Subleading soft dressings of asymptotic states in explicit QED processes

Stavros Christodoulou

University of Cyprus

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Content

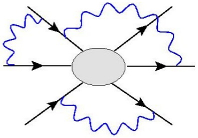
1. Soft physics & infrared (IR) divergence
   - IR divergences in QED
   - Soft photon theorems (leading & subleading)
   - Bloch-Nordsieck (BN) resolution & formal issues in scattering theory
2. The dressed state formalism
   - Faddeev-Kulish (FK) construction
   - Soft momentum expansion and extensions
3. QED scattering: a case study of explicit interactions
   - Fock space amplitudes
   - Applying subleading dressings to Fock states
   - Dressed amplitudes: soft-photon emission suppression & IR-finiteness
4. Summary & future directions

Content

Soft physics & IR divergences

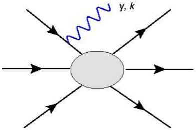
IR divergences in QED

- Virtual IR divergences $(\lambda \to 0)$



$$\Rightarrow \qquad S_{\beta\alpha} \sim e^{\ln(\frac{\lambda}{\Lambda})^{\mathcal{B}_{\beta\alpha}}}$$

- Real soft divergences $(\omega_k \to 0)$

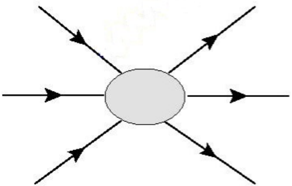


$\Rightarrow$ Soft photon theorems

Soft physics & IR divergences

Soft photon theorems (leading & subleading)

- Additional photon emission during $\alpha \to \beta$



Soft physics & IR divergences

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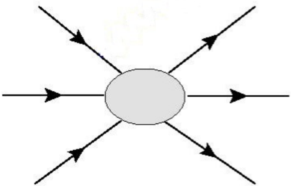

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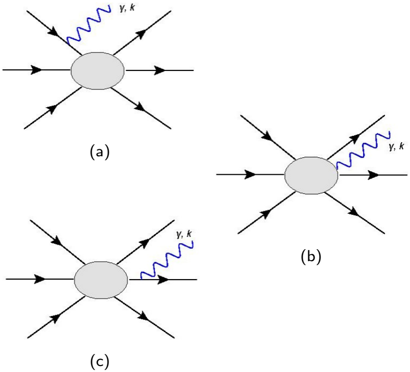

Soft physics & IR divergences

Soft photon theorems (leading & subleading)

- Additional photon production during $\alpha \to \beta$

  In the low energy limit $(\omega_k \to 0)$

  - Leading divergences: $\mathcal{O}(\omega_k^{-1})$ at any order in e
  - Dependence of leading soft factor on $\{(e_i,\ \vec{p}_i)\}$

  where ω_k is the photon energy.

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  where ω_k is the photon energy.

Soft photon theorem (leading & all loops):

Adding one soft photon (of momentum k and of polarization λ) in a scattering process $\alpha \to \beta$ results in

$$S_{\beta\gamma,\alpha} = \left[\sum_{i\in\beta} e_i \frac{p_i \cdot \epsilon^*_\lambda(k)}{p_i \cdot k} - \sum_{i\in\alpha} e_i \frac{p_i \cdot \epsilon^*_\lambda(k)}{p_i \cdot k}\right] S_{\beta\alpha} + \ldots$$

Soft physics & IR divergences

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  - Leading divergences: $\mathcal{O}(\omega_k^{-1})$ at any order in e
  - Dependence of leading soft factor on $\{(e_i, \vec{p}_i)\}$

  where ω_k is the emitted energy.

Soft photon theorem (subleading & tree):

Adding one soft photon (of momentum k and of polarization λ) in a scattering process $\alpha \to \beta$ results in

$$S^{(tree)}_{\beta\gamma,\alpha} = \left[\sum_{i\in\beta} e_i \frac{p_i \cdot \epsilon^*_\lambda(k)}{p_i \cdot k} - \sum_{i\in\alpha} e_i \frac{p_i \cdot \epsilon^*_\lambda(k)}{p_i \cdot k} \right] S_{\beta\alpha} + \mathcal{O}(\omega_k^0)$$

Soft physics & IR divergences

Soft photon theorems (leading & subleading)

- Additional photon production during $\alpha \to \beta$

  In the low energy limit $(\omega_k \to 0)$

  - Leading divergences: $\mathcal{O}(\omega_k^{-1})$ at any order in e
  - Dependence of leading soft factor on $\{(e_i,\ \vec{p}_i)\}$

  where ω_k is the photon energy.

