Chaos, averaging and chaotic precision holography in LLM geometries

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Black holes & chaos

- State of the art: fast scrambling (Susskind), MSS bound and near-horizon isometries
- Near-horizon AdS throat and its $\mathrm{SL}(2,\mathbb{R})$ isometry imply strong chaos in dual holographic quantum field theory (SYK model, Yang-Mills plasmas etc)
- The same symmetry arguments lead to *integrable* geodesics in the bulk (black hole geometry)

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- Qualitatively: integrable geodesics (AdS black hole) $\leftrightarrow$ maximum chaos (CFT)
- This talk and some other recent works: nonintegrable (chaotic) geodesics (microstate geometries) $\leftrightarrow$???

Black holes & averaging

- Big question: are black holes ensemble-averaged solutions?
- Hint 1: microstate solutions, fuzzballs etc.
- Hint 2: replica wormholes and the factorization problem: does AdS/CFT secretly perform ensemble averaging?
- JT gravity (Saad, Shenker, Stanford, Iliesiu) and AdS_3 gravity (Belin, Perlmutter): ensemble average over *theories*
- In higher dimension: unlikely, but perhaps ensemble average over *solutions* or *states*

LLM geometries & averaging

- Lin-Lunin-Maldacena (LLM) 1/2 BPS geometries: mapping to black and white patterns in the $x - y$ plane
- Lots of supersymmetry + mapping to the 2D plane $\Rightarrow$ very convenient for work
- But: this is clearly not a black hole!

- So why bother?

LLM geometries & averaging

- Lin-Lunin-Maldacena (LLM) 1/2 BPS geometries: mapping to black-and-white patterns in the $x - y$ plane
- Lots of supersymmetry + $\mathcal{N} = 4$ SYM CFT + mapping to the 2D plane $\Rightarrow$ very convenient for work
- But: this is clearly not a black hole!

- So why bother?
- Because it does provide us with a singularity: superstars and grayscale LLM geometries. And it remains more approachable than e.g. D1-D5 CFT

Idea: how is the singularity born out of bulk chaos and averaging?

- Study the dynamics of bulk probes in LLM geometries (for now mainly geodesics, can be upgraded to fields)
- Do we get more black-hollish behavior of geodesics after averaging over black-and-white ensembles? – Yes, but with caveats (Berenstein, Čubrović and Djukić 2508.09669)
- Can we identify the consequences of chaos and averaging on the CFT operators? – Yes, but they are probably system-dependent (Berenstein, Čubrović and Djukić, to appear)

1. LLM solutions
2. Geodesic chaos in black & white geometries
3. Weak geodesic chaos and averaging in grayscale geometries
4. CFT picture: toward the dictionary entry for chaos

1 LLM solutions

LLM solution

- Lin, Lunin & Maldacena 2004 "bubbling AdS" – 1/2 BPS solutions with symmetry $\mathrm{SO}(4) \times \mathrm{SO}(4) \times R$
- CFT interpretation: only one complex scalar in $\mathcal{N} = 4$ SYM on $\mathbb{S}^3 \times \mathbb{R}$ is excited $\Rightarrow$ extra R symmetry
- Ground state: $\mathrm{AdS}_5 \times \mathbb{S}^5$ (one $\mathbb{S}^3$ in AdS and one on $\mathbb{S}^5$)
- Gravity + RR 5-form field
- Metric:

$$ds^2 = \frac{1}{h^2}\Big[-(dt + V_a dx^a)^2 + h^4\left(d\xi^2 + dx_a dx^a\right) + + \left(\frac{1}{2} - z\right) d\tilde{\Omega}_3^2 + \left(\frac{1}{2} + z\right) d\Omega_3^2 \Big]$$

$$h^2 = \frac{1}{\xi}\sqrt{\frac{1}{4} - z^2}, \quad \partial_a V_b = \epsilon_{ab}\frac{\partial_\xi z}{\xi} + \partial_b V_a, \quad \partial_\xi V_a = \frac{\epsilon_{ab}\partial_b z}{\xi}$$

