Entanglement, scattering and NCQED

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Details can be found in

arXiv[hep-th]: 2506.15350 Carmelo P. Martin.

PLAN

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Some background.

Some background, I.

- It is well known by now that ENTANGLEMENT can be generated by scattering of particles.
- Indeed, put the case that we've got particles with two helicities (eg, electrons, gluons..), say,

+,-

- Then the scattering of one particle in the state **+** and another particle in the state **-** may give rise to the generation of an entangled state $|f\rangle$:

$$|i\rangle = |+-\rangle \quad \longrightarrow \quad |f\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle,$$

$$|c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2 = 1.$$

- To characterize the amount of entanglement of $|f\rangle$, one may use the CONCURRENCE:

$$\Delta = 2|c_{++}c_{--} - c_{+-}c_{-+}|$$

- $0 \leq \Delta \leq 1$. $\Delta = 0$, no entanglement. $\Delta = 1$, maximal entanglement.

Some background, II.

- In SciPost Phys. 3 (2017) 5, 036, Cervera-Lierta, Latorre, Rojo and Rottoli computed –at tree level– the CONCURRENCE for several scattering processes in QED.
- The CONCURRENCE depends on the scattering angle θ.
- Two remarkable results. In the very high energy limit, $E >> m_e$, and for opposite incoming helicities, they obtained

$$\Delta(e^- e^- \to e^- e^-) = \frac{8\sin^4\theta}{35+28\cos 2\theta+\cos 4\theta} = \frac{2\tan^4\theta/2}{1+\tan^8\theta/2},$$
$$\Delta(e^- e^+ \to e^- e^+) = \frac{8\sin^4\theta}{35+28\cos 2\theta+\cos 4\theta} = \frac{2\tan^4\theta/2}{1+\tan^8\theta/2}.$$

- MAXIMAL ENTANGLEMENT is obtained if and only if $\theta = \pi/2$.

Some background, III.

- In arXiv:2504.15353 [hep-th] Nunez, Cervera-Lierta and Latorre compute –at tree level– the CONCURRENCE for the scattering process of two gluons.
- Again, the CONCURRENCE depends on the scattering angle θ, if the incoming gluon have different helicities.
- Remarkable result: they obtained, when the colliding gluons have opposite helicities, the same result as for electrons in the case of massless QED:

$$\Delta(\text{colliding gluons} = +-) = \frac{2\tan^4\theta/2}{1+\tan^8\theta/2}.$$

- MAXIMAL ENTANGLEMENT is obtained, again, if and only if $\theta = \pi/2$.

My motivation

My motivation

- The previously displayed results were obtained for Minkowski spacetime.
- Then, I wondered what would happen to the entanglement phenomenon if the particles collided on noncommutative Moyal (canonical) spacetime: this my motivation.
- Of course, one could argue that there is not a single shred of experimental evidence which tells us that at short enough distances spacetime is to be described by a noncommutative manifold rather than by a differentiable manifold.
- And yet, one can supply the following hand-waving argument to justify the use of a noncommutative manifold as a model of spacetime:

  “Because of the uncertainty principle, to localize a particle with ever increasing precision one has to increase the energy correspondingly; so eventually one creates a black hole."
- A more refined argument is that field theories on Moyal noncommutative spacetime arise naturally in the framework of strings and branes.

NCQED with massless zero charge fermions.

NCQED with massless zero charge fermions I.

- Moyal noncommutative space: The noncommutative space obtained out of the algebra of complex functions on $\mathbb{R}^4$ for the Moyal product

$$(f\star g)(x)=\int\frac{d^4p}{(2\pi)^4}\,\frac{d^4q}{(2\pi)^4}\,e^{-\frac{i}{2}p_\mu\omega^{\mu\nu}q_\nu}\,e^{-i(p+q)_\mu x^\mu}\,\tilde{f}(p)\tilde{g}(q).$$

- For (well-known) technical reasons and connection with string theory, I shall assume that $\omega^{0i}=0$, $i=1,2,3$ in the Lab. reference frame.
- The action of the model reads

$$S=-\frac{1}{4}\int d^4x\,F_{\mu\nu}(x)F^{\mu\nu}(x)+i\int d^4x\,\bar{\psi}(x)\gamma^\mu D_\mu[A]\psi(x),$$

where

$$F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)-ie[A_\mu,A_\nu]_\star(x),\,D_\mu[A]\psi(x)=\partial_\mu\psi(x)-ie[A_\mu,\psi]_\star(x)$$