Soft photon theorem (subleading & 1-loop): [Sahoo & Sen '19]

Adding one soft photon (of momentum k and of polarization λ) in a scattering process $\alpha \to \beta$ results in

$$S^{(loop)}_{\beta\gamma,\alpha} = \left[\sum_{i\in\beta} e_i \frac{p_i \cdot \epsilon^*_\lambda(k)}{p_i \cdot k} - \sum_{i\in\alpha} e_i \frac{p_i \cdot \epsilon^*_\lambda(k)}{p_i \cdot k} \right] S_{\beta\alpha} + \mathcal{O}(\ln \omega_k)$$

Soft physics & IR divergences

Soft photon theorems (leading & subleading)

- Additional photon production radiation during $\alpha \to \beta$

  Beyond the strict zero energy limit, at the tree level

  - No divergences: $\mathcal{O}(\omega_k^0)$ at leading order in e
  - Dependence of subleading soft factor on $\{J_i\}$

  where ω_k is the photon energy.

Low-Burnett-Kroll (LBK) Theorem:

$$\mathcal{O}(\omega_k^0) = i\left[\sum_{i\in\beta} e_i \frac{\epsilon^*_{\lambda\mu}(\vec{k})\ k_\nu}{p_i \cdot k}\ \bar{J}_i^{\mu\nu} - \sum_{i\in\alpha} e_i \frac{\epsilon^*_{\lambda\mu}(\vec{k})\ k_\nu}{p_i \cdot k}\ J_i^{\mu\nu}\right] S_{\beta\alpha}$$

where $J_i^{\mu\nu} = L_i^{\mu\nu} + S_i^{\mu\nu}$ *and* $\bar{J}_i^{\mu\nu} = \bar{L}_i^{\mu\nu} + \bar{S}_i^{\mu\nu}$

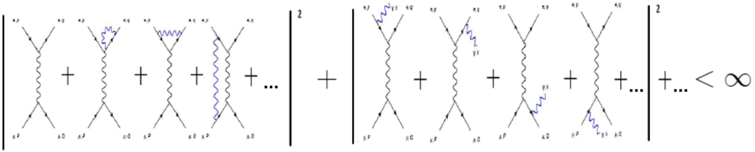
Soft physics & IR divergences

BN resolution & formal issues in scattering theory

- Bloch - Nordsieck (1937): All processes that can not be distinguished by experimental setups contribute to observable quantities, such as the inclusive cross sections.

  i.e.: soft bremsstrahlung radiation ($\omega_k < \Lambda$)

  Diagrammatically, we illustrate that as follows



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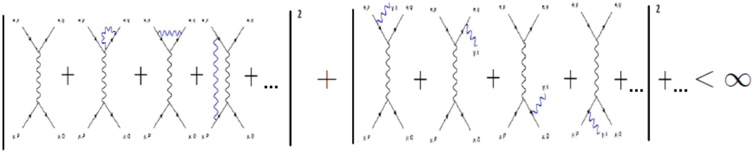

$\Rightarrow$ Cancellation works only at the level of inclusive probabilities (order by order), but the S-matrix itself is ill-defined!

Content

The dressed state formalism

Faddeev-Kulish (FK) construction

- Scattering theory in QED: Asymptotically H_0^I
  1. Fields obey free dynamics far before and far after the scattering
- Faddeev-Kulish (1970): Asymptotically $H_{\text{as}}^I = H_0^I + V_{\text{as}}^I \neq H_0^I$
  1. Coulombic interactions can not be neglected

$$V_{as}^I(t) = -e \int \widetilde{d^3p}\ \rho(\vec{p})\ p^\mu \int \widetilde{d^3k}\ \left[\alpha_\mu(\vec{k}) e^{ip\cdot kt/p^0} + \text{h.c.}\right]$$

where $\rho(\vec{p}) = \sum_s \left[b_s^\dagger(\vec{p}) b_s(\vec{p}) - d_s^\dagger(\vec{p}) d_s(\vec{p})\right],\ \alpha_\mu = \sum_\lambda \epsilon_{\lambda\mu}(k)\alpha_\lambda(k)$