- BPS condition to have a solution of type IIB SUGRA:

$$\partial_a \partial_a z + \xi \partial_x \left(\frac{\partial_\xi z}{\xi}\right) = 0.$$

Black & white patterns and bubbling AdS

- Everything determined by the single function $z(\xi, x, y)$
- Finite curvature requires $z(\xi = 0) = -1/2$ ("black") or $z(\xi = 0) = +1/2$ ("white")
- Exactly one 3-sphere vanishes at every point in the plane: in AdS / on $\mathbb{S}^5$ for black / white
- In the matrix model black / white corresponds to electrons/holes (Berenstein 2004)
- Geometry of black & white patterns:
  - Black disk – AdS
  - Multi-disk patterns – "bubbling AdS"
  - Black half-plane – pp-wave limit
  - Small deformations (rings, droplets etc) – small fluctuations
- Disk + concentric thin ring ≈ giant graviton excitation on AdS
- Charge and momentum (0th and 2nd moment of the blackness distribution in the LLM plane)

$$Q = \frac{1}{4\pi^2 \ell_P^4} int_{\mathcal{D}} d^2x, \quad J = \frac{1}{16\pi^3 \ell_P^8}\left[\int_{\mathcal{D}} d^2x(x_1^2 + x_2^2) - \frac{1}{2\pi}\left(\int_{\mathcal{D}} d^2x\right)^2\right]$$

Grayscale solutions

- Constructed as "superstars" (Myers & Tafjord 2001) even before LLM: naked singularity
- In LLM terms: $-1/2 < z(\xi = 0) < 1/2 \Rightarrow$ both 3-spheres shrink to zero
- $g_{\mu\nu}(\xi \to 0) \sim h^2 \sim \sqrt{1/4 - z^2}/\xi \Rightarrow R \sim 1/\xi^3$
- "Good" singularity a la Gubser (potentials remain finite $\Rightarrow$ enclose it by a horizon?)

- Matrix-wise: coarse-grained Young tableaux – smoothen the edges (Balasubramanian, Berenstein Levkowycz, Miller, Parrikar 2019) $\Rightarrow$ can be pictured as "grayscale" areas

- Natural arena for averaging: we expect to get grayscale physics by averaging over small deformations of black & white solutions

2 Geodesic chaos in black & white geometries

Why study geodesic chaos?

- As a toy model for fields (otherwise: solve time-dependent 3D PDEs $\Rightarrow$ forget it)
- To compare bulk dynamics in smooth geometries as opposed to the integrable behavior in black hole backgrounds
- To construct explicitly the effective averaging (i.e. coarse-graining) procedure that will yield grayscale singularities
- To gain insight into statistical properties of two-point functions for Δ large

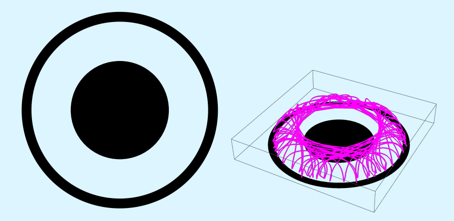
Equations of motion

- Geodesic Hamiltonian:

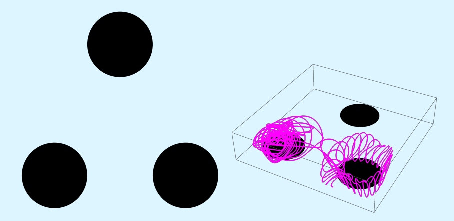
$$\mathcal{H} = \frac{1}{2h^2}\left[P_\xi^2 + (P_x + EV_x)^2 + (P_y + EV_y)^2 - h^4\left(E^2 - \frac{2L^2}{1-2z} - \frac{2\tilde{L}^2}{1+2z}\right)\right]$$

- $V_{x,y}$ – effective "magnetic" fields in the LLM plane
- In the LLM plane: magnetic billiard (Berenstein, Maderazo, Mancilla, Ramirez 2023)
- Ground state (black disk = AdS) integrable
- Two representative excited (CFT) and nonintegrable (bulk) configurations: disk+ring and 3-disk
- Disk+ring described by the Schur polynomials of $Z = \Phi_1 + \imath\Phi_2$
- Both are nonintegrable but disk+ring has P_ϕ as an extra integral of motion $\Rightarrow$ 2 degrees of freedom instead of 2 and a half

