# Supergravity breaking on Bieberbach manifolds

Based on arXiv:2507.02339, with Gianguido Dall'Agata
(overlap with Bento-Montero arXiv:2507.02037)

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Workshop on Quantum Gravity and Strings
EISA, Corfu, 10 September 2025

# Motivation and programme

Two outstanding unsolved hierarchy problems

$$G_F^{-1/2} \sim 10^{-16}\, M_P \quad \& \quad \rho_V \sim (10^{-30}\, M_P)^4$$

might be related with (broken) supersymmetry

- SUSY at the Fermi scale? Not borne out by experiment
- D=4 no-scale sugra models? UV corrections out of control

Motivated by superstrings, look at D+d supergravities broken (classically) by compactifications to D=4 Minkowski, explore spectrum & perturbative stability when the 1-loop potential $V_1$ is fully calculable (exponentially suppressed string corrections)

# Why sugra and not strings directly?

Many string constructions exist with fully broken susy

both at the string ($M_S$) and at the compactification ($M_C$) scale

where spectrum and perturbative stability were investigated*

- D+d sugra good EFT for $M_C << M_S$ (in contrast with D=4 reduced/truncated theory)
- Can avoid many complications with localized susy-breaking defects (orbifold fixed points, D-branes, O-planes, …)
- Can deal with d=7 perturbatively (d+D=11 sugra)
- Helps distinguishing string vs sugra properties and better identifying the reasons behind 1-loop UV behaviour
- Can suggest new string compactifications

## Incomplete list of references to the relevant string literature

- Susy breaking at $M_S$: Dixon-Harvey 86, AlvarezGaume-Ginsparg-Moore-Vafa 86, …
- Susy braking at $M_C$: Rohm 84, Ferrara-Kounnas-Porrati 88, Kounnas-Porrati 88, Ferrara-Kounnas-Porrrati-FZ 89, …
- Perturbative stability and string corrections: Itoyama-Taylor 87, Antoniadis 90, …
- Super no-scale models: Kounnas-Partouche 16, … (and many here)
- Misaligned supersymmetry: Dienes 94, …

# A well known toy example

[Scherk-Schwarz 1979, …, Dall'Agata-FZ, arXiv:2401.02480]

5-dim pure supergravity on $S^1$ with Scherk-Schwarz twists

$$m^2_{n,\alpha} = \frac{(n+s_\alpha)^2}{R^2}$$

n is KK level, R is $S^1$ radius

## D=4 reduced/truncated theory

$V_{1,red}$ divergent (finite) for N≤4 (N>4)

N=8: Str $M_0^{0,2,4,6} = 0$ , Str $M_0^8 > 0$

## Full D+d=5 compactified theory

$$\mathrm{Str}\,\mathcal{M}^p_n \equiv \sum_\alpha (-1)^{2J_\alpha}\,(2J_\alpha+1)\,m^p_{n,\alpha}$$

independent of n=0,1,…for any p

Non-local susy breaking => $V_1$ finite for N>0 [Rohm 1984]

Effective UV cutoff = compactification scale (1/R)

# Bieberbach manifolds

Bieberbach manifold $M_d$: freely acting quotient of $R^d$ with co-compact torsion-free discrete group $\boldsymbol{\Gamma} \subset O(d) \ltimes R^d$

Must be orientable to allow for a spin structure

$$M_d = \mathbb{R}^d/\Gamma \quad \Gamma \subset \mathrm{SO}(d) \ltimes \mathbb{R}^d$$

$$\Gamma \ni \gamma = (R, \vec{b}) \quad R \in \mathrm{SO}(d) \quad \vec{b} \in \mathbb{R}^d$$

$$\gamma = \begin{pmatrix} R & \vec{b} \\ 0 & 1 \end{pmatrix}$$

Can construct the lattice $\boldsymbol{\Lambda} = \boldsymbol{\Gamma} \cap \mathbb{R}^d$ basis $e_a = (\mathbf{1}, \vec{e}_a)$ and the dual

$$\Lambda^* \ni \vec{k}^* = n_a \vec{e}_a^{\,*} \quad n_a \in \mathbb{Z}$$

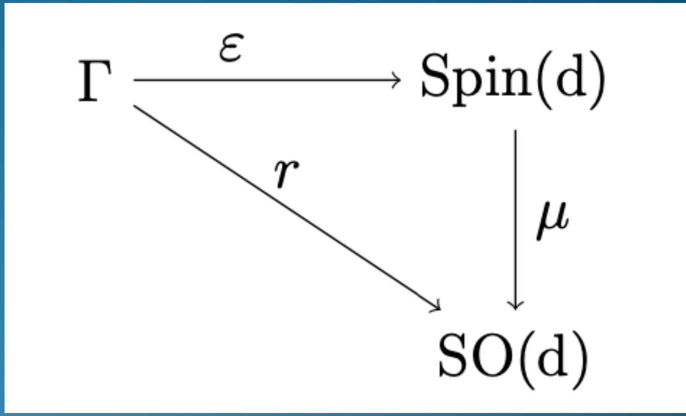
# Spin Bieberbach manifolds: orientable and more