$\Rightarrow$ Non-zero contribution due to low energy photons

$\Rightarrow$ Coupling with matter fields regardless their spin

The dressed state formalism

Faddeev-Kulish (FK) construction

- Multi-fermion FK states: let $\alpha = \{e_i, \vec{p}_i, s_i\}$. Then,

$$|\alpha\rangle_d = e^{R_f} |\alpha\rangle$$

where

$$R_f = \int \widetilde{d^3p}\ \rho(\vec{p}) \int_\lambda^{E_d} \widetilde{d^3k}\ \left[f(\vec{p},\vec{k}) \cdot \alpha^\dagger(\vec{k}) - f^*(\vec{p},\vec{k}) \cdot \alpha(\vec{k})\right]$$

and

$$f(\vec{p},\vec{k}) \cdot \alpha^\dagger(\vec{k}) = \sum_\lambda f^\mu(\vec{p},\vec{k}) \epsilon^*_{\lambda\mu}(\vec{k}) \alpha^\dagger_\lambda(\vec{k})$$

$$f^\mu(\vec{p},\vec{k}) = \left(\frac{p^\mu}{p \cdot k} - c^\mu\right) e^{-ip\cdot kt/p^0}, \quad c^\mu = \frac{1}{2k^0}(-1, \hat{k})$$

The dressed state formalism

Faddeev-Kulish (FK) construction

- Multi-fermion FK states: let $\alpha = \{e_i, \vec{p}_i, s_i\}$. Then,

$$|\alpha\rangle_d = e^{R_f} |\alpha\rangle$$

obtain a product of $|\alpha\rangle$ with a coherent photon state

$$|\alpha\rangle_d = |\alpha\rangle \times |f_\alpha\rangle$$

where

$$|f_\alpha\rangle = \mathcal{N}_\alpha \ e^{\int_\lambda^{E_d} \widetilde{d^3k} \ \left[f_\alpha(\vec{k})\cdot\alpha^\dagger(\vec{k})\right]} |0\rangle$$

and

$$\mathcal{N}_\alpha = e^{-\frac{1}{2}\int_\lambda^{E_d} \widetilde{d^3k} \ f_\alpha^*(\vec{k})\cdot f_\alpha(\vec{k})}, \quad f_\alpha^\mu(\vec{k}) = \sum_{i\in\alpha} e_i \left(\frac{p_i^\mu}{p_i\cdot k} - c^\mu\right) e^{-ip_i\cdot kt/p_i^0}$$

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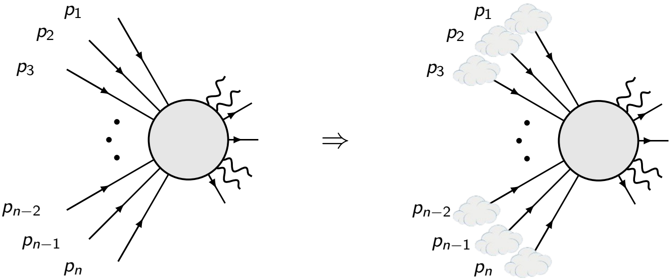
Diagrammatically...



Dressed State Formalism

Faddeev-Kulish (FK) construction

- Elastic FK amplitudes

$$\tilde{S}_{\beta\alpha} = \left(\frac{E_d}{\Lambda}\right)^{\mathcal{B}_{\beta\alpha}} e^{i\phi_{\beta\alpha}} \, S^{(\Lambda)}_{\beta\alpha} \; < \infty$$

where

$$\tilde{S}_{\beta\alpha} = {}_d\langle\beta|S|\alpha\rangle_d \,, \qquad \text{and} \qquad S_{\beta\alpha} = \langle\beta|S|\alpha\rangle$$

and

1. Λ : IR-cutoff scale
2. $\mathcal{B}_{\beta\alpha}$: positive kinematical factor
3. $\phi_{\beta\alpha}$: real phase (not contributing to observable quantities)
4. $S^{(\Lambda)}_{\beta\alpha}$: S-matrix element without contributions from virtual soft photons

