Quantum Poincaré groups as locally compact quantum groups

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Quantum Poincaré groups

Quantum Poincaré group $\rightarrow$ NC Minkowski as quantum homogeneous space.

Many different models:

- κ-Poincaré (Lukierski–Nowicki–Ruegg–Tolstoy ’91, ...);
- ρ-Poincaré (Lukierski–Woronowicz ’05, ...);
- Etc.

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Question: Can we construct these models at the C^*-level? $\rightarrow$ Locally compact quantum group (Kustermans–Vaes '00).

- C^*/von Neumann algebra, coproduct, quantum Haar measures;
- Interpretation as deformation of $\mathcal{P}$.

T-Poincaré groups

Notation: Minkowski: $M = \mathbb{R}^{1,3}$; Poincaré: $\mathcal{P} = M \rtimes \mathrm{SO}(1,3)$;
Lie alg: $\mathrm{Lie}(\mathcal{P}) = \mathfrak{p} = M \rtimes \mathfrak{so}(1,3)$.

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Today: triangular r-matrix of the form

$$r = \frac{-1}{2}\omega^{\mu\nu\rho} P_\mu \wedge M_{\nu\rho} \in \mathfrak{p} \wedge \mathfrak{p}.$$

Triangular $\rightarrow$ CYBE: $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$

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- Classified by Zakrzewski '97 (cases 7-18), see also Tolstoy '07;
- Include ρ-Poincaré and lightlike κ-Poincaré;
- Mercati '24: T-Poincaré (with $\theta = 0$);
- Maris–Požar–Wallet '25: $\star$-products on Minkowski, but few admit Poincaré symmetries.

Expected properties

Dual algebra: $\mathfrak{p}^* = \mathfrak{so}(1,3)^* + M^*$, $M^* = \langle x^\mu \rangle$,

$$[\xi, \eta]_r = \mathrm{ad}^\flat_{r^\sharp(\xi)}\eta - \mathrm{ad}^\flat_{r^\sharp(\eta)}\xi.$$

- $\mathfrak{so}(1,3)^*$ is abelian sub-algebra $\quad \Rightarrow \mathrm{SO}(1,3)$ remains classical;
- $[x^\mu, x^\nu] = c^{\mu\nu}{}_\rho x^\rho, \quad c^{\mu\nu}{}_\rho = \omega^{\mu\nu}{}_\rho - \omega^{\nu\mu}{}_\rho \quad \Rightarrow M^*$ Lie sub-algebra;
- $\mathfrak{p}^* = \mathfrak{so}(1,3)^* \rtimes M^*$;
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Strategy: Look inside the double $\mathfrak{d} = \mathfrak{p} + \mathfrak{p}^*$.
Lemma: Let $\mathfrak{so}(1,3) + M^* =: \mathfrak{d}_0 \subset \mathfrak{d} = \mathfrak{p} + \mathfrak{p}^*$, and

$$J : M^* \to \mathfrak{p}$$

$$x^\mu \mapsto \eta^{\mu\nu} P_\nu + r^\sharp(x^\mu) = \eta^{\mu\nu} P_\nu - \frac{1}{2}\omega^{\mu\rho\sigma} M_{\rho\sigma}.$$

Then $\mathfrak{d}_0$ is a sub-algebra of $\mathfrak{d}$, and $\varphi = \mathrm{id}_{\mathfrak{so}(1,3)} + J : \mathfrak{d}_0 \to \mathfrak{p}$ is an isomorphism of Lie algebras.

Integrability condition

Recall: $J : M^* \to \mathfrak{p} : x^\mu \mapsto \eta^{\mu\nu} P_\nu + r^\sharp(x^\mu)$.

Write $\mathfrak{h} = J(M^*)$. We have a decomposition $\mathfrak{p} = \mathfrak{so}(1,3) + \mathfrak{h}$.

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(IC) Suppose $\exists H \leq \mathcal{P}$ integrating $\mathfrak{h} \leq \mathfrak{p}$ such that $\tilde{\forall} g \in \mathcal{P}$,

$$g = \Lambda h$$

for *unique* $\Lambda \in \mathrm{SO}(1,3), h \in H$.
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Warning: There can be a set $X \subset \mathcal{P}$ of measure 0 for which there is no factorization.

Examples:

- ρ-Poincaré: $H \cong (\mathbb{R} \ltimes_{e^{i-}} \mathbb{C}) \times \mathbb{R}$ $\to$ Exact decomposition;
- Lightlike κ-Poincaré: $H \cong \mathbb{R}^\times \ltimes \mathbb{R}^3$ $\to$ Discrepancy of measure 0.

