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Noncommutative Riemannian Geometry

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Plan:

-present classical Gravity on NC space-times by developping a NC Riemmanian geometry on quantum algebras.

-provide a constructive method –via a NC Koszul formula– for Levi-Civita connections on a wide class of quantum algebras, including any algebra obtained from twist deformation of a commutative one:

Quantum phase space $xy - yx = i\hbar$,

NC Torus $uv = qvu$, $u^* = u^{-1}, v^* = v^{-1}$,

Connes-Landi spheres,..., ($\sum_{i=0}^{n} x_i^2 = 1$ with NC coordinates x_i)

Cotriangular quantum groups [Reshetikhin] (after [FRT]),

their quantum coset spaces.

[P.A. e-Print: 2006.02761v2]

-Physics Motivations

Classical Mechanics $\longrightarrow$ Quantum Mechanics functions (observables) on phase space become noncommutative (phase space noncommutativity)

General Relativity $\longrightarrow$ Quantum Gravity Spacetime structure itself becomes noncommutative.

This expectation is supported by Gedanken experiments suggesting that spacetime structure is not necessarily that of a smooth manifold (a continuum of points). Quantum spacetime effects at Planck scale $L_P \sim 10^{-33} cm$.

Below Planck scales it is then natural to conceive a more general spacetime structure where uncertainty relations and discretization naturally arise.
Space and time are then described by a *Noncommutative Geometry.*

• In string theory, study of string scatterings shows that generalized uncertainty principles where a minimal length occurs is natural. Also, because of T-duality, strings can be considered unable to test compactifications of spacetimes with radii smaller than the string scale.

• Noncommutative spacetimes arise in T-duality of open string theory in the presence of fluxes. Yang-Mills (and Born-Infeld) theories on NC spacetime have proven very fruitful
-they provide an exact low energy D-brane effective action (in a given $\alpha' \to 0$ sector of string theory where closed strings decouple).
-they allows to realize string theory T-duality symmetry within the low energy physics of Noncommutative (Super) Yang-Mills theories [Connes, Douglas, Schwartz 1997].

• T-dualities for closed strings in presence of fluxes suggests even more general nongeometric backgrounds that are NC and non associative geometries [Lüst, Blumenhagen, et al; Mylonas, Schupp, Szabo]

It is interesting to
1) understand the Riemannian geometry of these NC spacetimes,
2) see if one can consistently formulate a gravity theory. An effective theory that may capture some aspects of a quantum gravity theory.

What is the status of NC differential and Riemannian geometry?

Well established NC differential geometry ingredients
for A a NC algebra like a quantum affine variety (given by generators and relations)

- Differential calculus: $(\Omega^\bullet, \wedge, \mathrm{d})$ with $\Omega^0 = A$
- Connection: $\nabla\colon \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ satisfying $\nabla(\omega a) = \nabla(\omega)a + \omega \otimes_A \mathrm{d}a$
- Torsion: $\mathrm{Tor}_\nabla = \wedge \circ \nabla + \mathrm{d}$,
- Curvature: $\mathrm{R}_\nabla = \nabla^2$
- ...
- ... ... Equivalence of different formulations of Tor and R (forms versus vector fields)
- ... ... Bianchi identities?

But, coming to Riemmanian Geometry:

- What is a metric $\mathrm{g} \in \Omega^1 \otimes_A \Omega^1$?

- How to make sense of metric-compatibility $\nabla(\mathrm{g}) = 0$?

Two approaches to metrics on quantum algebras

- Metric structure compatible with the NC structure, e.g. *central metrics* $a\mathrm{g} = \mathrm{g}a$ for all $a \in A$ (i.e., metrics as A-bimodule maps $\mathrm{g} : \mathfrak{X} \otimes_A \mathfrak{X} \to A$)

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Similarly for connections

- The metric compatibility condition $\nabla(\mathrm{g}) = 0$ requires extending connections from Ω^1 to $\Omega^1 \otimes_A \Omega^1$.

  This is typically done by considering *Bimodule connections* [Dubois-Violette, Michor '96], [Dubois-Violette, Masson '96]

- A connection treated as a dynamical field is generally not a bimodule connection.