- $A_\mu(x)$ is the noncommutative photon field. $\psi(x)$ is the noncommutative massless zero charge fermion field.
- The action is invariant under noncommutative gauge transformations:

$$\delta A_\mu(x)=-\partial_\mu\chi(x)+ie[A_\mu,\chi]_\star(x)=-\partial_\mu\chi(x)+ie(A_\mu\star\chi(x)-\chi\star\mathrm{A}_\mu(x)),$$
$$\delta\psi(x)=ie[\psi,\chi]_\star(x)=ie(\psi\star\chi(x)-\chi\star\psi(x)).$$

NCQED with massless zero charge fermions II.

- Salient features of the model at tree-level:
  - Photons interact on noncommutative space:

$$F_{\mu\nu} \star F^{\mu\nu} \longrightarrow \partial_\mu A_\nu \star [A^\mu, A^\nu]_\star,\ [A^\mu, A^\nu]_\star \star [A_\mu, A_\nu]_\star.$$

  - Zero charge fermions interact with the photon on noncommutative space:

$$\bar{\psi}\gamma^\mu \star D_\mu[A]\psi \longrightarrow \bar{\psi}\gamma^\mu [A_\mu, \psi]_\star.$$

- These vertices give rise to tree-level scattering process which do not occur for Minkowski spacetime:
  - photon+photon $\longrightarrow$ photon+photon,
  - fermion + fermion $\longrightarrow$ fermion + fermion (for zero charge fermions)
- In the sequel, we shall study some entanglement phenomena in those scattering processes.

$$\gamma + \gamma \longrightarrow \gamma + \gamma.$$

$\gamma + \gamma \longrightarrow \gamma + \gamma$, I.

- The scattering process to consider is the following

$$|i\rangle \to |f\rangle$$

where

$$|i\rangle = |(p_1, h_1)(p_2, h_2)\rangle,$$

$$|f\rangle = C[h_1 h_2; ++]\,|(p_3,+)(p_4,+)\rangle + C[h_1 h_2; +-]\,|(p_3,+)(p_4,-)\rangle +$$
$$C[h_1 h_2; -+]\,|(p_3,-)(p_4,+)\rangle + C[h_1 h_2; --]\,|(p_3,-)(p_4,-)\rangle.$$

- Bear in mind that $h_i 1$, $h_i = \pm$, $i = 1,2$ are the helicities of the photons and p_i, $i = 1,2,3,4$ are the 4-momenta.
- The $C[h_1 h_2; \cdot\cdot]$ coefficients above are obtained from the appropriate $\mathscr{M}$-matrix elements $\mathscr{M}[h_1 h_2; \cdot\cdot]$:

$$C[h_1 h_2; ++] = \frac{1}{\mathscr{N}}\mathscr{M}[h_1 h_2; ++], \quad C[h_1 h_2; +-] = \frac{1}{\mathscr{N}}\mathscr{M}[h_1 h_2; +-],$$
$$C[h_1 h_2; -+] = \frac{1}{\mathscr{N}}\mathscr{M}[h_1 h_2; -+], \quad C[h_1 h_2; --] = \frac{1}{\mathscr{N}}\mathscr{M}[h_1 h_2; --],$$
$$\mathscr{N} = \left(|\mathscr{M}[h_1 h_2; ++]|^2 + |\mathscr{M}[h_1 h_2; +-]|^2 + |\mathscr{M}[h_1 h_2; -+]|^2 + |\mathscr{M}[h_1 h_2; --]|^2\right)^{1/2}.$$

$\gamma + \gamma \longrightarrow \gamma + \gamma$, II.

- We are interested in entanglement being generated at tree-level. But, not for every value of the (h_1, h_2) –the incoming helicities– is entanglement produced:.
- Indeed, it has been shown by Huang, Huang and Jia (J. Phys A 44 (2011) 42541) that

$$\mathscr{M}[(k_1,h_1),(k_2,h_2),(k_3,h_3),(k_4,h_4)] =$$
$$\sum_{\sigma\in S_3} \phi(k_1,k_{\sigma(2)},k_{\sigma(3)},k_{\sigma(4)})\,\tilde{\mathscr{M}}[(k_1,h_1),(k_{\sigma(2)},h_{\sigma(2)}),(k_{\sigma(3)},h_{\sigma(3)}),(k_{\sigma(4)},h_{\sigma(4)})],$$

where

$$\phi(q_1,q_2,q_3,q_4) = e^{-\frac{i}{2}\left(\sum_{1\le i<j\le 4} q_i^\mu\,\omega_{\mu\nu}\,q_j^\nu\right)}$$

