Homotopy theory of Lie algebroids

Damjan Pištalo

University of Luxembourg

damjan.pistalo@gmail.com

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Setting

- We work in *algebraic setting* because it is more suitable for homotopy theory. Some of the things should extend to the smooth setting, but not all. Technically, we speak of Lie Rinehart pairs.
- k- a field of characteristic zero.

What are Lie algebroids?

Definition 1

Lie Rinehart pair (from the next slide onward a Lie algebroid) (A, M) is:

- A – a commutative k-alg;
- M – a Lie alg./k + A-module
- A Lie algebra map (*anchor*) $\rho : M \to \mathrm{Der}(A)$
- Leibniz rule

$$[v, fw] = \rho(v)(f)w + f[v, w], \quad v, w \in M, f \in A.$$

Example 1. A vector bundle $\rho : E \to TM$ is a Lie algebroid exactly when $(C^\infty(M), \Gamma E)$ is a Lie Rinehart pair

Example 2. For an involutive distribution $\mathfrak{X}$ on a manifold M, $(C^\infty(M), \Gamma E)$ is a Lie Rinehart pair..

Example 3. A Lie Rinehart par $(k, \mathfrak{g})$ is equivalently a Lie algebra $\mathfrak{g}$ over k.

What are dg Lie algebroids?

Definition 1

dg Lie Rinehart pair (from the next slide onward a dg Lie algebroid) (A, M) is:

- A – a differential graded commutative k-alg;
- M – a differential graded Lie alg./k + A-module
- A Lie algebra map (*anchor*) $\rho : M \to \mathrm{Der}(A)$
- Leibniz rule

$$[v, fw] = \rho(v)(f)w + (-1)^{|f||a|} f[v, w], \quad v, w \in M, f \in A.$$

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Example 2. For an involutive distribution $\mathfrak{X}$ on a manifold M, $(C^\infty(M), \Gamma E)$ is a Lie Rinehart pair..

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What are morphisms of dg Lie algebroids?

- A morphism of dg lie algebroids $(f_0, f) : (B, M) \to (A, N)$ is a pair morphisms

$$f_0 : A \to B, \qquad f : M \to B \otimes_A N$$

which is

1. compatible with anchors:

$$\begin{array}{ccc} M & \xrightarrow{f} & B \otimes_A N \\ \downarrow & & \downarrow \\ \mathrm{Der}(B) & \xrightarrow{-\circ f_0} & \mathrm{Der}(A, B); \end{array}$$

2. and the brackets: For $f(m) = \sum_i b_i \otimes n_i$, $f(m') = \sum_j b'_j \otimes n'_j$,

$$f([m, m']) = \pm \sum_{i,j} b_i b'_j \otimes [n_i, n'_j] + \sum_j \Gamma(m)(b'_j) \otimes n'_j \mp \sum_i \Gamma(m')(b_i) \otimes n_i.$$

- A morphism (f_0, f) is a weak equivalence if f_0 and f are both quasi-isomorphisms.

How to do a homotopy theory

- Weak equivalences make sense only if $B \otimes_A N$ is of correct homotopy type, so we consider the (full sub)category of pairs (A, M) where M is cofibrant (projective as graded module + conditions on differential)
- Category with weak equivalences $\xrightarrow{\text{Dwyer Kan localization}}$ ∞- category
- ... but we know nothing about it! Does it have limits, colimits, is it a presentable ∞-category?
- Can we have computational tools ex. homotopies, lifting properties, etc.?

We need more structure to answer this!

More structure

- Under certain conditions the functor which associates to a dg Lie algebroid the underlying affine derived scheme is a **Cartesian fibration**, whose fibers (dg Lie algebroids over a fixed affine derived scheme) are presentable infinity categories **[J. Nuiten, '17]**.
- ∞-category of dg Lie algebroids is equivalent to that of Lie ∞-algebroids, and later is a **category of fibrant objects [P. '25]**. We get stuff like homotopy limits, homotopies... but no homotopy colimits ☹

Lie ∞-algebroids

Definition 2

dg Lie algebroid (A, M) is:

- A – a differential graded commutative k-alg;
- M – a differential graded Lie alg./k + A-module
- A dg Lie algebra map $\rho : M \to \mathrm{Der}(A)$
- Leinbiz rule

$$[v, fw] = \rho(v)(f)w + (-1)^{|f||a|} f[v, w], \quad v, w \in M, f \in A.$$

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Lie ∞-algebroids

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- For cofibrant A-module (dualizable in a weird sense) M, equivalently a differential d_{CE} on $\widehat{\mathrm{Sym}}_A M^\vee[1]$.
- A morphism $(B, M) \to (A, N)$ is dually a morphism of "dg algebras"

$$(\widehat{\mathrm{Sym}}_A N^\vee[1], d_{\mathrm{CE}}) \to (\widehat{\mathrm{Sym}}_B M^\vee[1], d_{\mathrm{CE}})$$

A morphism is weak equivalence if it is qis in weight zero

$$A \to B; \quad (B \otimes_A N^\vee \to M^\vee) \Leftrightarrow (M \to B \otimes_A N)$$