Results with compatible (central or H-equivariant) metrics:

- NC Riemannian geometry for central metrics on fuzzy spaces [Madore '93, '96]
- Bimodule connections and weak Levi-Civita condition [Majid '99], [Beggs, Majid '11,'14], [Beggs, Majid book '20]
- Connections on central modules ${}_{Z(A)}\mathcal{M}_{Z(A)}$ and 'tame' differential calculi [Bhowmick, Goswami, Landi '19,'20]
- LC connections for H-coinvariant metrics on algebras A with triangular Hopf algebra symmetry $(H, \mathcal{R})$ [Weber '19]
- Compatible metrics on
  - (cosemisimple) quantum groups H [Bhowmick, Mukhopadhyay '19] (strongly)
  - the NC 3-sphere S^3_q [Arnlind, Ilwale, Landi '20,'22] (weakly compatible)

Selected class of noncommutative algebras allows for arbitrary metrics:

- Moyal-Weyl noncommutativity [Wess et al. '05] (R^n_θ), [Rosenberg '13] (T^N_θ).
  (Here explicit LC connection construction)

- Abelian Drinfeld twist [Aschieri, Castellani '09] (here just existence result of LC)

- NC 3-sphere S^3_θ [Arnlind, Wilson '17]

  These LC connection results are based on existence of adapted coordinate systems (e.g. $f dx^\mu = dx^\mu f$ for $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ and derivations $\frac{\partial}{\partial x^\mu}$ generating the bimodule of vector fields.

We extend this list and present a canonical construction of Noncommutative Riemannian Geometry, including existence and uniqueness of the Levi-Civita connection, on a wide class of noncommutative algebras

Datum: An algebra A with a multiplication that is braided commutative:

$$ab = (\bar{R}^\alpha \triangleright b)(\bar{R}_\alpha \triangleright b) \ .$$

Here

$$a \otimes b \to (\bar{R}^\alpha \triangleright b) \otimes (\bar{R}_\alpha \triangleright b)$$

is a representation of the premutation group.

Examples:

- All NC algebras arising as Drinfeld twist (2-cocycle) deformations of commutative algebras are of this kind: e.g. NC-torus; Connes-Landi spheres, Connes–Dubois-Violette NC manifolds....

- Any cotriangular Hopf algebra, for example Sweedler Hopf algebra H_4.

In the present study there is no assumption on the existence of derivations of the algebra, and no use of special coordinates. Indeed we use a global, coordinate independent, approach.

We retrive the results in [Wess et al. 2005] [Rosenberg '13] by considering coordinates x^μ and partial derivatives ∂_μ. Similarly for [Rosenberg '13].

We complement the results in [Wess et al. 2006] where we used an arbitrary twist but we did not have an explicit formula for the Levi-Civita connection.

Differential and Cartan Calculus

[Gurevich '95] [T. Weber 2019]

(twist deformation case in

[P.A, Dimitrievich, Meyer, Wess '06])

Braided derivations

$$u(ab) = u(a)b + (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright u)(b) \ .$$

The commutator

$[\ ,\ ]_\mathcal{R} : \mathrm{Der}_\mathcal{R}(A) \otimes \mathrm{Der}_\mathcal{R}(A) \to \mathrm{Der}_\mathcal{R}(A)\ ,\ \ u \otimes v \mapsto uv - (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright u)$

structures $\mathrm{Der}_\mathcal{R}(A)$ as a quantum Lie algebra,

$$[u, v]_\mathcal{R} = -[\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u]_\mathcal{R}$$

$$[u, [v, z]_\mathcal{R}]_\mathcal{R} = [[u, v]_\mathcal{R}, z]_\mathcal{R} + [\bar{R}^\alpha \triangleright v, [\bar{R}_\alpha \triangleright u, z]_\mathcal{R}]_\mathcal{R} \ .$$

1-forms $\Omega(A)$ are dual to vector fields.