Cocycle twisting

Notation: G Lie group, $\lambda : G \times L^2(G) \to L^2(G) : (\lambda_g \xi)(g_0) = \xi(g^{-1}g_0)$.
Group von Neumann algebra: $W^*(G) = \overline{\langle \lambda_g : g \in G \rangle}^{\mathrm{WOT}} \subset \mathcal{B}(L^2(G))$,
Coproduct: $\hat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$.
$\to (W^*(G), \hat{\Delta})$ analogue of $U(\mathfrak{g})$.

Algebra of functions: $L^\infty(G)$, $\Delta(f)(g_1, g_2) = f(g_1 g_2)$.
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Goal: Construct $\Omega \in W^*(G) \hat{\otimes} W^*(G)$ unitary satisfying the *cocycle equation*

$$(\Omega \otimes 1)(\hat{\Delta} \otimes 1)(\Omega) = (1 \otimes \Omega)(1 \otimes \hat{\Delta})(\Omega).$$

$\Rightarrow \Omega$ is a *dual unitary 2-cocycle.*
$\Rightarrow (W^*(G), \Omega \hat{\Delta}(\cdot) \Omega^*)$ is a locally compact quantum group (de Commer '11).

Main theorem

Recall: **(IC)** $\mathcal{P} = \mathrm{SO}(1,3) \cdot H$.

Define

$$A : M \to \mathrm{SO}(1,3) : \quad x = A(x)h, \qquad x \in M, h \in H;$$

$$\Omega = \int_{M \times M} D \cdot e^{iy^\mu z_\mu} \lambda_{A(y)^{-1}} \otimes \lambda_z \, \mathrm{d}y \, \mathrm{d}z,$$

for D suitable normalization factor.

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Theorem

1. *$\Omega \in W^*(\mathcal{P}) \hat{\otimes} W^*(\mathcal{P})$ is a unitary 2-cocycle;*
2. $(W^*(\mathcal{P}), \Omega \hat{\Delta}(\cdot)\Omega^*) \cong L^\infty(H) \rtimes W^*(\mathrm{SO}(1,3))$ *(bicrossed product).*
3. *$\star$-product on M:*

$$(f \star g)(x) = \int_{M \times M} D \cdot e^{iy^\mu z_\mu} f(A(y)^{-1}x) \, g(x+z) \, \mathrm{d}y \, \mathrm{d}z.$$

Proof Sketch

Two decompositions $\mathcal{P} = \mathrm{SO}(1,3) \cdot H = \mathrm{SO}(3,1) \cdot M$.
$\to$ Two bicrossed products $L^\infty(H) \rtimes W^*(\mathrm{SO}(1,3))$ and $L^\infty(M) \rtimes W^*(\mathrm{SO}(1,3))$.

Fact: Fourier transform on M: $\mathcal{F}_M : L^\infty(M) \rtimes W^*(\mathrm{SO}(1,3)) \cong W^*(\mathcal{P})$.

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Fact: Fourier transform on M: $\mathcal{F}_M : L^\infty(M) \rtimes W^*(\mathrm{SO}(1,3)) \cong W^*(\mathcal{P})$.
Stachura's cocycle ('13): $(\mathrm{SO}(1,3), H)$ and $(\mathrm{SO}(1,3), M)$ share $\mathrm{SO}(1,3)$
$\Rightarrow$ Consider $\tilde{\Omega} \in \mathcal{B}(L^2(M \times \mathrm{SO}(1,3) \times M \times \mathrm{SO}(1,3)))$,

$$\tilde{\Omega}\xi(x_1, \Lambda_1, x_2, \Lambda_2) = \xi(A(x_2)x_1, A(x_2)\Lambda_1, x_2, \Lambda_2) \cdot D^{1/2}$$

for D a suitable Radon-Nikodym derivative. Then

- $\tilde{\Omega}$ is a unitary 2-cocycle on $L^\infty(M) \rtimes W^*(\mathrm{SO}(1,3))$;
- $(L^\infty(M) \rtimes W^*(\mathrm{SO}(1,3)))_{\tilde{\Omega}} \cong L^\infty(H) \rtimes W^*(\mathrm{SO}(1,3))$;
- $\mathcal{F}_M(\tilde{\Omega}) = \Omega$.

Concluding remarks

Nothing specific about $\mathcal{P}$: Can take any $\mathfrak{g} = V \rtimes \mathfrak{q}$,

$$r = \sum_i a_i v_i \wedge X_i \quad \text{triangular,} \qquad\qquad X_i \in \mathfrak{q}, v_i \in V.$$

Further research directions:

- Central extensions: $r = \frac{-1}{2}\theta^{\mu\nu} P_\mu \wedge P_\nu + \frac{-1}{2}\omega^{\mu\nu\rho} P_\mu \wedge M_{\nu\rho}$.
- Non-triangular case, such as κ-Poincaré.

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Thank you for your attention!