

Generalized Palatini formalism

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- 2 Classical vs. Generalized setting
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Such that for all $\phi, \psi, \psi' \in \Gamma(E)$, $f \in C^\infty(M)$:

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Generalized geometry

A **generalized metric** on a CA is an involution $\tau \in \text{Aut}(E)$ s.t.
 $\tau^2 = \hat{1}$ and

$$\mathbf{G}(\cdot, \cdot) := \langle \cdot, \tau(\cdot) \rangle_E \geq 0 \quad (1)$$

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A **generalized connection** is $\nabla : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ s. t.:

$$\nabla_{f\phi}\psi = f\nabla_\phi\psi \quad ; \quad \nabla_\phi f\psi = f\nabla_\phi\psi + \rho(\phi)f\psi \quad (2)$$

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Divergence $\text{div}_\nabla : \Gamma(E) \rightarrow C^\infty(M)$:

$$\text{div}_\nabla\psi := \langle \nabla_{e_i}\psi, e_E^i \rangle_E \quad (4)$$

Generalized geometry

Torsion 3-form [Gualtieri, 2007]:

$$T_{\nabla}(\psi, \psi', \phi) = \langle \nabla_{\psi}\psi' - \nabla_{\psi'}\psi - [\psi, \psi']_E, \phi \rangle_E + \langle \nabla_{\phi}\psi', \psi \rangle_E \quad (5)$$

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A generalized Levi-Civita connection ∇ is s.t.:

$$\nabla \mathbf{G} = 0 \quad (6)$$

$$T_{\nabla} = 0 \quad (7)$$

Leading to:

$$\text{LC}(E, \mathbf{G}) = \{ \text{ connections } \nabla \text{ on } E \mid \mathbf{G} = 0 ; T_{\nabla} = 0 \} \quad (8)$$

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Fixing a divergence of ∇ then yields:

$$\text{LC}(E, \mathbf{G}, \text{div}) = \{ \text{ connections } \nabla \text{ on } E \mid \mathbf{G} = 0 ; T_{\nabla} = 0 ; \text{div}_{\nabla} = \text{div} \} \quad (9)$$

Generalized geometry

Generalized curvature tensor $R \in \mathcal{T}_4^0(E)$:

$$\begin{aligned} R(\phi', \phi, \psi, \psi') = & \frac{1}{2} [R^0(\phi', \phi, \psi, \psi') + R^0(\psi', \psi, \phi, \phi')] + \\ & + \langle \nabla_{e_i} \psi, \psi' \rangle_E \langle \nabla_{e^i} \phi, \phi' \rangle_E \end{aligned} \tag{10}$$

where: $R^0(\phi', \phi, \psi, \psi') := \langle \phi', \nabla_\psi \nabla_{\psi'} \phi - \nabla_{\psi'} \nabla_\psi \phi - \nabla_{[\psi, \psi']} \phi \rangle_E$

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Scalar curvatures:

$$\mathbf{R}_E = \text{Ric}(e_E^i, e_i) \quad ; \quad \mathbf{R}_{\mathbf{G}} = \text{Ric}(\mathbf{G}^{-1}(e^i), e_i) \tag{12}$$

Ricci compatibility of the connection ∇ with the generalized metric $\mathbf{G} \leftrightarrow \tau$:

$$\text{Ric}(V_+, V_-) = 0 \quad (13)$$

Here V_+ and V_- are the $+1$ and -1 eigenbundles of τ .

Generalized Palatini action

Generalized Palatini action on a CA $E \rightarrow M$ [Moučka, Vysoký & Jurčo, 2023]:

$$S_{Pal.}[\mathbf{G}, \nabla, \omega] = \int_M (\mathbf{R}_E + \mathbf{R}_{\mathbf{G}}) \omega \quad (14)$$

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Extremalized iff:

- ∇ is Ricci compatible with \mathbf{G}
- $\mathbf{R}_E + \mathbf{R}_{\mathbf{G}} = 0$
- ∇ is s.t.:
 - (1.) $\nabla \mathbf{G} = 0$
 - (2.) $T_{\nabla} = 0$
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Classical vs. generalized setting

Classical setting	Generalized setting
<p>Pairs ∇, g extremalizing</p> $S_{Pal.}[g, \nabla] = \int_M g^{\mu\nu} R_{\mu\nu}(\nabla) \omega_g$ <p>where:</p> $T_\nabla = 0 \quad \text{or} \quad \nabla g = 0$	

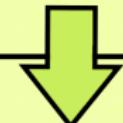
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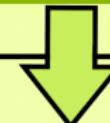
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Reduction-relevant triples $\mathbf{G}, \nabla, \omega$ on E induce $\mathbf{G}', \nabla', \omega'$ on E' [Vysoký, 2017].

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Does the generalized Palatini formalism commute with heterotic reductions ?

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