Pairing:

$$\langle\ ,\ \rangle : \mathfrak{X}(A) \otimes \Omega(A) \to A \quad , \quad u \otimes_A \omega \mapsto \langle u, \omega \rangle$$

Exterior derivative

$$\langle u, \mathrm{d}a \rangle = u(a) \ ,$$

Contraction operator

$$\mathrm{i}_u(\omega) = \langle u, \omega \rangle \ . \tag{1}$$

Generalize the pairing to the tensor algebra

$$\langle \nu \otimes_A u, \omega_1 \otimes_A \omega_2 \dots \omega_p \otimes_A v_1 \otimes_A \dots v_q \rangle = \langle \nu \, , \langle u_1, \omega_1 \rangle \, \omega_2 \dots \otimes_A v_1 \otimes_A v_q \rangle \ .$$

Exterior product

$$\omega \wedge \omega' := \omega \otimes_A \omega' - \bar{R}^\alpha \triangleright \omega' \otimes_A \bar{R}_\alpha \triangleright \omega \ , \tag{2}$$

is braided antisymmetric.

Lie derivative

$$\mathscr{L}_u(a) := u(a) \ , \quad \mathscr{L}_u(v) := [u, v] \ .$$

Extended to the tensor algebra via:

$$\mathscr{L}_u(v \otimes_A v') = \mathscr{L}_u(v) \otimes_A v' + \bar{R}^\alpha \triangleright v \otimes_A \mathscr{L}_{\bar{R}_\alpha \triangleright u}(v')$$

and on contravariant tensor fields is canonically defined by duality,

$$\mathscr{L}_u\langle v, \theta\rangle = \langle \mathscr{L}_u v, \theta\rangle + \langle \bar{R}^\alpha \triangleright v, \mathscr{L}_{\bar{R}_\alpha \triangleright u}\theta\rangle \tag{3}$$

Theorem (Braided Cartan calculus) [T. Weber]

$$\begin{aligned} [\mathscr{L}_u, \mathscr{L}_v] &= \mathscr{L}_{[u,v]_\mathcal{R}}, & [\mathsf{i}_u, \mathsf{i}_v] &= 0\,, \\ [\mathscr{L}_u, \mathsf{i}_v] &= \mathsf{i}_{[u,v]_\mathcal{R}}, & [\mathsf{i}_u, \mathsf{d}] &= \mathscr{L}_u, \\ [\mathscr{L}_u, \mathsf{d}] &= 0\,, & [\mathsf{d}, \mathsf{d}] &= 0\,, \end{aligned}$$

where $[L, L'] = L \circ L' - (-1)^{|L||L'|}\bar{R}^\alpha(L') \circ \bar{R}_\alpha(L)$ is the graded braided commutator of $\mathbb{k}$-linear maps L, L' on $\Omega^\bullet(A)$ of degree $|L|$ and $|L'|$.

Connections and Cartan equation

Def. A *right* connection on an A bimodule Γ is a $\mathbb{k}$-linear map

$$\nabla : \Gamma \to \Gamma \otimes_A \Omega \tag{4}$$

which satisfies the Leibniz rule, for all $s \in \Gamma$, $a \in A$,

$$\nabla(sa) = \nabla(s)a + s \otimes_A \mathrm{d}a \ . \tag{5}$$

A *left* connection on Γ is a $\mathbb{k}$-linear map

$$\nabla : \Gamma \to \Omega \otimes_A \Gamma \tag{6}$$

which satisfies the Leibniz rule,

$$\nabla(as) = \mathrm{d}a \otimes_A s + a\nabla(s) \ . \tag{7}$$

Extend ∇ to

$$\mathsf{d}_{\nabla} : \Gamma \otimes_A \Omega^{\bullet}(A) \longrightarrow \Gamma \otimes_A \Omega^{\bullet+1}(A) \quad ,$$

by

$$\mathsf{d}_{\nabla}(s \otimes_A \theta) = \nabla(s) \otimes_A \theta + s \otimes_A \mathsf{d}\theta \quad ,$$

d_{∇} satisfies the Leibniz rule,

$$\mathsf{d}_{\nabla}(\varsigma \wedge \vartheta) = \mathsf{d}_{\nabla}\varsigma \wedge \vartheta + (-1)^k \varsigma \wedge \mathsf{d}\vartheta$$