## Consistency condition

$$\varepsilon(e_a) = \delta_a \mathbf{1} \quad \delta_a = \pm 1$$

$$\psi(\vec{y} + \vec{e}_a) = \delta_a \psi(\vec{y})$$

$$\psi(\gamma(y)) = \varepsilon(\gamma)\psi(y)$$




Each manifold can admit zero, one or multiple spin structures (algorithmic approach in Lutowski-Putrycz arXiv:1411.7799)

d=3: 6 orientable, 6 spin

d=4: 27 orientable, 24 spin

d=5: 174 orientable, 88 spin

d=6: 3314 orientable, 760 spin

d=7: subclasses known but classification missing

# Example 1 (d=3): $\mathbf{T^3/Z_3}$

$$\alpha = \begin{pmatrix} R_\alpha & \vec{b}_\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} & 0 & 0 \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{L_3}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 & 0 & 0 & -\frac{L}{2} \\ 0 & 1 & 0 & \frac{\sqrt{3}\,L}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Two geometrical moduli: $L$, $L_3$

Two spin structures allowed: $\boldsymbol{\delta}_1=\boldsymbol{\delta}_2=1$, $\boldsymbol{\delta}_3=\pm1$

# Example 2 (d=3): Hantzsche-Wendt

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & \frac{L_1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} -1 & 0 & 0 & \frac{L_1}{2} \\ 0 & 1 & 0 & \frac{L_2}{2} \\ 0 & 0 & -1 & \frac{L_3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The holonomy is $\mathbb{Z}_2$ x $\mathbb{Z}_2$ and the lattice is generated by

$$e_1 = (\mathbf{1}_3, \vec{e}_1) = \alpha^2,\ e_2 = (\mathbf{1}_3, \vec{e}_2) = \beta^2,\ e_3 = (\mathbf{1}_3, \vec{e}_3) = (\beta\alpha)^2$$

Three geometrical moduli: $L_1$, $L_2$, $L_3$

Single spin structure allowed: $\boldsymbol{\delta}_1=\boldsymbol{\delta}_2=\boldsymbol{\delta}_3=-1$

# Harmonics and spectrum

Scalar harmonics are combinations of those on **T**$^d$ invariant under the holonomy group r(**Γ**)

$$Y_{\vec{k}^*} = \frac{1}{|r(\Gamma)|} \sum_{\gamma\in\Gamma/\Lambda} \exp\left[2\pi i\,\vec{k}^* \cdot \left(R_\gamma \vec{y} + \vec{b}_\gamma\right)\right] \qquad \vec{k}^* = n_a \vec{e}_a^{\,*} \in \Lambda^*$$

$$\Box Y_{\vec{k}^*} = -4\pi^2 \|\vec{k}^*\|^2\, Y_{\vec{k}^*} \equiv -M^2_{\vec{k}^*}\, Y_{\vec{k}^*}$$

*[Riazuelo-Weeks-Uzan-Lehoucq-Luminet 04]*

*[See also Kehagias-Tamvakis 04]*

Vector and tensor harmonics are simple generalizations

*[Andriot-Tsimpis 18, Peng-Lindblom-Zhang 19]*

Spinor harmonics may have a shifted dual lattice

*[Pfaffle 00, Miatello-Rossetti 01, Miatello-Podestà 03]*

$$\vec{k}^*_\varepsilon = \vec{k}^* + \vec{a}^*_\varepsilon = \left[n_a + \frac{1}{4}(1-\delta_a)\right] \vec{e}_a^{\,*}$$

# Important distinction

## depending on action of holonomy group

- No directions fixed (e.g. d-3 HW)

$\Rightarrow$ Symmetric spectrum

$\Rightarrow$ Susy-breaking mass splittings only from non-trivial spin structures

- Some direction fixed (e.g. $T^3/Z_3$)

$\Rightarrow$ Some asymmetric part of the spectrum

$\Rightarrow$ Susy-breaking mass splittings also with trivial spin structure

# The effective 1-loop potential $V_1$

$$V_1 = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D} \sum_I (-1)^{F_I} N_I \sum_{\vec{k}_I^*} \log\left(p^2 + M^2_{\vec{k}_I^*}\right)$$

$\mathrm{Mink}_D$, I=field label, $F_I$=fermion number, $N_I$=# d.o.f.