An algebraic model of B(F)V-BRST complex

- $\mathcal{A}^n = \mathrm{Spec}(k[x_1, \dots x_n])$– n-dimensional affine space
- $\Sigma = \mathrm{Spec}(k[x_1, \dots x_n]/I) \subseteq \mathcal{A}^n$ – affine variety
- $\mathfrak{X} \subseteq \mathrm{Der}(k[x_1, \dots x_n])$ – a distribution which
  1. restricts to Σ
  $$X(f) = 0 \qquad \forall X \in \mathfrak{X}, f \in I;$$
  2. the restriction is involutive $[\mathfrak{X}|_\Sigma, \mathfrak{X}|_\Sigma] \subseteq \mathfrak{X}|_\Sigma$
- Space of observables is the leaf space $\Sigma/\mathfrak{X}|_\Sigma$.
- Classical BV-BRST complex is the "derived algebra of functions on the leaf space $\mathcal{O}(\Sigma//\mathfrak{X}|_\Sigma)$".

An algebraic model of B(F)V-BRST complex

Step 1: ghosts

- $P^\bullet = (\mathcal{O}(\Sigma)\langle v_i\rangle, d) \xrightarrow{\text{qis}} \mathfrak{X}|_\Sigma$ – free resolution of the $\mathcal{O}(\Sigma)$-module $\mathfrak{X}|_\Sigma$.
- Lie_∞ algebroid structure on $P^\bullet$ s.t. $P^\bullet \xrightarrow{\text{qis}} \mathfrak{X}|_\Sigma$ is a morphism of Lie_∞ algebroids, i.e.
  1. it is a morphism of Lie_∞ algebras,
  2. anchor of $P^\bullet$ is the composition

$$P^\bullet \xrightarrow{\text{qis}} \mathfrak{X}|_\Sigma \hookrightarrow \operatorname{Der}\mathcal{O}(\Sigma).$$

- Ghosts are the generators v_i^* of $P^\vee$ in the Chevalley-Eilenberg complex $(\widehat{\operatorname{Sym}}_{\mathcal{O}(\Sigma)}\overline{P}^\vee[1], d_{\mathrm{CE}})$

An algebraic model of B(F)V-BRST complex

Step 2: anti-ghosts

- Koszul-Tate resolution $Q\mathcal{O}(\Sigma) \twoheadrightarrow \mathcal{O}(\Sigma)$: a semi-free differential graded commutative algebra

$$Q\mathcal{O}(\Sigma) = k[x_1, \dots x_n, y_j]$$

  acyclic in non-zero degree; and s.t $H_0(Q\mathcal{O}(\Sigma)) = \mathcal{O}(\Sigma)$.

- Lie$_\infty$-algebroid structure on $\overline{P}^\bullet = Q\mathcal{O}(\Sigma)\langle v_i \rangle$ i.e. differential s on the graded-commutative algebra

$$\widehat{\mathrm{Sym}}_{Q\mathcal{O}(\Sigma)} \overline{P}^\vee[1]$$

  such that tensoring

$$\mathcal{O}(\Sigma) \otimes_{Q\mathcal{O}(\Sigma)} - : \widehat{\mathrm{Sym}}_{Q\mathcal{O}(\Sigma)} \overline{P}^\vee[1] \to \widehat{\mathrm{Sym}}_{\mathcal{O}(\Sigma)} \overline{P}^\vee[1]$$

  respects the differential (is dual to a morphism of Lie$_\infty$ algebroids).

- Antighosts are generators of y_j of the KT resolution.

A question

- In summary, we got the following morphisms of Lie_∞ algebroids:

$$(Q\mathcal{O}(\Sigma), \overline{P}^\bullet) \xleftarrow{\text{step 2}} (\mathcal{O}(\Sigma), P^\bullet) \xrightarrow{\text{step 1}} (\mathcal{O}(\Sigma), \mathfrak{X}|_\Sigma).$$

- Questions:
  1. Are "BV resolutions" a consequence of a homotopy theory of for Lie algebroids?
  2. Do they satisfy the usual "uniqueness up to homotopy".

The answer

- For step 1 yes:

Theorem 1 (C. Laurent-Geneoux, R. Louis, '21)

Let $(\mathcal{O}(\Sigma), P^\bullet) \to (\mathcal{O}(\Sigma), \mathfrak{X}|_\Sigma)$ *be as in step 1, and let* $(\mathcal{O}(\Sigma), Q^\bullet) \to (\mathcal{O}(\Sigma), \mathfrak{X}|_\Sigma)$ *any morphism of* Lie$_\infty$ *algebroids. There exists an* ∞*-morphism unique up to homotopy* $(\mathcal{O}(\Sigma), Q^\bullet) \to (\mathcal{O}(\Sigma), P^\bullet)$ *such that the diagram*

$$
\begin{array}{ccc}
(\mathcal{O}(\Sigma), Q^\bullet) & \xrightarrow[\quad]{\exists!\ \text{up to homotopy}} & (\mathcal{O}(\Sigma), P^\bullet) \\
& \searrow \qquad \swarrow_{\sim}^{\text{step 1}} & \\
& (\mathcal{O}(\Sigma), \mathfrak{X}|_\Sigma) &
\end{array}
$$

commutes.