Backgrounds and geodesics



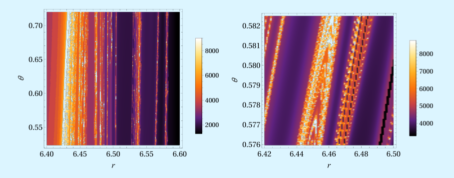
Disk+ring background and geodesic

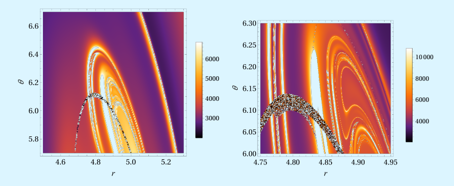


3-disk background and geodesic

- Two representative configurations: disk+ring and 3-disk
- Both are nonintegrable but disk+ring has P_ϕ as an extra integral of motion $\Rightarrow$ 2 degrees of freedom instead of 2 and a half
- Nonintegrability from normal variational equation or simply by inspection of orbits

Chaos in disk+ring case



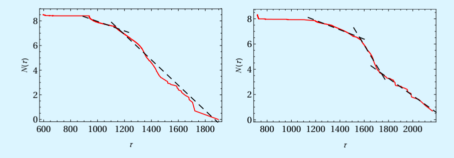


Disk+ring: typical mixed phase-space with remnants of KAM tori and the chaotic sea

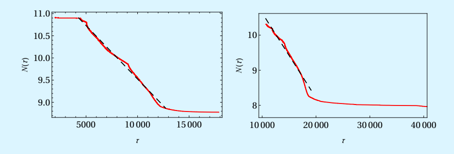
3 disks: KAM tori still present but do not present a barrier (3 degrees of freedom)

- Nonintegrable geodesics, unlike (most) black holes
- But: sticky trajectories provide trapping and mimic the black hole behavior
- Very different from the trapping in superstrata etc: stickynes (KAM tori remnants), not long (but eventually capped) throat as in superstrata

Escape rates and the fractal structures



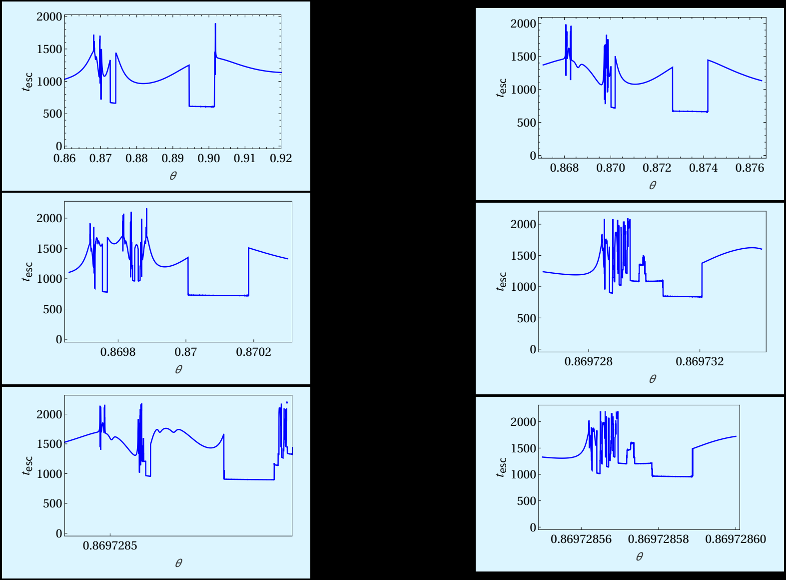
Disk+ring: several populations with different escape rates $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, plus sticky trajectories with very slow (subexponential) escape.



3 disks: uniform escape rate γ. Sticky trajectories are still present but do not divide the phase space into disjoint populations.