- $\tilde{\mathscr{M}}[(q_1,h_1),(q_2,h_2),(q_3,h_3,(q_4,h_4)]$ is the standard colour-ordered tree-level amplitude for four incoming gluons.
- It is known (see any textbook on the spinor helicity formalism) that
  - $\tilde{\mathscr{M}}[(q_1,h_1),(q_2,h_2),(q_3,h_3,(q_4,h_4)]$=0 if the four helicities are equal,
  - $\tilde{\mathscr{M}}[(q_1,h_1),(q_2,h_2),(q_3,h_3,(q_4,h_4)]$=0 if three helicities are equal.
- Then, one can show that [next frame]

$\gamma + \gamma \longrightarrow \gamma + \gamma$, III.

- the only non vanishing $\mathscr{M}$-Matrix elements are the following

$$\begin{aligned}
\mathscr{M}[++;++] &= \mathscr{M}[(p_1,+),(p_2,+);(p_3,+),(p_4,+)],\\
\mathscr{M}[+-;+-] &= \mathscr{M}[(p_1,+),(p_2,+);(p_3,+),(p_4,-)],\\
\mathscr{M}[+-;-+] &= \mathscr{M}[(p_1,+),(p_2,-);(p_3,-),(p_4,+)],\\
\mathscr{M}[--;--] &= \mathscr{M}[(p_1,-),(p_2,-);(p_3,-),(p_4,-)],\\
\mathscr{M}[-+;+-] &= \mathscr{M}[(p_1,-),(p_2,+);(p_3,+),(p_4,-)]\\
\mathscr{M}[-+;-+] &= \mathscr{M}[(p_1,-),(p_2,+);(p_3,-),(p_4,+)].
\end{aligned}$$

- Hence, entanglement is generated only when the incoming photons have opposite helicities:

$$\begin{aligned}
|(p_1,+)(p_2,-)\rangle &\to C[+-;+-]\,|(p_3,+)(p_4,-)\rangle + C[+-;-+]\,|(p_3,-)(p_4,+)\rangle,\\
|(p_1,-)(p_2,+)\rangle &\to C[-+;+-]\,|(p_3,+)(p_4,-)\rangle + C[-+;-+]\,|(p_3,-)(p_4,+)\rangle.
\end{aligned}$$

- Indeed

$$\begin{aligned}
|(p_1,+)(p_2,+)\rangle &\to |(p_3,+)(p_4,+)\rangle,\\
|(p_1,-)(p_2,-)\rangle &\to |(p_3,-)(p_4,-).
\end{aligned}$$

$\gamma+\gamma \longrightarrow \gamma+\gamma$, IV.

- The computation of $\mathscr{M}[+-;+-]$ and $\mathscr{M}[+-;-+]$ yields

$$\mathscr{M}[+-;+-] = \left[\mathscr{C}_1 + \mathscr{C}_2 + (\mathscr{C}_1 - \mathscr{C}_2)\cos\theta\right] e^{-2i\phi}\cot[\theta/2],$$

$$\mathscr{M}[+-;+-] = \left[\mathscr{C}_1 + \mathscr{C}_2 + (\mathscr{C}_1 - \mathscr{C}_2)\cos\theta\right] e^{-2i\phi}\tan[\theta/2],$$

  with

$$\mathscr{C}_1 = 4e^2\sin[\tfrac{1}{2}\tilde{p}_1\tilde{\omega}\tilde{p}_3]\sin[\tfrac{1}{2}(\tilde{p}_1\tilde{\omega}\tilde{p}_2 + \tilde{p}_2\tilde{\omega}\tilde{p}_3)]$$

$$\mathscr{C}_2 = 4e^2\sin[\tfrac{1}{2}\tilde{p}_2\tilde{\omega}\tilde{p}_3]\sin[\tfrac{1}{2}(\tilde{p}_1\tilde{\omega}\tilde{p}_3 - \tilde{p}_1\tilde{\omega}\tilde{p}_2)].$$

- $\tilde{p}_i$, $i=1,2,3,4$ and $\tilde{\omega}^{\mu\nu}$ are, respectively, the momenta of the photons and the noncommutativity matrix in the zero momentum frame of the incoming particles.
- θ and ϕ are, respectively, the polar and azimuth angle with regard to $\vec{\tilde{p}}_1$.