Dressed State Formalism

Faddeev-Kulish (FK) construction

- Radiation-emitting FK amplitudes

$$\tilde{S}_{\beta\gamma,\alpha} = F_{\beta\alpha}\big(\vec{q}_\gamma, \epsilon_r^*(\vec{q}_\gamma)\big)$$

where

$$\tilde{S}_{\beta\gamma,\alpha} = {}_d\langle\beta\gamma|S|\alpha\rangle_d$$

and

$$F_{\beta\alpha}\big(\vec{q}_\gamma, \epsilon_r^*(\vec{q}_\gamma)\big)$$

is a function free of

1. IR-virtual divergences, and
2. the leading (i.e. ω_k^{-1}) contribution in the soft expansion.

Dressed State Formalism

Subleading extensions

- Dressed states: [Choi & Akhoury '19]

$$|\alpha\rangle_{\tilde{d}} = |\alpha\rangle \times e^{\int \widetilde{d^3k}\ [f_\alpha(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]} \left(1 + \int \widetilde{d^3k}\ [g_\alpha(\vec{k}) \cdot a^\dagger(\vec{k}) - h.c.]\right) |0\rangle$$

where

$$g^\mu_\alpha(\vec{k}) = i \sum_{i\in\alpha} e_i \frac{k_\nu}{p_i \cdot k} J_i^{\mu\nu} e^{-ip_i\cdot kt/p_i^0}, \qquad J_i^{\mu\nu} = L_i^{\mu\nu} + S_i^{\mu\nu}$$

Dressed State Formalism

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- Elastic FK amplitudes:

$$\tilde{\tilde{S}}_{\beta\alpha} < \infty, \qquad \text{where} \qquad \tilde{\tilde{S}}_{\beta,\alpha} = {}_{\tilde{d}}\langle\beta|S|\alpha\rangle_{\tilde{d}}$$

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- Elastic FK amplitudes:

$$\tilde{\tilde{S}}_{\beta\alpha} < \infty, \qquad \text{where} \qquad \tilde{\tilde{S}}_{\beta,\alpha} = {}_{\tilde{d}}\langle\beta|S|\alpha\rangle_{\tilde{d}}$$

- Radiation-emitting FK amplitudes:

$$\tilde{\tilde{S}}_{\beta\gamma,\alpha} \sim \mathcal{O}(E_d), \qquad \text{(tree-level result)}$$

Content

QED scattering: a case study of explicit interactions

Fock space amplitudes

- Explicit process: $e^-(p_1) + \gamma(p_2) \to e^-(q_1) + \gamma(q_2)$

$$S_{\text{tree}}^{(\gamma)} = e \left\{ \left[\frac{q_2 \cdot \epsilon_r^*(\vec{k})}{q_2 \cdot k} - \frac{p_1 \cdot \epsilon_r^*(\vec{k})}{p_1 \cdot k} \right] + i\epsilon_{r\mu}^*(\vec{k})\ k_\nu \left[\frac{\bar{J}_{q_2}^{\mu\nu}}{q_2 \cdot k} - \frac{J_{p_1}^{\mu\nu}}{p_1 \cdot k} \right] \right\} S_0^{(\gamma)} + \mathcal{O}(\omega_k)$$

where $S_0^{(\gamma)}$ is the elastic tree-level amplitude, and

$$J_{p_1}^{\mu\nu} = L_{p_1}^{\mu\nu} + S_{p_1}^{\mu\nu}, \qquad \bar{J}_{q_1}^{\mu\nu} = \bar{L}_{q_1}^{\mu\nu} + \bar{S}_{q_1}^{\mu\nu}$$

with

$$L_{p_1}^{\mu\nu} = i\left(p_1^\mu \frac{\partial}{\partial p_{1\nu}} - p_1^\nu \frac{\partial}{\partial p_{1\mu}} \right), \qquad \bar{L}_{q_1}^{\mu\nu} = -i\left(q_1^\mu \frac{\partial}{\partial q_{1\nu}} - q_1^\nu \frac{\partial}{\partial q_{1\mu}} \right)$$

$$S_{p_1}^{\mu\nu} = +\frac{i}{4}[\gamma^\mu, \gamma^\nu]\ u(\vec{p}_1)\ \circ\ \frac{\partial}{\partial u(\vec{p}_1)}, \qquad \bar{S}_{q_1}^{\mu\nu} = +\bar{u}(\vec{q}_1)\ \frac{i}{4}[\gamma^\mu, \gamma^\nu]\ \circ\ \frac{\partial}{\partial \bar{u}(\vec{q}_1)}$$