- Expect multifractal scaling for disk+ring
- Escaping geodesics $\leftrightarrow$ geodesics from boundary to boundary $\leftrightarrow$ eikonal limit of two-point functions
- What is the holographic dictionary entry for chaotic scattering in the bulk?

Escape rates and the fractal structures



Multifractal spectrum for disk+ring with 4 exponents $\gamma_1, \gamma_2, \gamma_3, \gamma_4$: will be important for CFT correlators

Photon ring?

- For a black hole: the only unstable periodic orbit at $r = r_*$, positive Lyapunov exponent but no chaos (no skeleton of unstable periodic orbits)
- Cardoso et al: Lyapunov exponent on the photon ring determines the imaginary part of the quasi-normal mode spectrum for $n \gg 1$
- Here: chaotic dynamics, infinite skeleton of unstable periodic orbits $\Rightarrow$ photon ring has no special significance
- Effective potential: $Q = \mathcal{H}|_{P_\xi = P_x = P_y = 0}$
- Find the turning points r_* with $Q(E, r_*) = \partial_r Q(E, r_*) = 0$
- Wave quantization condition in the WKB approximation:

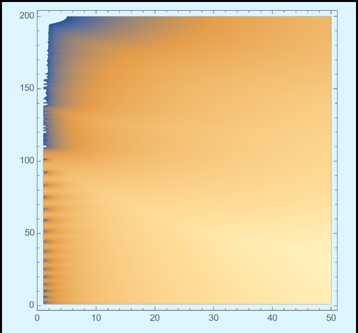
$$\frac{Q(E_*, r_*)}{\sqrt{2Q''(r_*, E_*)}} = -\imath\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}$$

$$E_n = -(2n+1)\frac{\sqrt{Q''(r_*, E_*)}}{\partial_E Q(r_*, E_*)\sqrt{2}}$$

- When the dust settles: $E_n = E_* - \imath(2n+1)\lambda$ – the Cardoso relation

Photon ring? – Yes but who cares

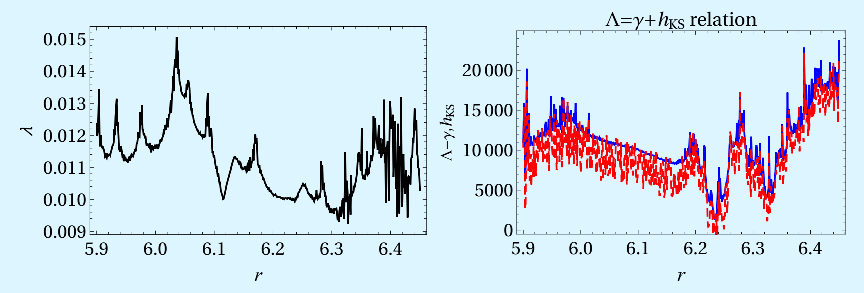
- Scalar wave energy E_n in the WKB approximation exactly determined by the energy E_* and position r_* of the photon ring geodesic
- Nontrivial: would not expect this for a horizonless metric
- But again, the meaning is very different: the photon ring orbit does *not* imply quasinormal modes: confirmed by the numerics
- No poles in the bottom complex plane, just the branch cut along $\Re\omega = 0$



Photon ring insignificant for the Lyapunov spectrum

Lyapunov exponent (left) and the Pesin relation for the sum of positive Lyapunov exponents Λ, Kolmogorov-Sinai entropy h_{KS} and escape rate γ (right):

$$\Lambda \equiv \sum_{\lambda_i > 0} \lambda_i = h_{\mathrm{KS}} + \gamma$$



- At the photon ring we have $\lambda_* \approx 0.001$ – much less than the typical exponent
- Cardoso relation remains but it does not influence dynamics and presumably observable quantities

Take-home message 1

Smooth geometries indeed show geodesic chaos, contrary to black holes, which always have integrable geodesics (modulo some pathological cases).

Take-home message 2

We know that LLM geometries are not black holes. So just because in sufficiently complicated configurations there is some trapping and some photon rings and a Cardoso relation does not mean the microstate comes close to a black hole.