$\gamma + \gamma \longrightarrow \gamma + \gamma$, V.

- As a stated at the beginning, to quantify the amount of entanglement in the outgoing state of the scattering process I shall use the so-called CONCURRENCE. In the case at hand it reads

$$\Delta = 2\,\frac{|\mathscr{M}[+-;+-]|\,|\mathscr{M}[+-;-+]|}{|\mathscr{M}[+-;+-]|^2 + |\mathscr{M}[+-;-+]|^2} = \Delta = \frac{2\tan^4[\theta/2]}{1+\tan^8[\theta/2]}$$

- This very same result obtained in arXiv:2504.15353 [hep-th] by Nunez, Cervera-Lierta and Latorre for gluons on Minkowski spacetime.
- Hence, maximal entanglement occurs if and only if $\theta = \pi/2$.
- Question: Does this mean that the scattering of massless gauge bosons associated to a nonabelian gauge symmetry –noncommutative $U(1)$ is nonabelian– always yields the previous result?

fermion + fermion $\longrightarrow$ fermion + fermion.

fermion + fermion $\longrightarrow$ fermion + fermion, I.

- We saw that when two photons of opposite helicity collided for massless QED, we get the same concurrence as with gluons: Will this result also hold for NCQED with massless zero charge fermions?
- We shall analyze the scattering process of two fermions with opposite helicities:

$$|i\rangle \longrightarrow |f\rangle,$$

$$\begin{aligned} |i\rangle &= |(p_1, +1/2)(p_2, -1/2)\rangle, \\ |f\rangle &= C[+-;++]\,|(p_3, +1/2)(p_4, +1/2)\rangle + C[+-;+-]\,|(p_3, +1/2)(p_4, -1/2)\rangle + \\ &\quad C[+-;-+]\,|(p_3, -1/2)(p_4, +1/2)\rangle + C[+-;--]|(p_3, -1/2)(p_4, -1/2)\rangle, \end{aligned}$$

- The $C[+-;\cdot\cdot]$ are given by the corresponding $\mathscr{M}$-matrix elements:

$$\begin{aligned} C[+-;++] &= \frac{1}{\mathscr{N}}\mathscr{M}[+-;++], \quad C[+-;+-] = \frac{1}{\mathscr{N}}\mathscr{M}[+-;+-], \\ C[+-;-+] &= \frac{1}{\mathscr{N}}\mathscr{M}[+-;-+], \quad C[+-;--] = \frac{1}{\mathscr{N}}\mathscr{M}[+-;--], \\ \mathscr{N} &= \left(|\mathscr{M}[+-;++]|^2 + |\mathscr{M}[+-;+-]|^2 + |\mathscr{M}[+-;-+]|^2 + |\mathscr{M}[+-;--]|^2\right)^{1/2}. \end{aligned}$$

fermion + fermion $\longrightarrow$ fermion + fermion, II.

- In the zero-momentum frame of the incoming fermions, one gets

$$\mathscr{M}[+-;+-] = 2\mathscr{C}_1\, e^{i\phi} \cot^2[\theta/2], \quad \mathscr{M}[+-;-+] = 2\mathscr{C}_2\, e^{i\phi} \tan^2[\theta/2],$$
$$\mathscr{M}[+-;++] = 0, \quad \mathscr{M}[+-;--] = 0,$$

with

$$\mathscr{C}_1 = 4e^2 \sin[\tfrac{1}{2}\tilde{p}_1\tilde{\omega}\tilde{p}_3] \sin[\tfrac{1}{2}(\tilde{p}_1\tilde{\omega}\tilde{p}_2 + \tilde{p}_2\tilde{\omega}\tilde{p}_3)]$$
$$\mathscr{C}_2 = 4e^2 \sin[\tfrac{1}{2}\tilde{p}_2\tilde{\omega}\tilde{p}_3] \sin[\tfrac{1}{2}(\tilde{p}_1\tilde{\omega}\tilde{p}_3 - \tilde{p}_1\tilde{\omega}\tilde{p}_2)].$$

- Hence, in the case at hand, the CONCURRENCE is given by

$$\Delta = 2|C[+-;++]\, C[+-;--] - C[+-;+-]\, C[+-;-+]| = \frac{2|\mathscr{C}_1\mathscr{C}_2|}{(\mathscr{C}_1\cot^2[\theta/2])^2 + (\mathscr{C}_2\tan^2[\theta/2])^2}.$$

- Bear in mind that

$$\tilde{p}_3^{\mu} = \tilde{E}(1, \cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

so that, in general, here, the CONCURRENCE depends on θ and ϕ: $\Delta(\theta,\phi)$.