Must sum only on the independent d-momenta $k_I^*$

Standard manipulations and ζ function approach give

$$V_1 = -\frac{\pi^{D/2}}{2}\Gamma(-D/2) \sum_I (-1)^{F_I} \frac{N_I}{|r(\Gamma_I)|} Z_{\Lambda_I^*}(-D, \vec{a}_I^*, 0)$$

Euler gamma

$$Z_\Lambda[s, \vec{a}, \vec{b}^*] = \sum_{\vec{z}\in\Lambda} \frac{e^{-2\pi i \vec{z}\cdot\vec{b}^*}}{\|\vec{z}+\vec{a}\|^s}$$

Epstein zeta

# Equivalent expression for $V_1$

$$V_1 = -\frac{1}{2^{D+1}\pi^{D/2}} \sum_{\vec{k}^*} \frac{d}{ds}\left[\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)} \mathrm{Str}\mathcal{M}^{D-2s}\right]_{s=0}$$

$$\mathrm{Str}\,\mathcal{M}^{2p} \equiv \sum_I (-1)^{F_I}\, N_I\, M^{2p}_{\vec{k}^*_I}$$ sums over the labels I associated with a given k* in the dual lattice

We will see later a non trivial replica of what was discussed initially in the toy example of d=1, D=4:

$V_1$ finite for any N>0 (no need for modular invariance and string cutoff) after summing over the dual lattice

Still, high-N supertrace formulae valid level-by-level before lattice sum: **"REALIGNED SUPERSYMMETRY"**

# Example 1 (d=3): Type IIA on $T^3/Z_3$

Symmetric part of the spectrum for trivial spin structure

$$\|\vec{k}^*\|^2 = \left[\frac{n_1^2}{L^2} + \frac{(n_1+2n_2)^2}{3L^2} + \frac{n_3^2}{L_3^2}\right] \qquad (\; n_1, n_2, n_3 \in \mathbb{Z},\; n_1 \geq 1,\; n_2 > -n_1)$$

For each $(n_1,n_2,n_3)$ 128B & 128F completely degenerate =>
vanishing contributions to supertraces and to $V_1$ (here D=7)

Asymmetric part of the spectrum for $n_1=n_2=0$ & $n_3 \neq 0$

| state | dofs | $[n_3]$ – level |
|---|---|---|
| $g_{\mu\nu}$ | 20 | $\|3[n_3]\|$ |
| $\psi_\mu$ | 20 | $2\times\|3[n_3]+1\|,\; 2\times\|3[-n_3]+1\|,$ |
| $a_\mu$ | 6 | $2\times\|3[n_3]\| \oplus 2\times\|3[n_3-1]+1\| \oplus 2\times\|3[-n_3-1]+1\|$ |
| $b_{\mu\nu}$ | 15 | $\|3[n_3]\| \oplus \|3[n_3-1]+1\| \oplus \|3[-n_3-1]+1\|$ |
| $c_{\mu\nu\rho}$ | 20 | $\|3[n_3]\|$ |
| $\chi$ | 4 | $2\times\|3[n_3+1]\| \oplus 2\times\|3[-n_3+1]\| \oplus 4\times\|3[n_3]+1\| \oplus 4\times\|3[-n_3]+1\|$ |
| $\phi$ | 1 | $3\times\|3[n_3]\| \oplus \|3[-n_3+1]+1\| \oplus \|3[n_3-1]+1\| \oplus \|3[-n_3-1]+1\| \oplus \|3[n_3+1]+1\|$ |

Reconstruct 128B & 128F dof of maximal supermultiplet for each $n_3$ after a suitable re-alignment of the KK levels ($n_3$=0 subtle but OK)

$$\mathrm{Str}(\mathcal{M}^2) = \mathrm{Str}(\mathcal{M}^4) = \mathrm{Str}(\mathcal{M}^6) = 0 \quad \mathrm{Str}(\mathcal{M}^8) = 40320\frac{\pi^8}{L_3^8}$$