3 Weak geodesic chaos and averaging in grayscale geometries

Grayscale geodesics

- Same Hamiltonian ⇒ still nonintegrable. But different z function leads to a potential well which is never present in black & white:

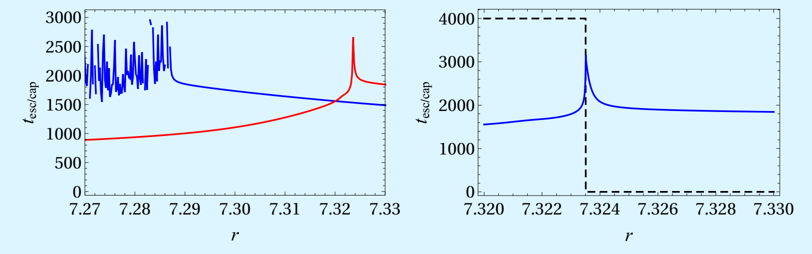
$$V_{\mathrm{eff;BW}}(\xi) = \frac{J_-^2\Theta(\rho - R_i) + J_+^2\Theta(R_i - \rho)}{\xi^2} \geq 0$$

$$V_{\mathrm{eff;gray}}(\xi) = \frac{-(\frac{E}{2})^2\left(1 - g^2\right) + \frac{J_-^2+J_+^2}{2}\frac{g}{2}\left(J_-^2 - J_+^2\right)\mathrm{sgn}(\rho - R_i)}{\xi^2}$$

- Now both escapes ($\gamma > 0$) and captures by the singularity ($\gamma_s > 0$) are possible
- Dynamics is now much simpler. In terms of Kolmogorov-Sinai entropy: $h_{\mathrm{KS}} = \sum_{\lambda_i > 0} \lambda_i - \gamma - \gamma_s \approx 0$
- Still nonintegrable, but $h_{\mathrm{KS}} \approx 0$ just like for black holes!

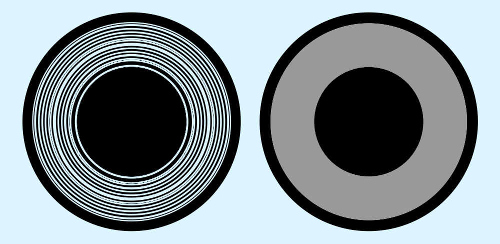
Grayscale escape rates

- Left: escape/capture rate for black & white (blue) vs gray (red)
- Right: the photon ring separates captures ($r < r_*$) and escapes ($r > r_*$)



- Smooth escape rate dependence, no fractal structure
- The photon ring is again observable and crucial: separates captures from escapes
- More black-hollish than black & white

Averaged black & white vs. gray backgrounds



Disk + multiring + ring vs. disk + gray area + ring

- Gray background = average of black & white backgrounds with the same total flux

$$\sum_{i=1}^{2N+1} (-1)^{i+1} R_i^2 = R_1^2 - R_{2N}^2 + R_{2N+1}^1 + g\left(R_{2N}^2 - R_1^2\right)$$

- Does the same hold for geodesics?
- Idea: generate an ensemble of disk + multiring + ring backgrounds, compute geodesics, average them over the ensemble

Averaged black & white vs. gray potential

- Quadratic fluctuations yield an ensemble of disk + multiring + ring solutions with (random Gaussian) radii R_i centered at $R_{i;0}$

$$V_{\mathrm{BW}}(\xi,\rho) = \frac{1}{\xi^2}\left[J_-^2 \sum_{j=1}^{2N+1} (-1)^j \Theta(\rho - R_j) + J_+^2 \sum_{j=1}^{2N+1} \Theta(R_j - \rho) \right]$$

- Distribution with the flux conservation constraint:

$$P(R_2, \ldots R_{2N-2}) = \mathcal{N} e^{-\sum_{j=2}^{2N-1} \frac{(R_j - R_{j;0})^2}{2\sigma^2}} \delta\left(\sum_{j=1}^{2N+1} R_j^2 - \frac{\mathcal{A}_0}{\pi} \right)$$

