# Emergence of $F^4$-couplings in heterotic/type IIA string theory

## Predictions from the Swampland Program

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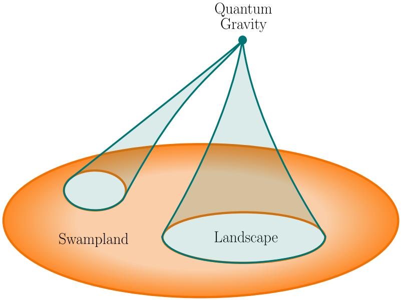
Based on **2504.05392** with M. Artime, R. Blumenhagen

# The Swampland Program

## Conceptual framework

Consistent set of conjectures motivated mainly (but not exclusively) by string theory.
Example reviews [Palti ’18, van Beest, Calderón-Infante, Mirfendereski, Valenzuela ’22, van Riet, Zoccarato ’23, Anchordoqui, Antoniadis, Lüst ’24].




- No Global Symmetries Conjecture
- **Distance Conjecture**
- **Emergent String Conjecture**
- Weak Gravity Conjecture
- (A)dS Distance Conjecture

⋮

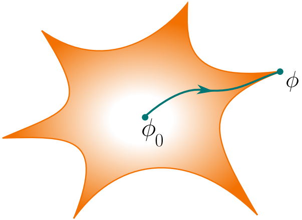
- **The Emergence Proposal**

# The Swampland Program

## Why are we always talking about light states?

- **Swampland Distance Conjecture:** [Ooguri, Vafa '06] At an infinite distance in moduli space, a tower of exponentially light states appears in our EFT, with masses given by

$$\mathrm{M(p)} \sim M(p_0)e^{-\alpha d(p_0,p)}\,, \quad \alpha \sim \mathcal{O}(1)\,.$$




- **Emergent String Conjecture:** [Lee, Lerche, Weigand '19] A QG theory at an infinite distance limit either **decompactifies**, or reduces to an asymptotically tensionless, **weakly coupled** string theory. $\rightarrow$ restriction on the lightest towers!

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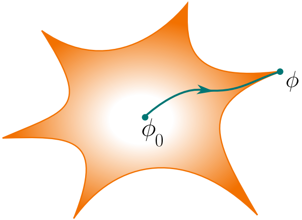


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What about the effects of these light states?

# The Species Scale

## Overview

[Dvali '07]: The UV cut-off in the presence of many light fields is the **species scale**

$$\Lambda_{\rm sp} = \frac{M_{{\rm pl},d}}{N_{\rm sp}^{1/(d-2)}} \, .$$

EFT approach to QG [Vafa et al '23-'24]

$$S_{{\rm corr.},d} \subset \frac{M_{{\rm pl},d}^{d-2}}{2} \int d^d x \sqrt{-g} \left[ \sum_n a_n(\phi_i) \frac{\mathcal{O}_n(\mathcal{R})}{M_{{\rm pl},d}^{2n-2}} \right] \, , \qquad \frac{1}{\Lambda_{\rm sp}(\phi_i)^{2n-2}} \simeq \frac{a_n(\phi_i)}{M_{{\rm pl},d}^{2n-2}} \, .$$

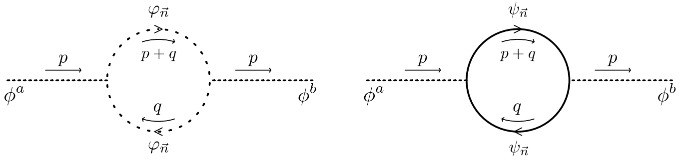
- Initial example: 4D, $\mathcal{N}=2$ [van de Heisteeg, Vafa, Wiesner, Wu '22].
- Thermodynamics interpretation [(Basile),Cribiori, Lüst, Montella '23('24)], Herráez, Lüst, Masias, Scalisi '24].
- Ongoing research: separating it from other energy scales (e.g. [Bedroya, Vafa, Wu '24]), identifying precisely which terms it suppresses [Calderón-Infante, Castellano, Herráez '25], which black holes probe it [Calderón-Infante, Delgado, Li, Lüst, Uranga '25], ...

# The Emergence Proposal

## Initial Formulation

**Emergence Proposal (Strong):** The dynamics (kinetic terms) for all fields are emergent in the infrared by integrating out towers of states down from an ultraviolet scale $\Lambda$ [Palti '19].

EFT intuition:




See [Grimm, Palti, Valenzuela '18, Castellano, Herráez, Ibáñez '22] and related applications in [Casas, Ibáñez, Marchesano '24]. Recently, emergence was connected with black hole [Calderón-Infante, Li, Lüst, Uranga '25] and localized gravity applications [Anastasi, Angius, Huertas, Uranga, Wang '25].