Finite contribution to $V_1$ from each realigned KK level

After resumming:

$$V_1 = -\frac{3936}{35}\frac{\pi^4}{L_3^7}$$

Our result agrees in the large $L_3$ limit with the string computation [Acharya-Aldazabal-Andrés-Font-Narain-Zadeh 2010] modulo a harmless universal multiplicative factor

# Example 2 (d=3): Type IIA on HW

Interesting because it cannot be related to twisted tori reductions a la Scherk-Schwarz (no fixed direction under holonomy group)

Realigning the KK states to reconstruct maximal supermultiplets

| state | dofs | $(n'_1, n'_2, n'_3)$ |
|---|---|---|
| $g_{\mu\nu}$ | 20 | $(n_1, n_2, n_3)$ |
| $\psi_\mu$ | 20 | $2\times\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}\right)\,,\ 2\times\left(n_1-\frac{1}{2}, n_2-\frac{1}{2}, n_3-\frac{1}{2}\right)\,,$ |
| $a_\mu$ | 6 | $2\times(n_1, n_2, n_3)\,,\ 2\times(n_1+1, n_2+1, n_3+1)\,,\ 2\times(n_1-1, n_2-1, n_3-1)$ |
| $b_{\mu\nu}$ | 15 | $(n_1, n_2, n_3)\,,\ (n_1+1, n_2+1, n_3+1)\,,\ (n_1-1, n_2-1, n_3-1)$ |
| $c_{\mu\nu\rho}$ | 20 | $(n_1, n_2, n_3)$ |
| $\chi$ | 4 | $2\times\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}\right)\,,\ 2\times\left(n_1-\frac{1}{2}, n_2-\frac{1}{2}, n_3-\frac{1}{2}\right)\,,$ |
| | | $4\times\left(n_1+\frac{3}{2}, n_2+\frac{3}{2}, n_3+\frac{3}{2}\right)\,,\ 4\times\left(n_1-\frac{3}{2}, n_2-\frac{3}{2}, n_3-\frac{3}{2}\right)$ |
| $\phi$ | 1 | $3\times(n_1, n_2, n_3)\,,\ (n_1+1, n_2+1, n_3+1)\,,\ (n_1-1, n_2-1, n_3-1)\,,$ |
| | | $(n_1+2, n_2+2, n_3+2)\,,\ (n_1-2, n_2-2, n_3-2)$ |

Straight lattice with mass eigenvalues controlled by

$$\|\vec{k}^*\|^2 = \left[\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right]$$

Again, at each realigned KK level the 1st non-zero supertrace is

$$\mathrm{Str}\mathcal{M}^8 = 40320\,\frac{\pi^8\left(L_1^2L_2^2 + L_2^2L_3^2 + L_1^2L_3^2\right)^4}{L_1^8L_2^8L_3^8}$$

For the simple choice $L_1=L_2=L_3=L$, the resulting D=7 $V_1$ is

$$V_1 = -\frac{384}{\pi^5 L^7}\left[Z_{\mathbb{Z}^3}(10,0,\vec{0}) - Z_{\mathbb{Z}^3}\left(10,0,\left\{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\}\right) - 6\left\{\zeta(10)+\eta(10)\right\}\right]$$

$$\sim -\frac{0.1}{L^7}$$

# General supertrace formulae

Realigned maximal supermultiplets and supertrace formulae can be "explained" for any D,d extending D=4, N=8 results

[Dall'Agata-FZ arXiv:1205.4711]

Assign $U(1)_A$ charges (A=1,…,n≤7) to the supercharges $Q_i$ (i=1,…,N=8):

$$\vec{q}_i \equiv \{q_i^A\}_{A=1,\ldots,n} \equiv (q_i^1, \ldots, q_i^n)$$

Charges of all 128B+128F helicity states are then determined

$$\begin{aligned}
|+2\rangle : &\quad \vec{0}\,, \\
|+3/2\,,\,i\rangle = Q_i|+2\rangle : &\quad \vec{q}_i\,, \\
|+1\,,\,[ij]\rangle = Q_i Q_j|+2\rangle : &\quad \vec{q}_i + \vec{q}_j\,, \\
|+1/2\,,\,[ijk]\rangle = Q_i Q_j Q_k|+2\rangle : &\quad \vec{q}_i + \vec{q}_j + \vec{q}_k\,, \\
|0\,,\,[ijkl]\rangle = Q_i Q_j Q_k Q_l|+2\rangle : &\quad \vec{q}_i + \vec{q}_j + \vec{q}_k + \vec{q}_l
\end{aligned}$$