Averaged black & white vs. gray potential

- Without the constraint just a bunch of Gaussian integrals but the constraint introduces effective interactions:

$$\langle V_{\mathrm{eff}} \rangle = \frac{1}{Z\xi^2} \prod_{j=2}^{2N-1} \int dR_j \int d\lambda V_{\mathrm{BW;eff}} \exp\left[-\mathbf{R} \cdot \hat{\mathbf{M}} \cdot \mathbf{R} - \mathbf{K} \cdot \mathbf{R} - \imath \lambda \Sigma_- \right]$$

$$\hat{\mathbf{M}} = \mathrm{diag}\left(\frac{1}{2\sigma^2} + (-1)^j \imath \lambda \right), \quad \mathbf{K} = \left(\frac{R_{j;0}}{\sigma} \right), \quad j = 2, \ldots 2N-1$$

- Mean field: fixed $\hat{\mathbf{M}}$. Nonlinear fluctuations: make $\hat{\mathbf{M}}$ dynamical.
- In mean field exactly solvable

Averaged black & white vs. gray potential

- Partition function for $2N+1$ disks:

$$Z = \frac{\pi^{N-1}}{(M_+ M_-)^{\frac{N-1}{2}}} \prod_{j=2}^{2N-1} e^{\frac{R_{j;0}^2}{4\sigma^2 M_{(-1)^j}}} \left[1 - \mathrm{Erf}\left(\frac{R_{j;0}}{2\sigma\sqrt{M_{(-1)^j}}} \right) \right]$$

- Averaged effective potential:

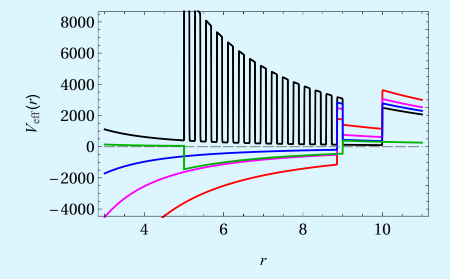
$$\langle V_{\mathrm{eff}}(\xi,\rho) \rangle = \frac{1}{2\xi^2} \int d\lambda e^{-\imath\lambda\Sigma_-} \sum_{j=2}^{2N-1} \frac{A_j}{Z_{1;(j)}}$$

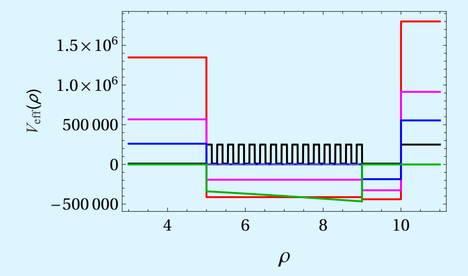
$$\frac{A_j}{Z_{1;(j)}} = \frac{\left(J_-^2 + J_+^2\right) \mathrm{Erf}\left(x_j + \rho\sqrt{M_{(-1)^j}}\right) + J_+^2 - J_-^2 \mathrm{Erf}(x_j)}{1 - \mathrm{Erf}(x_j)}$$

$$x_j \equiv \frac{R_{j;0}}{2\sigma\sqrt{M_{(-1)^j}}}$$

- λ-integral doable but the outcome is a few pages long

Averaged black & white vs. gray potential



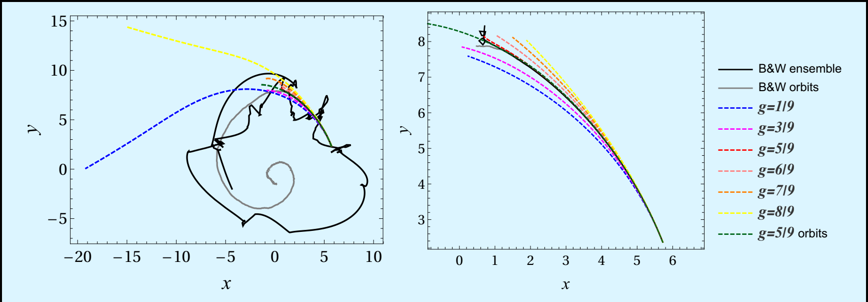


Effective potential: black – microscopic; blue, magenta, red – grayscale with different g; green – ensemble average

