Braidings in BV

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Intro

Introduction

(Spacetime) noncommutativity and physical models have a long history of mutual interaction:

- open strings ending on branes give rise to a noncommutative geometry, $[x^i, x^j] = \mathrm{i}\theta^{ij}$
  Seiberg–Witten '99

Noncommutative quantum field theories have been developed too:

- UV/IR mixing are a typical feature
- homotopical methods on the rise: braided QFTs
  Dimitrijević-Ćirić et al.
  - suitable for perturbative quantization (integration theory of noncommutative spaces is not known)

End goal

Develop the homotopical framework (Batalin–Vilkovisky) with a coboundary structure

w/ Krutov and Weber

to overcome the case of a braided field theory with **triangular R-matrix**

[Nguyen–Schenkel–Szabo '21]

Classical finite dim field theory

Batalin–Vilkovisky

A Batalin–Vilkovisky theory consists of:

- a (co)chain complex E with differential Q;
- a bilinear symmetric form

$$\langle -, - \rangle : E \times E \mapsto \mathbb{K}[-1]\,;$$

- Q is self-adjoint w.r.t. $\langle -, - \rangle$.

Costello, Gwilliam '11

$\mathfrak{su}(2)$ example:

$$E = \Lambda^0\mathfrak{su}(2)^* \xrightarrow{\mathrm{d}} \Lambda^1\mathfrak{su}(2)^* \xrightarrow{\mathrm{d}} \Lambda^2\mathfrak{su}(2)^* \xrightarrow{\mathrm{d}} \Lambda^3\mathfrak{su}(2)^*$$

with the obvious differential $\mathrm{d} := \epsilon_{ij}{}^k\, \tilde{e}^i \tilde{e}^j\, e_k$, $\langle \tilde{e}^j, e_k\rangle_{\mathfrak{su}(2)} = \delta^j_k$.

The cohomology is [Whitehead lemmas]:

$$\mathbf{H}^i(E) = \begin{cases} \mathbb{C}\,, & i = 0, 3\,, \\ \emptyset\,, & i = 1, 2\,. \end{cases}$$

The pairing is

$$\langle \alpha, \beta\rangle = \int \alpha \wedge \beta.$$

SDR to cohomology

A *special deformation retract* to cohomology is

$$\overset{h}{\circlearrowright}(E,Q)\underset{e}{\overset{\pi}{\rightleftarrows}}(\mathbf{H}^{\bullet}(E),0).$$

with $h^2=0$, $h\circ e=0=\pi\circ h$.

- e,π are quasi-isomorphisms and $\pi\circ e=\mathbb{1}$;
- h is the homotopy inverse to Q: $h\circ Q+Q\circ h=\mathbb{1}-e\circ\pi$.

SDR for $\mathfrak{su}(2)$

$$\begin{array}{ccccccc} \Lambda^0\mathfrak{su}(2)^* & \overset{h}{\leftarrow}\!\!\!\!\underset{}{\to} & \Lambda^1\mathfrak{su}(2)^* & \overset{h}{\leftarrow}\!\!\!\!\underset{}{\to} & \Lambda^2\mathfrak{su}(2)^* & \overset{h}{\leftarrow}\!\!\!\!\underset{}{\to} & \Lambda^3\mathfrak{su}(2)^* \\ \int\star=\pi\downarrow\uparrow e & & 0\downarrow\uparrow e & & 0\downarrow\uparrow e & & \int=\pi\downarrow\uparrow e \\ \mathbb{C} & & \emptyset & & \emptyset & & \mathbb{C} \end{array}$$

with $h := \mathrm{d}^* \frac{1}{\Delta}(1 - e \circ \pi)$.

Nguyen–Schenkel–Szabo '21

Lift to the Symmetric algebra

Request from physics: **Physical observables** are rather in $S^\bullet(E^*) := T^\bullet(E^*)/ < x \otimes y - y \otimes x >$.

Firstly, the differential Q of the complex E is extended as a derivation on $(S^\bullet(E^*), \otimes_S)$ (denoted with Q_S). Then,

$$\overset{H}{\circlearrowright} (S^\bullet(E^*), Q_S) \underset{\mathcal{E}}{\overset{\Pi}{\rightleftarrows}} (S^\bullet \mathbf{H}^\bullet(E^*), 0).$$

with

$$\Pi \equiv \otimes_S \pi \,,\; \mathcal{E} = \otimes_S e \,,$$
$$H^k = \sum_{l+p=k-1} \mathbb{1}^{\otimes_S^l} \otimes_S h \otimes_S (e \circ \pi)^{\otimes_S^p}.$$

Known also as *tensor trick*, *bar* construction.