- Notice that $\Delta(\theta,\phi)$ depends, in general, on the noncommutativity matrix ω^{ij}.

fermion + fermion $\longrightarrow$ fermion + fermion, III.

- In Summary, in the fermion+fermion$\longrightarrow$ fermion+fermion scattering considered here, the amount of entanglement in the outgoing state is not the same, in general, as in the tree-level scattering of two photons with opposite helicity.
- Since the general expression for the CONCURRENCE is rather cumbersome, it is advisable to further analyze the entanglement phenomenon for two important examples:
  - head-on collision of fermions in the Lab. frame.
  - Collision at right angles in the Lab. frame.
- This we shall do next

HEAD-ON COLLISION OF FERMIONS IN THE LAB. FRAME.

- Now,
$$E_1 = E_2, \quad \vec{p}_1 = -\vec{p}_2,$$
so that
$$\Delta = \frac{2\tan^4[\theta/2]}{1+\tan^8[\theta/2]}.$$
- So, the ω^{ij} dependence has gone away, yielding the same result as for photons: a peculiarity of the head-on collision.

FERMION SCATTERING AT RIGHT ANGLES IN THE LAB. FRAME.

- For later use, let us express ω^{ij} as follows

$$\omega^{ij} = \frac{c^{ij}}{\Lambda_{nc}^2}, c^{12} = c_3, c^{23} = c_1, c^{13} = c_2, (c_1)^2 + (c_2)^2 + (c_3)^2 = 1$$

- Let us choose the colliding fermions with four-momentum $p_1^\mu = (E, \vec{p}_1)$ and $p_2^\mu = (E, \vec{p}_2)$, respectively, where $\vec{p}_1 = E(0, 1/\sqrt{2}, 1/\sqrt{2})$ and $\vec{p}_2 = E(0, 1/\sqrt{2}, -1/\sqrt{2})$.
- Then,

$$\Delta = \frac{2|\mathscr{C}_1 \mathscr{C}_2|}{(\mathscr{C}_1 \cot^2[\theta/2])^2 + (\mathscr{C}_2 \tan^2[\theta/2])^2}$$

with

$$\mathscr{C}_1 = -4e^2 \sin\left[\tfrac{1}{4}x^2\left[(c_2+c_3)\sin\phi\sin\theta + c_1(1-\cos\theta+\sqrt{2}\sin\phi\sin\theta)\right]\right]\times \sin\left[\tfrac{1}{4}x^2\left[c_1(\cos\theta-1)+(\sqrt{2}c_1+c_2-c_3)\sin\theta\sin\phi\right]\right],$$

$$\mathscr{C}_2 = 4e^2 \sin\left[\tfrac{1}{4}x^2\left[(c_2-c_3)\sin\phi\sin\theta + c_1(1+\cos\theta+\sqrt{2}\sin\phi\sin\theta)\right]\right]\times \sin\left[\tfrac{1}{4}x^2\left[c_1(1+\cos\theta)-(\sqrt{2}c_1+c_2+c_3)\sin\theta\sin\phi\right]\right].$$

$$x = \frac{E}{\Lambda_{nc}}$$

- An involved expression! Let us particularize it.

fermion + fermion $\longrightarrow$ fermion + fermion, VI.

- By rotating the Lab. frame, one may set

$$c_1 = 1/\sqrt{3}, \quad c_2 = -1/\sqrt{3}, \quad c_3 = 1/\sqrt{3}.$$

- Let set $\theta = \pi/2$ (maximal entanglement for photons and head-on collision).Then,

$$\mathscr{C}_1(x,\phi) = -4e^2 \sin\left[x^2\, \tfrac{1+\sqrt{2}\sin\phi}{4\sqrt{3}}\right] \sin\left[x^2\, \tfrac{-1+(-2+\sqrt{2})\sin\phi}{4\sqrt{3}}\right]$$
$$\mathscr{C}_2(x,\phi) = -4e^2 \sin\left[x^2\, \tfrac{-1+\sqrt{2}\sin\phi}{4\sqrt{3}}\right] \sin\left[x^2\, \tfrac{1+(-2+\sqrt{2})\sin\phi}{4\sqrt{3}}\right], \quad x = \frac{E}{\Lambda_{nc}}.$$