- Effective potential for Gaussian ensemble (left) vs uniform ensemble (right), for black-and-white (black), grayscale with different g (blue, magenta, red), averaged black-and-white (green)
- Averaged potential has a potential well of the form $-c/\xi^2 \Rightarrow$ but it eventually truncates and remains finite: falling into the center for times up to t_a

Averaged black & white geodesics vs. gray geodesics

- At short timescales ($t < t_a$): perfect agreement $x^\mu_{\rm gray}(t) \approx \langle x^\mu_{\rm BW}(t)\rangle$
- Averaging over orbits is roughly equivalent: $x^\mu_{\rm gray}(t) \approx \bar{x}^\mu_{\rm BW}(t)$, also for $t < t_a$
- At longer timescales: no averaging
- Compare to the results on 5-brane stars in Martinec & Zigdon 2023, 2024



Take-home message 3:

Take-home message 3: The same averaging procedure that leads to the superstar singularity also leads to black-hollish geodesic dynamics and proper trapping. However, this is only true for short times/high excitations.

4 CFT picture: toward the dictionary entry for chaos

Chaos in the bulk vs the boundary

- Naiveley: black hole $\Rightarrow$ integrable geodesics and maximal CFT chaos; microstate $\Rightarrow$ usually nonintegrable geodesics and submaximal CFT chaos
- Check this and make it precise!
- Study the statistics of the CFT operators corresponding to the LLM patterns
- Match the escape rates and fractal dimensions to the CFT correlators
- Make use of the precision holography results in Turton & Trukov 2024, 2025, Giusto, Russo, Rosso, Aprile 2023, 2024, 2025

LLM precision holography

- At linear order, light CFT primaries $\mathrm{Tr}(Z^k)$ are given by the ripple deformation of the black disk (AdS) black-white boundary $R(\phi)$:

$$R(\phi) = R_0 \mapsto R(\phi) = R_0 \left(1 + \epsilon \cos k\phi\right)$$

- Heavy primaries $\mathcal{O}_{k,p} = \left(\mathrm{Tr} Z^k\right)^p$ with $p \gg 1$ have a SUGRA description only if combined into coherent states of the form

$$\mathcal{O}_{\mathrm{k,p_0;SUGRA}} = \sum_p c_{k,p} \mathrm{O}_{k,p}$$

- The distribution of p's peaks at some $p_0 \propto \epsilon^2$
- Generally we expect these (recently also rigorously shown) to be given by

$$R^2(\phi) = R_0^2 \left(1 + \epsilon \cos k\phi + d_2 \epsilon^2 \cos 2k\phi + \ldots\right)$$

- Rings are conjectured in the LLM paper to correspond to symmetric polynomials $S_{k,p}(\mathrm{Tr} Z, \mathrm{Tr} Z^2 \ldots \mathrm{Tr} Z^k)$.
- Ripples are easier to work with and importantly *we get the same gray disk after averaging over ripples*

Averaging over ripples: the CFT side

- Ensemble of ripple deformations $R^2(\phi)$ with $k = \mathrm{const.} \gg 1$ (geodesic regime – heavy field), $\epsilon = \mathrm{const.} < 1$ (ripple radius i.e. grayscale ring radius) and thus $p_0 = p_0(k, \epsilon) = \mathrm{const.}$
- Average over the coefficients $c_{k,p}$ drawn from a distribution $P\left(c_{k,p}\right)$
- Constraint: $\max P\left(c_{k,p}\right) = p_0$
- Highly non-unique. Reduce by requiring the conservation of charge Q and momentum J

- This is the CFT picture. But we start from solutions in the bulk so we know $\mathcal{O}_{\mathrm{k,p_0;SUGRA}}$ and want to invert this to find $c_{k,p}$