+ CPT Conjugates

⇩

$$\sum_{i=1}^{8} \vec{q}_i = \vec{0}$$

In all cases considered here, KK spectrum can be written as:

$$\begin{aligned}
|2\rangle : &\quad M^2 = \vec{n}^2 , \\
|3/2, i\rangle : &\quad M_i^2 = (\vec{n} + \vec{q}_i)^2 , \\
|1, [ij]\rangle : &\quad M_{ij}^2 = (\vec{n} + \vec{q}_i + \vec{q}_j)^2 , \\
|1/2, [ijk]\rangle : &\quad M_{ijk}^2 = (\vec{n} + \vec{q}_i + \vec{q}_j + \vec{q}_k)^2 , \\
|0, [ijkl]\rangle : &\quad M_{ijkl}^2 = (\vec{n} + \vec{q}_i + \vec{q}_j + \vec{q}_k + \vec{q}_l)^2
\end{aligned}$$

scalar products with suitable field-dependent real diagonal metric

$$\mathrm{Str}\,\mathcal{M}^{2k} = \sum_a \epsilon_a \left(\vec{M}_a^{\,2}\right)^k = \sum_a \epsilon_a \left[\sum_A \left(M_a^A\right)^2 \mu_A^2\right]^k$$

$$\vec{M}_a = \{\vec{n}, \vec{n} + \vec{q}_i, \vec{n} + \vec{q}_i + \vec{q}_j, \vec{n} + \vec{q}_i + \vec{q}_j + \vec{q}_k, \vec{n} + \vec{q}_i + \vec{q}_j + \vec{q}_k + \vec{q}_l\}$$

$$(\vec{M}_a)^2 = \sum_{A=1}^{n} (M_a^A)^2 \, \mu_A^2$$

$\in_{a:}$= (1,-2,2,-2,2) for spin=(0,1/2,1,3/2,2)

Then always, for each KK realigned maximal supermultiplet

$$\mathrm{Str}\,\mathcal{M}^2 = \mathrm{Str}\,\mathcal{M}^4 = \mathrm{Str}\,\mathcal{M}^6 = 0$$

For a single U(1) with non-zero charges (e.g. $T^3/Z_3$ case)

$$\mathrm{Str}\,\mathcal{M}^8 = 40320\,\left(\prod_{i=1}^{8} q_i\right)\,\mu^8$$

For A-independent $U(1)_A$ charges (e.g. d=3 HW case)

$$\mathrm{Str}\,\mathcal{M}^8 = 40320\,\left(\prod_{i=1}^{8} q_i\right)\left(\sum_A \mu_A^2\right)^4$$

# Connections with Scherk-Schwarz

Some compactifications on Riemann-flat spin manifolds admit a Scherk-Schwarz interpretation if there are submaximal orbits of holonomy group (e.g. $T^3/Z_3$, any other $Z_k$ freely acting orbifold)
Otherwise (e.g. d=3 HW) no such interpretation is possible.

Full spectrum in sugra Sch-Sch compatifications rarely computed
[partial results in Gkountumis-Hull-Stemerdink-Vandoren 23]

Otherwise, reductions/truncations to D dimensions may miss important physical properties and are not genuine EFTs
[Dall'Agata-Prezas 05, Grana-Minasian-Triendl-VanRiet 13]

e.g. quantization of the susy-breaking parameters, unbroken susy on torus but broken in the reduced/truncated theory in D dim, incomplete (or even divergent for N≤4) result for the one-loop effective potential

# Conclusions

Derived general results for compactifications of D+d supergravity on Riemann-flat manifolds with full (classical) breaking on $Mink_D$:

- Consistency conditions
- KK spectrum and harmonics
- 1-loop effective potential (Casimir energy)

Equivalent results, with slightly different techniques, also by Bento-Montero. We agree with them and previous literature [Fabinger-Horava 00, Bagger-Feruglio-FZ 01]

Observed the presence of a realigned D-dim broken sugra in the KK spectrum, derived supertrace formulae valid at each KK level

# Outlook

In all simple examples considered so far:

$$V_1^{(D)} = -\frac{k}{f(L_i)}$$

With $k>0$ dimensionless and $f(L_i)>0$ function of the geometrical moduli of dimension $(\text{length})^D$, with $f(L_i)\to\infty$ for $L_i \to \infty$

However, no systematic investigation yet.

If this happens to be always the case…

How to introduce additional structure and more interesting physics in the potential? Open, possibly controversial question

Combine with fluxes [Bento-Montero]? Re-introduce string effects?