- A similar result holds for step 2, and for the whole thing **(P. '25)**.

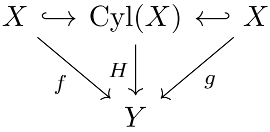
What is a homotopy?

In topology homotopy is usually defined using cylinder,

- Fold map factors through cylinder into a cofibration (nice inclusion, ex. cell attachment) followed by a weak (homotopy) equivalence

$$X \coprod X \hookrightarrow \mathrm{Cyl}(X) \xrightarrow{\sim} X.$$

- Maps $f, g : X \to Y$ are homotopic if there exists $H : \mathrm{Cyl}(X) \to Y$ such that the diagram



commutes.

What is a homotopy?

but it can also be defined using the path space!

- Diagonal map factors through path space into a weak (homotopy) equivalence followed by a fibration (nice surjection, ex. top. fiber bundles)

$$Y \xrightarrow{\sim} \mathrm{Path}(Y) \twoheadrightarrow Y \times Y.$$

- Maps $f, g : X \to Y$ are homotopic if there exists $H : X \to \mathrm{Path}(Y)$ such that the diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & \downarrow{\scriptstyle H} & \searrow^{g} & \\ Y & \twoheadleftarrow & \mathrm{Path}(Y) & \twoheadrightarrow & Y \end{array}$$

commutes.

Abstract homotopy theory

Homotopy theory (of chain complexes and topological spaces) is axiomatized by D. Quillen in 60s:

Model category is a complete and cocomplete category, with 3 classes of morphisms (fibrations, cofibrations and weak equivalences) satisfying axioms

1. 2-out-of-3 for weak equivalences (f, g, $f \circ g$)
2. Factorizations $* \hookrightarrow * \xrightarrow{\sim} *$, and $* \xrightarrow{\sim} * \twoheadrightarrow *$
3. Lifting properties

Path object, cylinder object, homotopy limits, homotopy colimits, presentable ∞-category...

- X is *cofibrant* if $\emptyset \hookrightarrow X$ (ex. relative cell complex)
- *cofibrant replacement* (ex. projective resolution) is a factorization $\emptyset \hookrightarrow QX \xrightarrow{\sim} X$.

What do we have?

Category of fibrant objects is a category with 2 classes of morphisms (fibrations and weak equivalences) with

- 2-out-of-3 for weak equivalences (f, g, $f \circ g$)
- path object (homotopies)
- homotopy limits

Theorem 2 (P '24)

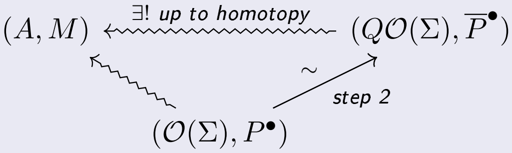
Full subcategory of Lie_∞ Algoid *consisting of pairs* (A, M) *where* A *is a cofibrant dgca, and* M *is a cofibrant* A*-module is a category of fibrant objects.*

We also have cofibrations, lifting properties, "cofibrant replacements" (step 1 and step 2).

Step 2

Proposition 1

Map $\imath : (\mathcal{O}(\Sigma), P^\bullet) \hookrightarrow (Q\mathcal{O}(\Sigma), \overline{P}^\bullet)$ *be as in step 2, and let* $f : (\mathcal{O}(\Sigma), P^\bullet) \to (A, M)$ *be an map of Lie*$_\infty$ *algebroids with* A *and* M *cofibrant. Then there exists an* ∞*-morphism unique up to homotopy* $l : (A, M) \to (Q\mathcal{O}(\Sigma), \overline{P}^\bullet)$ *such that the diagram*



commutes.

Step 2: Proof

Map $\imath$ of step 2 is a cofibration and a weak equivalence

$$\imath_0 : Q\mathcal{O}(\Sigma) \twoheadrightarrow \mathcal{O}(\Sigma), \qquad \imath_1 = \mathrm{id} : P \to P$$

$$\begin{array}{ccc}
(\mathcal{O}(\Sigma), P^\bullet) & \xrightarrow{\ f\ } & (A, M) \\
{\scriptstyle \sim}\big\downarrow{\scriptstyle \imath} & \overset{l}{\nearrow} & \big\downarrow \\
(Q\mathcal{O}(\Sigma), \overline{P}^\bullet) & \longrightarrow & (k, 0)
\end{array}$$

Assume l_1 and l_2 are two such lifts:

$$\begin{array}{cccc}
(\mathcal{O}(\Sigma), P^\bullet) & \xrightarrow{\ f\ } (A, M) & \longrightarrow & \mathrm{Path}(A, M) \\
{\scriptstyle \sim}\big\downarrow{\scriptstyle \imath} & & \overset{H}{\nearrow} & \big\downarrow \\
(Q\mathcal{O}(\Sigma), \overline{P}^\bullet) & & \xrightarrow{\ (l_1.l_2)\ } & (A, M) \times (A, M).
\end{array}$$