QED scattering: a case study of explicit interactions

Applying subleading dressings to Fock states

- Use dressed states, as suggested by CA:

$$|q_1, q_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_q(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_q(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]\right)|q_1, q_2\rangle$$

$$|p_1, p_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]\right)|p_1, p_2\rangle$$

where $|q_1, q_2\rangle, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

1. leading dressing functions

$$f_q^\mu(\vec{k}) = e\ e^{-iq_2\cdot k t_0/q_2^0}\left(\frac{q_2^\mu}{q_2\cdot k} - c^\mu\right)$$

$$f_p^\mu(\vec{k}) = e\ e^{-ip_1\cdot k t_0/p_1^0}\left(\frac{p_1^\mu}{p_1\cdot k} - c^\mu\right)$$

QED scattering: a case study of explicit interactions

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where $|q_1, q_2\rangle\,, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

2. subleading dressing functions

$$g_q^\mu(\vec{k}) = iek_\nu\ e^{-iq_2\cdot k t_0/q_2^0}\, \frac{\bar{J}_{q_2}^{\mu\nu}}{q_2 \cdot k}$$

$$g_p^\mu(\vec{k}) = iek_\nu\ e^{-ip_1\cdot k t_0/p_1^0}\, \frac{J_{p_1}^{\mu\nu}}{p_1 \cdot k}$$

QED scattering: a case study of explicit interactions

Dressed amplitudes: soft-emission suppression & IR-finiteness

- The radiation-emitting amplitude, defined as

$$\tilde{\tilde{S}}^{(\alpha)}_{\text{tree}} = {}_{\tilde{d}}\langle q_1, q_2; \gamma | S^{(1)}_{(\alpha)} | p_1, p_2 \rangle_{\tilde{d}}\,, \qquad \alpha \in \{\gamma,\ \mu,\ e\}$$

vanishes upon taking $E_d \to \Lambda$:

$$\tilde{\tilde{S}}^{(\alpha)}_{\text{tree}} = \mathcal{O}(\Lambda)$$

Subleading corrections of dressings in tree-level amplitudes

$\Rightarrow$ suppress soft photon emission

$\Rightarrow$ renders dressed state formalism equivalent to the BN method

QED scattering: a case study of explicit interactions

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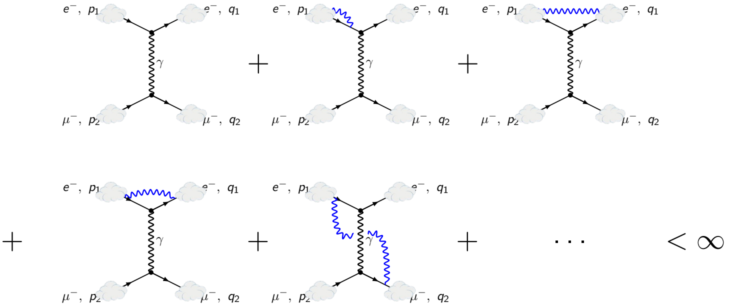
QED scattering: a case study of explicit interactions

Dressed amplitudes: soft-emission suppression & IR-finiteness

- The elastic amplitude for each of these processes is finite and equal to the corresponding undressed amplitude without virtual soft photons, i.e. for the electron-muon interaction



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Summary & future directions

- Summary of results:

  1. Dressed states are the eigenstates of H_{as} ($\neq H_0$)

     $\Rightarrow$ appropriate for the definition of S-matrix elements

Summary & future directions

- Summary of results:

  1. Dressed states are the eigenstates of H_{as} $(\neq H_0)$

     $\Rightarrow$ appropriate for the definition of S-matrix elements

  2. Application of FK dressings to Fock states

     $\Rightarrow$ removes IR-divergences due to virtual photons

     $\Rightarrow$ removes leading real soft divergences

     $\Rightarrow$ suppresses additional soft radiation (extended dressings)

$$\tilde{\tilde{S}}_{\text{tree}} \sim \mathcal{O}(E_d), \qquad \text{(tree-level result)}$$

Extensions are integrable, they do not affect elastic amplitudes

Summary & future directions

- Limitation:

  1. Tree-level result.

     $\Rightarrow$ What happens at the loop-level (or at higher perturbative orders)?