Homological perturbation

The deformation retract can be deformed by a *small perturbation* δ_Q ($\Longrightarrow \exists\,(Q \circ \delta_Q)^{-1}$),

$$(Q + \delta_Q)^2 = 0\,.$$

It allows us to obtain another SDR.

If E is an L_∞-algebra, its multiproducts are possible deformations $\Longleftrightarrow$ codifferential for a coalgebra.

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Seeking a deformation for $S^\bullet(\Lambda^\bullet \mathfrak{su}(2)^*)$

The obvious wedge product is inconsistent. Idea: work with $\Lambda^\bullet(\mathfrak{su}(2)^*) \otimes \mathfrak{g}$ with $\mathfrak{g}$ a Lie algebra instead! Then a deformation to $Q = \mathrm{d}^* \otimes \mathbb{1}$ is

$$\delta_Q = \mathbb{1} \otimes [-,-]_{\mathfrak{g}}.$$

Deformation

A particularly important outcome of the *homological perturbation lemma* is the differential on the cohomology:

$$D' = \Pi \circ \delta_Q \circ \mathcal{E} + \Pi \circ \delta_Q \circ (H \circ \delta_Q) \circ \mathcal{E} + \mathcal{O}((H \circ \delta_Q)^2) \quad (1)$$

It gives rise to tree level Feynman diagram on the physics' side!

Action functional

If Q is compatible with $(\!(-,-)\!)$, then on contracted functions $\varphi \in S^\bullet(E^*) \otimes E[-1]$

$$(\!(\varphi, Q\varphi)\!) = (-1)^{|\varphi|}(\!(Q\varphi, \varphi)\!)$$

which is an element of $S^\bullet(E^*)_0$.

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Free BV field theory of $\Lambda\mathfrak{su}(2) \otimes \mathfrak{g}$
Assuming $\mathfrak{g}$ has an invariant pairing,

$$(\!(\omega, \mathrm{d}\omega)\!)$$

Flat connections, like Chern–Simons

Jurčo–Raspollini–Saemann–Wolf '18

S-matrix for $\mathfrak{su}(2) \otimes \mathfrak{g}$

w/ L. Ravas, Master's Thesis, and J. Pulmann

The BV theory $(S^\bullet(\Lambda\mathfrak{su}(2)^* \otimes \mathfrak{g})^*, Q_S, (\!(,)\!))$ admits only the cubic vertices.

Proof: The symmetrizer of the cohomology of $\mathfrak{su}(2)^* \otimes \mathfrak{g}$ gives:

$$S^\bullet(\mathfrak{g}[0] \oplus \mathfrak{g}[-3]).$$

Since degree -1 and -2 are not present, only some of the contributions to D' can be non-zero: if m_{n+2} is

$$D'^{(n)} = \Pi \circ \underbrace{\delta_Q \circ (H \circ \delta_Q)^{\otimes^n}}_{m_{n+2}} \circ \mathcal{E} \tag{2}$$

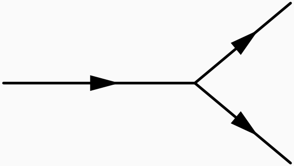
then only $m_2, m_5, m_8 ...$ are non-zero. Moreover, actually only $S^{d-1}(\mathfrak{g}[0]) \otimes \Lambda\mathfrak{g}[-3]$ and $S^d(\mathfrak{g}[0])$ embed in the complex.

H behaves like a weighted derivation. So we can follow what happens on the generators of $S^\bullet(E^*)$. In the diagrams they are intended as outputs of δ_Q

$$
\begin{array}{ccccccc}
(\Lambda^0 \mathfrak{su}_2^* \otimes \mathfrak{g})^* & \overset{h}{\underset{\leftarrow}{\rightarrow}} & (\Lambda^1 \mathfrak{su}_2^* \otimes \mathfrak{g})^* & \overset{h}{\underset{\leftarrow}{\rightarrow}} & (\Lambda^2 \mathfrak{su}_2^* \otimes \mathfrak{g})^* & \overset{h}{\underset{\leftarrow}{\rightarrow}} & (\Lambda^3 \mathfrak{su}_2^* \otimes \mathfrak{g})^* \\
\int * = \pi \downarrow \uparrow e & & 0 \downarrow \uparrow e & & 0 \downarrow \uparrow e & & \int = \pi \downarrow \uparrow e \\
\mathbb{C} \otimes \emptyset & & \emptyset \otimes \emptyset & & \emptyset \otimes \emptyset & & \mathbb{C} \otimes \emptyset
\end{array}
$$

h on the green cochain is zero; δ_Q is degree preserving and then on the coalgebra 1-form π is zero. Hence all m_{2+i} with $i > 0$ are null! The only surviving tree diagram is the 3-pt.