However, there are **counterexamples** to its full realization both in emergent string [Blumenhagen, Gligovic, AP '23] and decompactification limits [Lee, Lerche, Weigand '21].

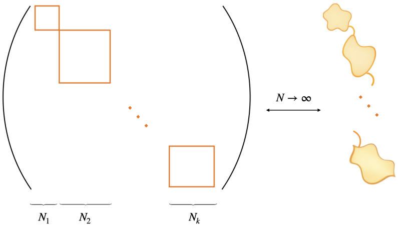
# The M-Theoretic Emergence Proposal

## Refinement

**Emergence Proposal (Strong):** The dynamics (kinetic terms) for all fields are emergent in the infrared by integrating out towers of states down from an ultraviolet scale $\Lambda$.[Palti '19]

**M-theoretic Refinement:** In the isotropic **M-theory limit** with $M_{\rm pl}^{(d)}$ fixed, a QG theory arises whose effective description emerges by integrating out the **full infinite** towers of states with mass scale not larger than $M_*$ [Blumenhagen, Cribiori, Gligovic, AP '24].

**Conceptual Similarities:** BFSS matrix model [Banks, Fischler, Shenker, Susskind '97].



# Testing the M-theoretic Refinement

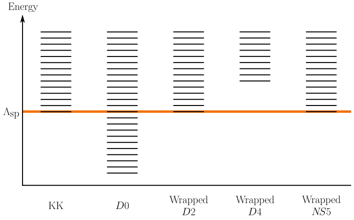
## Higher curvature corrections

- **Coscaled** decompactification limit: $\mathbf{M_{pl,d} \simeq const}$ and in M-theory quantities:

$$r_{11} \to \lambda^{\frac{2}{3}} r_{11}\,, \qquad M_* \to \frac{M_*}{\lambda^{\frac{2}{3(d-2)}}}\,, \qquad r_I \to r_I\,.$$

So that $\Lambda_{sp} \simeq M_*$ and the light states are

$$M_{D2,NS5} \simeq \frac{M_s}{g_s^{1/3}} \simeq \frac{M_{\rm pl}^{(d)}}{\lambda^{\frac{2}{3(d-2)}}} \simeq M_*\,, \quad M_{D0} \simeq \frac{M_s}{g_s} \simeq \frac{M_{\rm pl}^{(d)}}{\lambda^{\frac{2(d-1)}{3(d-2)}}}\,.$$




Using an appropriate **regularization** the refinement was succesfully tested via calculating

- the pert. contributions to the prepotential in 4D N=2 type IIA building on [Gopakumar, Vafa '98] in the resolved conifold [Blumenhagen, Cribiori, Gligovic, AP '23] and in compact examples [Blumenhagen, Gligovic '25]→ **Ralph's talk!**
- the **full** $R^4$ term in M-theory on $T^k$ in $d \geq 7$, providing physical interpretation for its mathematical properties (e.g. [Obers, Pioline '99]) in $d \geq 4$ [Blumenhagen, Cribiori, Gligovic, AP '24], extending [Green, Gutperle, Vanhove '97]

$$\mathcal{E}^{E_{k+1(k+1)}}_{\Lambda_{E_{k+1}},s=\frac{k}{2}-1} = \mathcal{E}^{E_{k(k)}}_{\Lambda_{E_k}\oplus 1,s=\frac{k}{2}-1}\,.$$

# Heterotic/Type IIA duality

## 6D setup

Twofold motivation:

- make emergence-inspired predictions ,
- use dualities to get a string theoretic result .

We will use the six dimensional duality [Sen ’96]

$$\frac{\text{Het}\,.\,SO(32)}{T^4} \xrightarrow{S} \frac{\text{Type I}}{T^4} \xrightarrow{T_{1234}} \frac{\text{Type IIB}}{T^4/\Omega\mathbb{Z}_2} \xrightarrow{S} \frac{\text{Type IIB}}{T^4/(-1)^{F_L}\mathbb{Z}_2} \xrightarrow{T_1} \frac{\text{Type IIA}}{T^4/\mathbb{Z}_2}$$

$$l_{\rm H} = g_{\rm IIA}^{(6)}\, l_{\rm IIA}\,, \qquad g_{\rm H}^{(6)} = \left(g_{\rm IIA}^{(6)}\right)^{-1}, \qquad \left(R_1^{\rm H}\right)^2 := \left(\frac{R_1}{l_{\rm H}}\right)^2 = \frac{V_{\rm K3}}{l_{\rm IIA}^4}\,.$$