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  1. Construct loop-corrected subleading dressings by studying loop-level radiation-emitting amplitudes for these interactions.

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Summary & future directions

- Expectations regarding loop-corrections: [Krishna & Sahoo '23]

perturbation theory ↓ / soft expansion →	1-st	2-nd	. . .	m-th
1-st	ω_k^{-1}	ω_k^{0}	$\cdots$	ω_k^{m}
2-nd	ω_k^{-1}	$\log \omega_k$	$\cdots$	ω_k^{m-1}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n-th	ω_k^{-1}	$\omega_k^{n-1} \times (\log \omega_k)^n$		

Summary & future directions

- Expectations regarding loop-corrections: [Krishna & Sahoo '23]

soft expansion →

perturbation theory ↓	1-st	2-nd	. . .	m-th
1-st	$\cancelto{0}{\omega_k^{-1}}$	ω_k^0	$\cdots$	ω_k^m
2-nd	$\cancelto{0}{\omega_k^{-1}}$	$\log \omega_k$	$\cdots$	ω_k^{m-1}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n-th	$\cancelto{0}{\omega_k^{-1}}$	$\omega_k^{n-1} \times (\log \omega_k)^n$		

FK dressings

Summary & future directions

- Expectations regarding loop-corrections: [Krishna & Sahoo '23]

perturbation theory \ soft expansion	1-st	2-nd	...	m-th
1-st	ω_k^{-1} → 0	ω_k^{0} → 0	$\mathcal{O}(E_d)$	ω_k^{m}
2-nd	ω_k^{-1} → 0	$\log \omega_k$	$\cdots$	ω_k^{m-1}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n-th	ω_k^{-1} → 0	$\omega_k^{n-1} \times (\log \omega_k)^n$		

FK dressings

CA dressings

perturbation theory

Summary & future directions

- Expectations regarding loop-corrections: [Krishna & Sahoo '23]

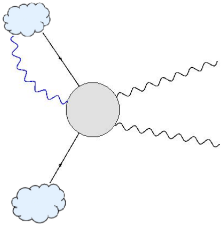
perturbation theory ↓ / soft expansion →	1-st	2-nd	...	m-th
1-st	$\cancelto{0}{\omega_k^{-1}}$	$\cancelto{0}{\omega_k^{0}}$	...	ω_k^{m}
2-nd	$\cancelto{0}{\omega_k^{-1}}$	$\cancelto{0}{\log \omega_k}$	$\mathcal{O}(\omega_k^0)$	ω_k^{m-1}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n-th	$\cancelto{0}{\omega_k^{-1}}$	$\omega_k^{n-1} \times (\log \omega_k)^n$		

FK dressings

CA dressings

Loop-corrected

Thank you for your attention



QED scattering: a case study of three interactions

Fock space amplitudes

- Explicit process: $e^-(p_1) + \mu^-(p_2) \to e^-(q_1) + \mu^-(q_2)$

$$S^{(\mu)}_{\text{tree}} = e\Big\{ \Big[\frac{q_1\cdot\epsilon^*_r(\vec{k})}{q_1\cdot k} + \frac{q_2\cdot\epsilon^*_r(\vec{k})}{q_2\cdot k} - \frac{p_1\cdot\epsilon^*_r(\vec{k})}{p_1\cdot k} - \frac{p_2\cdot\epsilon^*_r(\vec{k})}{p_2\cdot k} \Big]$$
$$+ \frac{2k\cdot(p_1-q_1)}{(p_1-q_1)^2} \Big[\frac{q_1\cdot\epsilon^*_r(\vec{k})}{q_1\cdot k} - \frac{q_2\cdot\epsilon^*_r(\vec{k})}{q_2\cdot k} - \frac{p_1\cdot\epsilon^*_r(\vec{k})}{p_1\cdot k} + \frac{p_2\cdot\epsilon^*_r(\vec{k})}{p_2\cdot k} \Big]$$
$$+ i\epsilon^*_{r\mu}(\vec{k})\; k_\nu \Big[\frac{\bar{S}^{\mu\nu}_{q_1}}{q_1\cdot k} + \frac{\bar{S}^{\mu\nu}_{q_2}}{q_2\cdot k} - \frac{S^{\mu\nu}_{p_1}}{p_1\cdot k} - \frac{S^{\mu\nu}_{p_2}}{p_2\cdot k} \Big] \Big\}\; S^{(\mu)}_0 + \mathcal{O}(\omega_k)$$