□



Braided QFT with $\mathfrak{sl}_q(2)$

BV data for $\mathfrak{sl}_q(2)$

Cochain complex is $\Lambda^\bullet \mathfrak{sl}_q(2)$ with differential

$$\mathrm{d}_{\wedge_q} = \frac{q^2}{1+q^4}\left(\frac{1}{c} v_{-2} L_{X_2} + \frac{q^3}{(1+q^2)c} v_0 L_{X_0} + \frac{q^2}{c} v_2 L_{X_{-2}}\right)$$

Krutov–Pandžić '24

and pairing $\langle -, - \rangle : E \times E \to \mathbb{K}[-1]$ so $E^* \cong E[1]$:

$$\langle x, y \rangle = \int x \wedge y.$$

Our definition for the symmetric tensors is

$$S(E^*) \equiv T(E^*)/\langle x \otimes y - \tilde{\sigma}_{\mathcal{R}}(x \otimes y)\rangle \tag{3}$$

where $\tilde{\sigma}_{\mathcal{R}} : E_1^* \otimes E_2^* \to E_2^* \otimes E_1^*$ is the normalised braiding.

$S^\bullet(E^*)$ is a complex with differential d_S, obtained extending $\mathrm{d}^*_{\wedge_q}$ by Leibniz property:

$$\begin{cases} \mathrm{d}_S(e) = \mathrm{d}^*_{\wedge_q} e \\ \mathrm{d}_S(e \otimes f) = \mathrm{d}^*_{\wedge_q}(e) f + (\mathrm{id} \otimes \mathrm{d}^*_{\wedge_q})(e \otimes f) \\ \mathrm{d}_S(e \otimes f \otimes g) = \mathrm{d}^*_{\wedge_q} e \otimes f \otimes g + (\mathrm{id} \otimes \mathrm{d}^*_{\wedge_q} \otimes \mathrm{id})(e \otimes f \otimes g) \\ \qquad\qquad + (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{d}^*_{\wedge_q})(e \otimes f \otimes g) \end{cases}$$

and noticing that

$$\ker S^k := \bigcap_{m+n=k-2} T^m \otimes \Lambda^2 \otimes T^n,$$

is respected by d_S.

Some preliminary results

- There is a strong deformation retract

$$\overset{H}{\circlearrowright}(S^\bullet(E^*), \mathrm{d}_S) \underset{\mathcal{E}}{\overset{\Pi}{\rightleftarrows}} (S^\bullet(\mathbf{H}^\bullet(E^*)), 0). \qquad (4)$$

  This follows from the fact that $\tilde{\sigma}_{\mathcal{R}}$ is a natural transformation (and that Π and $\mathcal{E}$ are algebra maps).
- On contracted functions $\varphi \in S^\bullet(E^*) \otimes E[-1]$, one can define a pairing that leaves the first input untouched

$$(\!(-,-)\!) : E[-1] \otimes E[-1] \to \mathbb{K}[-3], \quad (\!(-,-)\!) := \langle -,- \rangle[-2].$$

Some preliminary results

- We showed the compatibility with the differential

$$(\!(\varphi, \mathrm{d}_{\otimes}\varphi')\!) = (\!(\mathrm{d}_{\otimes}\varphi, \varphi')\!)\,, \qquad \mathrm{d}_{\otimes} := \mathsf{id} \otimes \mathrm{d}^{*}_{\wedge_q}$$

  because $e, e' \in E[-1]$, $(\!(e, \mathrm{d}^{*}_{\wedge_q} e')\!) = (\!(\mathrm{d}^{*}_{\wedge_q} e, e')\!)$.
- We suggested the action functional for the *free field theory*,

$$\mathcal{S}_q(\varphi) = (\!(\varphi, \mathrm{d}_S \varphi)\!) \in S(E^{*})^{(0)}. \tag{5}$$

Odd Poisson bracket

On the symmetric algebra we have not yet identified a Poisson/BV bracket.

We think we have

$$\{x, y\} := (x, y)\, \mathbb{1}_S, \tag{6}$$

that satisfies Jacobi up to hexagonators.

Summary

With Krutov and Weber we are attempting to arrange the BV/homotopical framework for (finite dim.) field theory with a coboundary structure.

- ✓ Focus is on a concrete example, $\Lambda\,\mathfrak{sl}_q(2)$;
- ✓ Deformation retract and a functional $\mathcal{S}_q$ are at hand;
- × Odd Poisson bracket is still missing ($\rightsquigarrow$ is $\mathcal{S}_q$ a Hamiltonian for it?)
- × Interacting theory?

$\epsilon\upsilon\chi\alpha\rho\iota\sigma\tau\omega$ for your attention!