We focus on the **same** type IIA strong coupling limit

$$g_{\rm IIA} \to \lambda^{\frac{6}{5}} g_{\rm IIA}\,, \quad l_{\rm IIA} \to \lambda^{-\frac{1}{5}} l_{\rm IIA}\,, \quad g_{\rm IIA}^{(6)} \to \lambda^{\frac{2}{5}} g_{\rm IIA}^{(6)}\,, \quad R_i^{\rm IIA} \to \lambda^{\frac{2}{5}} R_i^{\rm IIA}\,,$$

dual to the heterotic decompactification limit with light strings present

$$g_{\rm H} \to g_{\rm H}\,, \quad l_{\rm H} \to \lambda^{\frac{1}{5}} l_{\rm H}\,, \quad g_{\rm H}^{(6)} \to \lambda^{-\frac{2}{5}} g_{\rm H}^{(6)}\,, \quad R_{2,3,4}^{\rm H} \to R_{2,3,4}^{\rm H}\,, \quad R_1^{\rm H} \to \lambda^{\frac{4}{5}} R_1^{\rm H}\,.$$

# $F^4$-coupling

## Emergence predictions

The 1/2-BPS $F^4$-term for the diagonal linear combination of the sixteen gauge fields

$$A_i = \int_{S_i^2} C_3$$

is given by [Kiritsis, Obers, Pioline '00]

$$A_{F^4} \sim l_{\mathrm{H}}^2\, \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\mathcal{V};1} = l_*^2\, \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\mathcal{C};1} = l_*^2\, \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\mathcal{S};1}\,,$$

which is a manifestation of triality conjectured in [Obers, Pioline '99].

- "Hidden" motivation: similarity with $R^4$ expressions

$$\mathcal{E}^{\mathcal{G}}_{\mathcal{R};s} = \hat{\sum_{m_i\in\mathbb{Z}}} \left[\sum_{i,j} m_i \mathcal{M}_{ij} m_j\right]^{-s} = \frac{\pi^s}{\Gamma(s)} \hat{\sum_{m_i\in\mathbb{Z}}} \int_0^\infty \frac{dt}{t^{s+1}} e^{-\frac{\pi}{t}\sum_{i,j} m_i\mathcal{M}_{ij}m_j}\,,$$

$$\text{e.g.}\qquad \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\mathcal{V};1} = \pi \hat{\sum_{n_i,m_i}} \int_0^\infty \frac{dt}{t^2}\, \delta(\mathrm{BPS})\, e^{-\frac{\pi}{t}\left(\sum_i (n_i R_i^{\mathrm{H}})^2 + \sum_i (m_i/R_i^{\mathrm{H}})^2\right)}\,.$$

**Emergence Prediction:** $\mathcal{E}^{SO(4,4,\mathbb{Z})}_{\mathcal{V};1} \sim \mathcal{E}^{SO(3,3,\mathbb{Z})}_{\mathcal{V}\oplus 1;1}$,

# Particle Content

## Technical issues

We can map the particles states of the untwisted sector using

$$R_1^{\mathrm{H}} = \sqrt{R_1^{\mathrm{IIA}} R_2^{\mathrm{IIA}} R_3^{\mathrm{IIA}} R_4^{\mathrm{IIA}}}\,, \qquad R_2^{\mathrm{H}} = \sqrt{\frac{R_3^{\mathrm{IIA}} R_4^{\mathrm{IIA}}}{R_1^{\mathrm{IIA}} R_2^{\mathrm{IIA}}}}\,,$$

$$R_3^{\mathrm{H}} = \sqrt{\frac{R_2^{\mathrm{IIA}} R_4^{\mathrm{IIA}}}{R_1^{\mathrm{IIA}} R_3^{\mathrm{IIA}}}}\,, \qquad R_4^{\mathrm{H}} = \sqrt{\frac{R_2^{\mathrm{IIA}} R_3^{\mathrm{IIA}}}{R_1^{\mathrm{IIA}} R_4^{\mathrm{IIA}}}}\,,$$

as [Bergman, Gaberdiel '99]

$$(m_1; m_2, m_3, m_4; n_2, n_3, n_4; \mathbf{n_1}) = (m; n_{12}, n_{13}, n_{14}; n_{34}, n_{42}, n_{23}; \mathbf{n_{D4}})\,,$$

satisfying the equivalent BPS conditions

$$\sum_i n_i m_i = 0 \leftrightarrow \sum_{i,j,k,l} n_{ij} n_{kl} + m n_{D4} = 0\,.$$

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Technical difficulties:

- Impose BPS conditions within summations.
- Find organizing principle for the various terms$\to$ heterotic theory.
- Simplify as much as possible.