where $S^{(\mu)}_0$ is the elastic tree-level amplitude, and

$$S^{\mu\nu}_p = +\frac{i}{4}[\gamma^\mu, \gamma^\nu]\; u(\vec{p}) \;\circ\; \frac{\partial}{\partial u(\vec{p})}, \qquad \bar{S}^{\mu\nu}_q = +\bar{u}(\vec{q})\; \frac{i}{4}[\gamma^\mu, \gamma^\nu] \;\circ\; \frac{\partial}{\partial \bar{u}(\vec{q})}$$

with $p \in \{p_1,\ p_2\}$ and $q \in \{q_1,\ q_2\}$.

QED scattering: a case study of three interactions

Fock space amplitudes

- Explicit process: $e^-(p_1) + e^+(p_2) \to e^-(q_1) + e^+(q_2)$

$$S^{(e)}_{\text{tree}} = e \left\{ \left[\frac{p_2 \cdot \epsilon^*_r(\vec{k})}{p_2 \cdot k} - \frac{p_1 \cdot \epsilon^*_r(\vec{k})}{p_1 \cdot k} \right] + i\epsilon^*_{r\mu}(\vec{k})\ k_\nu \left[\frac{J^{\mu\nu}_{\bar{p}_2}}{p_2 \cdot k} - \frac{J^{\mu\nu}_{p_1}}{p_1 \cdot k} \right] \right\} S^{(e)}_0 + \mathcal{O}(\omega_k)$$

where $S^{(e)}_0$ is the elastic tree-level amplitude, and

$$J^{\mu\nu}_{p_1} = L^{\mu\nu}_{p_1} + S^{\mu\nu}_{p_1}, \qquad J^{\mu\nu}_{\bar{p}_2} = L^{\mu\nu}_{\bar{p}_2} + S^{\mu\nu}_{\bar{p}_2}$$

with

$$L^{\mu\nu}_{p_1} = i\left(p_1^\mu \frac{\partial}{\partial p_{1\nu}} - p_1^\nu \frac{\partial}{\partial p_{1\mu}} \right), \qquad L^{\mu\nu}_{\bar{p}_2} = i\left(p_2^\mu \frac{\partial}{\partial p_{2\nu}} - p_2^\nu \frac{\partial}{\partial p_{2\mu}} \right)$$

$$S^{\mu\nu}_{p_1} = +\frac{i}{4}[\gamma^\mu, \gamma^\nu]\ u(\vec{p}_1)\ \circ\ \frac{\partial}{\partial u(\vec{p}_1)}, \quad S^{\mu\nu}_{\bar{p}_2} = -\bar{u}(\vec{p}_2)\ \frac{i}{4}[\gamma^\mu, \gamma^\nu]\ \circ\ \frac{\partial}{\partial \bar{u}(\vec{p}_2)}$$

QED scattering: a case study of three interactions

Applying subleading dressings to Fock states

- Use dressed states, as suggested by CA:

$$|q_1, q_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_q(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_q(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]\right)|q_1, q_2\rangle$$

$$|p_1, p_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]\right)|p_1, p_2\rangle$$

where $|q_1, q_2\rangle, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

1. for the electron-muon interaction: leading dressing functions

$$f_q^\mu(\vec{k}) = e\left[e^{-iq_1\cdot kt_0/q_1^0}\left(\frac{q_1^\mu}{q_1\cdot k} - c^\mu\right) + e^{-iq_2\cdot kt_0/q_2^0}\left(\frac{q_2^\mu}{q_2\cdot k} - c^\mu\right)\right]$$

$$f_p^\mu(\vec{k}) = e\left[e^{-ip_1\cdot kt_0/p_1^0}\left(\frac{p_1^\mu}{p_1\cdot k} - c^\mu\right) + e^{-ip_2\cdot kt_0/p_2^0}\left(\frac{p_2^\mu}{p_2\cdot k} - c^\mu\right)\right]$$

