} \quad (8)$$

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Fixing a divergence of ∇ then yields:

$$\mathrm{LC}(E, \mathbf{G}, \mathrm{div}) = \{ \text{ connections } \nabla \text{ on E } | \mathbf{G} = 0 \; ; \; T_\nabla = 0 \; ; \; \mathrm{div}_\nabla = \mathrm{div} \} \quad (9)$$

Generalized curvature tensor $R \in \mathcal{T}_4^0(E)$:

$$R(\phi', \phi, \psi, \psi') = \frac{1}{2}\left[R^0(\phi', \phi, \psi, \psi') + R^0(\psi', \psi, \phi, \phi') + \right. \\ \left. + \left\langle \nabla_{e_i}\psi, \psi' \right\rangle_E \left\langle \nabla_{e^i}\phi, \phi' \right\rangle_E \right] \tag{10}$$

where: $R^0(\phi', \phi, \psi, \psi') := \left\langle \phi', \nabla_\psi \nabla_{\psi'}\phi - \nabla_{\psi'}\nabla_\psi \phi - \nabla_{[\psi,\psi']_E}\phi \right\rangle_E$

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$$\mathrm{Ric}(\psi, \psi') = R(e_E^i, \psi, \psi', e_i) \tag{11}$$

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Generalized Ricci tensor $\mathrm{Ric} \in \mathcal{T}^0_2(E)$:

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Scalar curvatures:

$$\mathbf{R}_E = \mathrm{Ric}(e^i_E, e_i) \quad ; \quad \mathbf{R}_{\mathbf{G}} = \mathrm{Ric}(\mathbf{G}^{-1}(e^i), e_i) \tag{12}$$

Ricci compatibility of the connection ∇ with the generalized metric $\mathbf{G} \leftrightarrow \tau$:

$$\mathrm{Ric}(V_+, V_-) = 0 \tag{13}$$

Here V_+ and V_- are the $+1$ and -1 eigenbundles of τ.

Generalized Palatini action

Generalized Palatini action on a CA $E \to M$ [Moučka, Vysoký & Jurčo, 2023]:

$$S_{Pal.}[\mathbf{G}, \nabla, \omega] = \int_M (\, \mathbf{R}_E + \mathbf{R}_{\mathbf{G}} \,)\, \omega \tag{14}$$

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Extremalized iff:

- ∇ is Ricci compatible with $\mathbf{G}$
- $\mathbf{R}_E + \mathbf{R}_\mathbf{G} = 0$
- ∇ is s.t.: **(1.)** $\nabla \mathbf{G} = 0$
  (2.) $T_\nabla = 0$
  (3.) $\mathrm{div}_\nabla = \mathrm{div}_\omega$

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Classical vs. generalized setting

Classical setting	Generalized setting
Pairs ∇, g extremalizing $S_{Pal.}[\,g\,,\,\nabla\,] = \int_M g^{\mu\nu} R_{\mu\nu}(\nabla)\,\omega_g$ where: $T_\nabla = 0$ or $\nabla g = 0$	

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where: $T_\nabla = 0$ or $\nabla g = 0$	where: *all the fields are arbitrary*
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Classical vs. generalized setting

Classical setting

Pairs ∇, g extremalizing

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Euler-Lagrange equations:

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Tripples $\mathbf{G}, \nabla, \omega$ extremalizing

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Euler-Lagrange equations:

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- $\nabla \in \mathrm{LC}(E, \mathbf{G}, \mathrm{div}_\omega)$

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Generalized Palatini formalism and heterotic reductions

An exact CA E over the total space P of a principal G-bundle $\pi : P \to M$ may be reduced to a heterotic CA E' over M [Bursztyn, Cavalcanti & Gualtieri, 2007] [Jurčo & Vysoký, 2016].

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Reduction-relevant tripples $\mathbf{G}, \nabla, \omega$ on E induce $\mathbf{G}', \nabla', \omega'$ on E' [Vysoký, 2017].

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Does the generalized Palatini formalism commute with heterotic reductions ?

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