Curvature

The curvature of $\nabla \in {}_A\mathrm{Con}(\Gamma)$ is

$$\mathsf{d}_{\nabla}^2 = \mathsf{d}_{\nabla} \circ \mathsf{d}_{\nabla} \; .$$

It is a left $\Omega^{\bullet}(A)$-linear map, $\Omega^{\bullet}(A) \otimes_A \Gamma \to \Omega^{\bullet+2} \otimes_A \Gamma$

Torsion For $\Gamma = \Omega(A)$,

$$\theta \mapsto (\mathsf{d} - \wedge \circ \nabla)\theta \; .$$

Def. Connection along vector field is

$$\mathbb{\nabla}_u := i_u \circ \mathbb{\nabla}$$

It is the composition of $\mathbb{\nabla}$ acting from the *right* and i_u acting from the *left*.

More in general:

$$\mathrm{d}_{\mathbb{\nabla}_u} := i_u \circ \mathrm{d}_{\mathbb{\nabla}} + \mathrm{d}_{\mathbb{\nabla}} \circ i_u \ , \tag{8}$$

Theorem Braided Cartan relation for $\mathrm{d}_{\mathbb{\nabla}_u}$

$$\mathrm{d}_{\mathbb{\nabla}_u} \mathrm{i}_v - \mathrm{i}_{\bar{R}^\alpha \triangleright v} \mathrm{d}_{\mathbb{\nabla}_{\bar{R}_\alpha \triangleright u}} = \mathrm{i}_{[u,v]} \ .$$

All other expression of curvature and torsion are equivalent due to the above Cartan relation.

Dual connections & Cartan structure equation for curvature and torsion

Let $\triangledown$ now denote the connection dual to ∇, i.e.

$$\mathsf{d}\langle u, \theta\rangle = \langle \nabla u, \theta\rangle + \langle u, \triangledown\theta\rangle \ .$$

Def.

$$R_{\triangledown}(u, v, z) := (\nabla_u \circ \nabla_v - \nabla_{\bar{R}^\alpha \triangleright v} \circ \nabla_{\bar{R}_\alpha \triangleright u} - \nabla_{[u,v]})(z) \ .$$

$$T_{\triangledown}(u, v) := \nabla_u v - \nabla_{\bar{R}^\alpha \triangleright v} \bar{R}_\alpha \triangleright u \ - [u, v] \ .$$

Proposition

$$\langle R_{\triangledown}(u, v, z), \theta\rangle = \langle u \otimes_A v \otimes_A z, \mathsf{d}_{\triangledown}^2 \theta\rangle$$

$$\langle T_{\triangledown}(u, v), \theta\rangle = -\langle u \otimes_A v, (\mathsf{d} + \wedge \circ \triangledown)\, \theta\rangle$$

Braided Riemaniann geometry

Let $\mathrm{g} \in \Omega(A) \otimes_A \Omega(A)$.

Def. g is braided symmetric if invariant under the action of $\bar{R}^\alpha \otimes \bar{R}_\alpha$.

Example: $\omega \otimes \omega' + (\bar{R}^\alpha \triangleright \omega') \otimes (\bar{R}_\alpha \triangleright \omega)$ is braided symmetric.

Def. A pseudo-Riemannian metric on $\mathfrak{X}(A)$ is a braided symmetric nondegenerate element

Let $\mathrm{g} \in \Omega(A) \otimes_A \Omega(A)$ be a pseudo-Riemannian metric. A connection $\nabla \in \mathrm{Con}_A(\Omega(A))$ is metric compatible if it satisfies $\nabla(\mathrm{g}) = 0$. It follows

$$\mathrm{d}\langle v \otimes_A z, \mathrm{g}\rangle = \langle \nabla(v \otimes_A z), \mathrm{g}\rangle$$

A Levi-Civita connection is a metric compatible and torsion free connection.

Existence and uniqueness of Levi-Civita connection is proven, similarly to the classical case, via a **braided Koszul formula**.