- It can be seen that the CONCURRENCE, Δ, obtained from the previous $\mathscr{C}$'s always (ie, any value of x) vanishes if

$$\phi = 0.78, \quad 2.35, \quad 3.92, \quad 5.49.$$

- And these are its only zeros if $x < x_c = 26^{1/4}\sqrt{\pi/(2+\sqrt{2})} = 3.0026...$
- At x_c, new zeros of the CONCURRENCE pop up.
- There are values of ϕ at which there is maximal entanglement.
- Let us illustrate this with some figures.

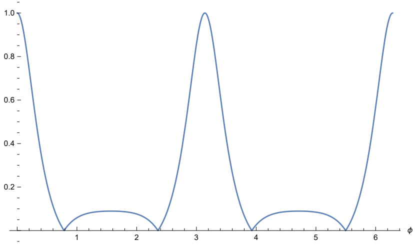

Figure: $x = 10^{-7}$ and $\theta = \pi/2$.

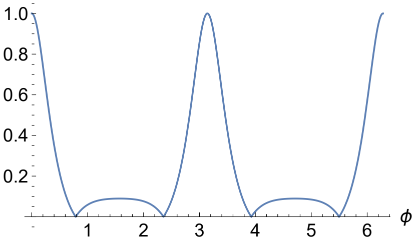

Figure: $x = 10^{-3}$ and $\theta = \pi/2$.

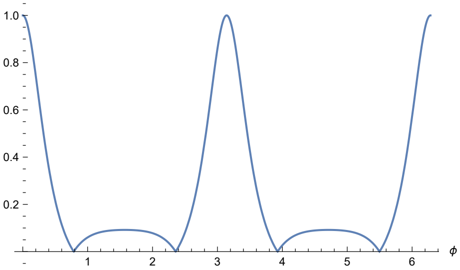

Figure: $x = 1$ and $\theta = \pi/2$.

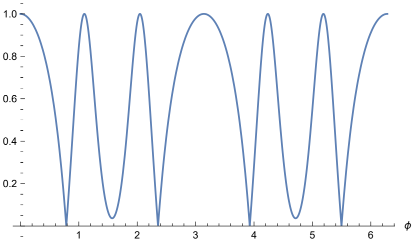

Figure: $x = 3$ and $\theta = \pi/2$.

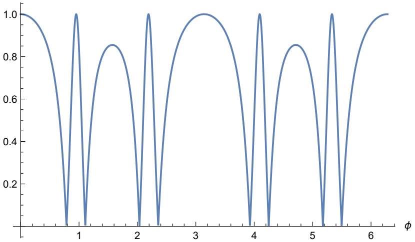

Figure: $x = 3.1$ and $\theta = \pi/2$.

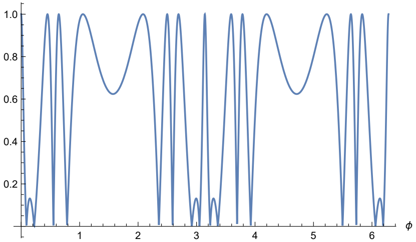

Figure: $x = 5$ and $\theta = \pi/2$.

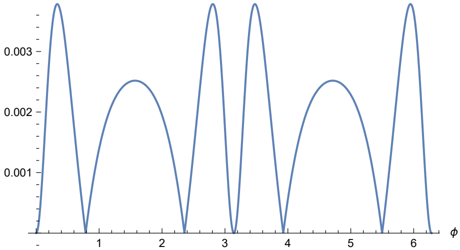

Figure: $\Delta(1,\phi) - \Delta(10^{-7},\phi)$.

Conclusions.

CONCLUSIONS.

- As far as the CONCURRENCE is concerned, the outcome of the entanglement phenomenon in NC spacetime sometimes cannot be distinguished from that in Minkowski spacetime (gluons and photons) and some other times it can (zero charge fermions at colliding at right angles).
- Much work remains to be done, eg, to see whether the maximal entanglement at $\theta = \pi/2$ found for gluons and NC photons has a deep origin.
- It would appear that the noncommutative character of spacetime leaves an imprint in the outcome of the entanglement phenomenon.