Averaging over ripples: $p = 2$ in CFT

- Bulk equations of motion and the basis for the ripples:

$$\Box_{\mathrm{AdS}_5} b_k = k(k-4) b_k, \quad \Box_{\mathbb{S}^5} Y_k = -k(k+4) Y_k$$
$$b_k = \cosh^{-k} \rho e^{\imath k \tau}, \;\; Y_k = \cos^k \theta e^{\imath k(\phi - t)}$$

- The known case: Giusto et al result for $\mathcal{O}_{\mathrm{k},2}$ (valid also in classical supergravity when operators do not mix different p's, e.g. the energy)

$$R_2(\phi) = R_0 \left(1 - \frac{\epsilon}{2} \cos 2\phi + \frac{3}{16} \epsilon^2 \cos 4\phi\right) + O(\epsilon^3)$$

- Use these as building blocks when inverting the distribution of $\mathcal{O}_{\mathrm{k,p_0;SUGRA}}$ to find $P(c_{k,p})$

Averaging over geodesics in the bulk LLM

- In the eikonal approximation we know $\mathcal{O}_{\mathrm{k,p;disk+ring}}(\phi, t)$
- From these find $c_{k,p}$
- Grayscale values $\mathcal{O}_{\mathrm{k,p;disk+ring}}(\phi, t)$ yield the moments of $P(c_{k,p})$
- From the escape rates we find the scaling of escape times map (see e.g. Dorfman's book):

$$t_{\mathrm{esc}} \sim \sum_{n=0}^{\infty} \left(\frac{|\mathbf{x}_1|}{|\mathbf{x}_2|} \right)^n \sum_{m=1}^{M} \cos\left(\gamma_m^n \theta\right)$$

- We find four populations of orbits with $\gamma_{1,2,3,4}$
- The scaling exponents /escape rates γ_m are the only ingredients from the numerics

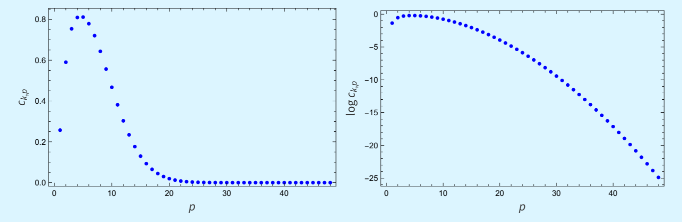
Statistical properties of $\mathcal{O}_{\mathrm{k,p;disk+ring}}(\phi, t)$

- When the dust settles:

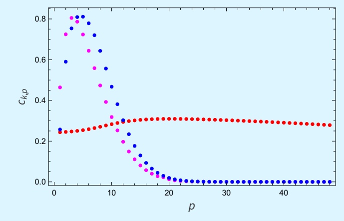
$$\langle \mathcal{O}_{\mathrm{k,p;disk+ring}}(\phi, t) \rangle \sim R_1^{2\Delta} \sum_{n=0}^{\infty} \sum_{m} \left(\frac{\gamma_m}{R_2 - R_1} \right)^n \frac{\sinh\left(R_1 \gamma_m^n t\right)}{\sin\left(R_1 \gamma_m^n \phi\right)}$$

- Weierstrass-function-like behavior – very different both from AdS and thermal correlators
- Take an ensemble of geodesics with different escape rates γ_m and perform the linear fit to $\sum_p c_{k,p}$ with 3 constraints: p_0, Q, J
- Compare grayscale $c_{k,p}$ with an average over black and white $c_{k,p}$

Statistical properties of $c_{k,p}$



Black and white: well-defined maximum, good coherent state



Black and white (blue), average over black and white (magenta) and grayscale (red) coefficients: in grayscale a broad distribution, more complex, no clear exponential decay. The maximum is shifted toward *heavier* states!

Conclusions

- Black & white LLM microstates vs gray LLM states vs black holes: no horizon and geodesic chaos vs naked singularity and weak chaos vs horizon and integrable geodesics
- Averaging to singular grayscale geometries works directly on orbits and correlation functions
- Strongly chaotic black-and-white orbits have parametrically lighter CFT operators than the weakly chaotic grayscale orbits
- Can we redo this e.g. in D1-D5 and show that averaging over chaotic operators yields a heavy operator in a thermal CFT?