For all $u, v, z \in \mathrm{Der}(A)$, (braiding omitted)

$$\mathscr{L}_u \langle v \otimes_A z, \mathrm{g} \rangle = \langle \nabla_u (v \otimes_A z), \mathrm{g} \rangle$$

$$= \langle z \otimes_A \nabla_v u \, , \mathrm{g} \rangle \; + \; \langle [u, v] \otimes_A z \, , \mathrm{g} \rangle \; + \; \langle v \otimes_A \nabla_u z \, , \mathrm{g} \rangle$$

Summing $\mathscr{L}_u \langle v \otimes_A z, \mathrm{g} \rangle - \mathscr{L}_z \langle u \otimes_A v, \mathrm{g} \rangle + \mathscr{L}_v \langle z \otimes_A u, \mathrm{g} \rangle$ (braiding omitted) we obtain

$$2 \langle {}^{\alpha} v \otimes_A \nabla_{{}_{\alpha}u} z \, , \mathrm{g} \rangle = \mathscr{L}_u \langle v \otimes_A z, \mathrm{g} \rangle - \mathscr{L}_{{}^{\alpha}v} \langle {}_{\alpha}u \otimes_A z, \mathrm{g} \rangle + \mathscr{L}_{{}^{\alpha\beta}z} \langle {}_{\alpha}u \otimes_A {}_{\beta}v, \mathrm{g} \rangle$$

$$- \langle [u, v] \otimes_A z \, , \mathrm{g} \rangle + \langle u \otimes_A [v, z] \, , \mathrm{g} \rangle + \langle [u, {}^{\beta}z] \otimes_A {}_{\beta}v \, , \mathrm{g} \rangle \; .$$

were ${}^{\alpha}v := \bar{R}^{\alpha} \triangleright v$ and ${}_{\alpha}u := \bar{R}_{\alpha} \triangleright u$. Now, since u, v, z are arbitrary, the pairing is nondegenerate and the metric is also nondegenerate, knowledge of the l.h.s. uniquely defines the Levi-Civita connection.

Conclusions

- Given a wide class of algebras A:
all those admitting an action of a triangular Hopf agebra, including all those obtained form Drinfeld twist (2-cocycle deform.) of commutative manifolds

- Given an arbitray braided symmetric metric g on A

We have shown existence and uniqueness of the Levi-Civita connection ∇.

This gives Einstein equations on A.

Ricci tensor (trace of Riemann tensor):

$$Ric(u,v) = \langle \omega^i, R_{\nabla}(e_i, u, v) \rangle' \ .$$

Einstein equations in vaccuum

$$Ric(u,v) = \lambda \langle\, u \otimes_A u \,, \mathrm{g} \rangle \ , \quad (\lambda \in \Bbbk) \ .$$

NC (speudo)Remannian manifolds (M_q, g) that satisfy this equation are NC Einstein spaces.

Example Riemannian geometry on $K \otimes K$, where K is Sweedler Hopf algebra.

K is the algebra generated by g and θ and defining relations

$$g^2 = 1 \ , \ \ \theta^2 = 0 \ , \ \ \theta g = -g\theta \ .$$

A vector space basis is given by $(1, g, \theta, g\theta)$. It it cotriangular, hence it is a braided commutative algebra (w.r.t. $K^{\circ\, op} \otimes K^{\circ}$).

The space of left invariant braided vector fields is 1-dimensional and spanned by u, with

$$u(1) = 0 \ , \ \ u(g) = 0 \ , \ \ u(\theta) = 1 \ , \ \ u(\theta g) = g \ .$$

The dual left invariant 1-form is $\omega = \mathrm{d}\theta$. (This is Woronowicz bic. diff. calc.).

Consider $K \otimes K$ generated by $1, g, \theta, g', \theta'$. We have the Einstein metric

$$\mathrm{g} = \mathrm{d}\theta \otimes_S \mathrm{d}\theta' \, (1 + \theta + \theta')$$

It is neither central nor equivariant. Its Levi-Civita connection is not a bimodule connection, the scalar curvature is $S = 12$.