QED scattering: a case study of three interactions

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$$|p_1, p_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_p(\vec{k}) \cdot a^\dagger(\vec{k}) - h.c.]\right) |p_1, p_2\rangle$$

where $|q_1, q_2\rangle, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

1. for the electron-muon interaction: subleading dressing functions

$$g_q^\mu(\vec{k}) = iek_\nu \left[e^{-iq_1\cdot k t_0/q_1^0} \frac{\bar{J}_{q_1}^{\mu\nu}}{q_1 \cdot k} + e^{-iq_2\cdot k t_0/q_2^0} \frac{\bar{J}_{q_2}^{\mu\nu}}{q_2 \cdot k} \right]$$

$$g_p^\mu(\vec{k}) = iek_\nu \left[e^{-ip_1\cdot k t_0/p_1^0} \frac{J_{p_1}^{\mu\nu}}{p_1 \cdot k} + e^{-ip_2\cdot k t_0/p_2^0} \frac{J_{p_2}^{\mu\nu}}{p_2 \cdot k} \right]$$

QED scattering: a case study of three interactions

Applying subleading dressings to Fock states

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$$|p_1, p_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_p(\vec{k}) \cdot a^\dagger(\vec{k}) - h.c.]\right)|p_1, p_2\rangle$$

where $|q_1, q_2\rangle, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

2. for the electron-photon interaction: leading dressing functions

$$f_q^\mu(\vec{k}) = e\ e^{-iq_2\cdot kt_0/q_2^0}\left(\frac{q_2^\mu}{q_2\cdot k} - c^\mu\right)$$

$$f_p^\mu(\vec{k}) = e\ e^{-ip_1\cdot kt_0/p_1^0}\left(\frac{p_1^\mu}{p_1\cdot k} - c^\mu\right)$$

QED scattering: a case study of three interactions

Applying subleading dressings to Fock states

- Use dressed states, as suggested by CA:

$$|q_1, q_2\rangle_d = e^{\int \widetilde{d^3k}\, [f_q(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\, [g_q(\vec{k}) \cdot a^\dagger(\vec{k}) - h.c.]\right)|q_1, q_2\rangle$$

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$$g_p^\mu(\vec{k}) = iek_\nu\; e^{-ip_1\cdot k t_0/p_1^0}\,\frac{J^{\mu\nu}_{p_1}}{p_1 \cdot k}$$

QED scattering: a case study of three interactions

Applying subleading dressings to Fock states

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$$|p_1, p_2\rangle_d = e^{\int \widetilde{d^3k}\ [f_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\ [g_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]\right)|p_1, p_2\rangle$$

where $|q_1, q_2\rangle, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

3. for the electron-positron interaction: leading dressing functions

$$f_q^\mu(\vec{k}) = (0,0,0,0)$$

$$f_p^\mu(\vec{k}) = e\left[e^{-ip_1\cdot kt_0/p_1^0}\left(\frac{p_1^\mu}{p_1\cdot k} - c^\mu\right) - e^{-ip_2\cdot kt_0/p_2^0}\left(\frac{p_2^\mu}{p_2\cdot k} - c^\mu\right)\right]$$

QED scattering: a case study of three interactions

Applying subleading dressings to Fock states

- Use dressed states, as suggested by CA:

$$|q_1, q_2\rangle_d = e^{\int \widetilde{d^3k}\, [f_q(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\, [g_q(\vec{k}) \cdot a^\dagger(\vec{k}) - h.c.]\right)|q_1, q_2\rangle$$

$$|p_1, p_2\rangle_d = e^{\int \widetilde{d^3k}\, [f_p(\vec{k})\cdot a^\dagger(\vec{k}) - h.c.]}\left(1 + \int \widetilde{d^3k}\, [g_p(\vec{k}) \cdot a^\dagger(\vec{k}) - h.c.]\right)|p_1, p_2\rangle$$

where $|q_1, q_2\rangle, |p_1, p_2\rangle \in \mathcal{H}_{\text{Fock}}$, and

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$$g_q^\mu(\vec{k}) = (0, 0, 0, 0)$$

$$g_p^\mu(\vec{k}) = iek_\nu \left[e^{-ip_1\cdot kt_0/p_1^0} \frac{J_{p_1}^{\mu\nu}}{p_1 \cdot k} - e^{-ip_2\cdot kt_0/p_2^0} \frac{J_{\bar{p}_2}^{\mu\nu}}{p_2 \cdot k} \right]$$