# Emergence of $F^4$ couplings

## Technical Issues

We can interpret the various contributions as instantonic actions after Poisson resummations using [Kiritsis, Pioline '97]

$$\int_0^\infty \frac{dx}{x^{1-\nu}}\, e^{-\frac{b}{x}-cx} = 2\left|\frac{b}{c}\right|^{\frac{\nu}{2}} K_\nu\left(2\sqrt{|b\,c|}\right),$$

focusing on the **winding sector** of the heterotic theory to perform an analysis of worldsheet instantons.

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focusing on the **winding sector** of the heterotic theory to perform an analysis of worldsheet instantons. Still the task is daunting, as the most general solution is

$$A_4^{\mathrm{H}} = 2^4\pi\, l_{\mathrm{H}}^2 V_4^{\mathrm{H}} \sum_{N>0} \sum_{\tilde{\mathbf{n}}_4} \hat{\sum_{\mu_1,\mu_2,\mu_3\in\mathbb{Z}}} \frac{e^{-2\pi N L_{\mathrm{H}}\sqrt{\mu_i \mathcal{M}_{ij}^{-1}\mu_j}}}{L_{\mathrm{H}}\sqrt{\mu_i \mathcal{M}_{ij}^{-1}\mu_j}},$$

with $\left(L_{\mathrm{H}}\sqrt{\mu_i\mathcal{M}_{ij}^{-1}\mu_j}\right)^2 = \sum_{i<j}\vartheta_{ij}^2\, C_{ij}^2$, given by

$$\begin{aligned}
C_{12} &= g_2 g_3 \mu_1\,, & C_{34} &= (Y_0X_1 - X_0Y_1)\mu_1 + (\hat{n}_1X_1 + \hat{n}_2Y_1)\mu_2 + \hat{n}_3\mu_3\,,\\
C_{23} &= g_3(X_0\mu_1 - \hat{n}_2\mu_2)\,, & C_{14} &= (Y_0Z_1 - g_2Y_1)\mu_1 + \hat{n}_1Z_1\mu_2 - \hat{n}_1 g_2\mu_3\,,\\
C_{24} &= -(X_0Z_1 - g_2X_1)\mu_1 + \hat{n}_2Z_1\mu_2 - g_2\hat{n}_2\mu_3\,, & C_{13} &= g_3(Y_0\mu_1 + \hat{n}_1\mu_2)\,.
\end{aligned}$$

# Emergence of $F^4$ couplings

**[Artime, Blumenhagen, Paraskevopoulou ’25]**

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However, things are not so depressing

$$C_{12}\, C_{34} + C_{14}\, C_{23} + C_{13}\, C_{24} = 0\,.$$

and we can scan through all solutions

| # winding numbers | # worldsheet instantons |
|---|---|
| 1 | 1, 2, 3 |
| 2 | 1, 2, 3, 4, 5 |
| 3 | 2, 3, 5, 6 |
| 4 | 3, 4, 5, 6 |

To finally obtain the surprisingly simple expression (for vanishing axions)

$$A^{\mathrm{H}}_{\mathrm{inst}} = 2\,\pi\, l^2_{\mathrm{H}} V^{\mathrm{H}}_4 \sum_{N>0} \hat{\sum_{C_{ij}|\mathrm{BPS}}} \frac{e^{-N S_{\mathrm{inst}}(\vec{C})}}{S_{\mathrm{inst}}(\vec{C})}\,, \qquad S_{\mathrm{inst}}(\vec{C}) := \sqrt{\sum_{i<j} \vartheta^2_{ij}\, C^2_{ij}}\,.$$

# Emergence of $F^4$ couplings

## Examples and $NS5$-branes in 5D

Simplest case: no instantons

$$\mathrm{C}_{\mathrm{H},n_1} = -\frac{\pi^2}{3}\frac{l_*^2 r_{11}}{r_1^2}\sum_{n_{D4}\neq 0}\sum_{|\alpha|=0}^{3}(-1)^{|\alpha|}\binom{3}{|\alpha|} = 0\,, \quad \mathrm{C}_{\mathrm{H},n_1=0} = \frac{\pi^2}{3}\frac{l_*^2 r_{11}}{r_1^2}\,.$$

Simplest instanton example: A single instanton coming from

- Single winding mode: $n_i \neq 0$ or $n_j \neq 0$ ,
- Two winding modes: $n_i, n_j \neq 0$ .

so that in total

$$A^{\mathrm{H}}_{EF1_{ij}} = A[n_i] + A[n_j] - A[n_i, n_j]\,,$$

verifying that we could neglect any $n_1$ contributions.

This simplistic example can also be used to motivate why we expect our considerations to apply also in $5D$, where

$$n_5 \leftrightarrow n_{NS5}\,,$$

and we would need to keep the $NS5$- branes while “throwing away” the $D4$-branes.

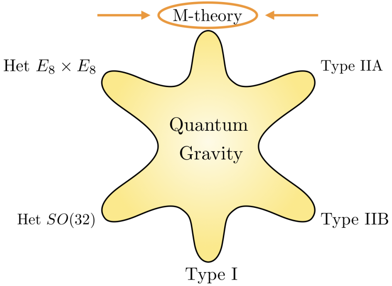
# Summary and Outlook

We used the **M-theoretic Emergence Proposal** in making a **microscopic analysis** for a string theoretic amplitude in 6D strongly coupled type IIA on K3 dual to heterotic string theory on $T^4$ leading to the **mathematical prediction**

$$\mathcal{E}^{SO(4,4,\mathbb{Z})}_{\mathcal{V};1} \sim \mathcal{E}^{SO(3,3,\mathbb{Z})}_{\mathcal{V}\oplus 1;1} \, .$$

**Future directions:**

- Non 1/2-BPS quantities? Non BPS?
- Connections with other approaches e.g. [Hattab, Palti '23-24]?
- M(-atrix) model implications? e.g. [Blumehagen, AP, Raml '25]



# Thank you

# $R^4$ terms: examples in emergent string limit

Usual starting point:

$$a_{10-k}^{1-\mathrm{loop}} \simeq 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_1^2} \sum_{m^I, n^I \in \mathbb{Z}^k} e^{-\frac{\pi}{\tau_1} \sum_{I,J=1}^{k} (m^I + n^I \tau) G_{IJ} (m^J + n^J \bar{\tau})} \,,$$

can also be expressed as a particular Eisenstein series [Obers, Pioline '99, Angelantonj, Florakis, Pioline '12]

$$\mathcal{E}^{SO(k,k)}_{V; s=\frac{k}{2}-1} \simeq \frac{2\pi}{\vartheta_{(k)}} \int_0^\infty \frac{dt}{t^{(k/2-1)+1}} \hat{\sum_{m_I, n^I \in \mathbb{Z}^k}} \delta(\mathrm{BPS})\, e^{-\frac{\pi}{t} M^2} \,,$$

with the BPS conditions taking the form of a simple Diophantine equation*

$$\sum_{I=1}^{k} m_I \, n^I = 0 \,.$$

# $R^4$ terms: examples in emergent string limit

- $d = 10$: Our method seems to not work? $\int_\epsilon^\infty dt/t^2 = 1/\epsilon$ .
- $d = 9$: We are back in the game

$$\int_\epsilon^\infty \frac{dt}{t^{\frac{3}{2}}} e^{-tA} = \frac{2}{\sqrt{\epsilon}} - 2\sqrt{\pi A} + \mathcal{O}(\sqrt{\epsilon})\,,$$

$$\text{BPS conditions} \rightarrow \begin{cases} a_{9,m=0}^{1-\text{loop}} \simeq \frac{2\pi}{\rho_1} \sum_{n\neq 0} \int_0^\infty \frac{dt}{t^{3/2}} e^{-\pi t \rho_1^2 n^2} = \frac{2\pi^2}{3}\,, & \text{winding,} \\ a_{9,n=0}^{1-\text{loop}} \simeq \frac{2\pi}{\rho_1} \sum_{m\neq 0} \int_0^\infty \frac{dt}{t^{3/2}} e^{-\pi t \frac{m^2}{\rho_1^2}} = \frac{2\pi^2}{3} \frac{1}{\rho_1^2}\,, & \text{KK.} \end{cases}$$

  We can get the full **one-loop** result and decompactify to get the 10D one.
- $d \leq 8$: The pattern remains the same, we can get the full **one-loop** result, as long as we are able to solve the BPS conditions.
- **Key Observation**: constant term $\leftrightarrow$ extended objects (strings).

# $R^4$ terms: Example of an $NS5$-brane contribution

The full set of BPS conditions is [Obers, Pioline '99]

$$\sum_J n_{IJ} m_J = 0\,, \quad n_{[IJ}\, n_{KL]} + \sum_P m_P\, n_{PIJKL} = 0\,, \quad n_{I[J}\, n_{KLMNP]} = 0\,. \tag{1}$$

An example of such a solution is the configuration

$$(n_{45}, m_1) = P(-\tilde{\nu}_5, \tilde{n}_{23})\,, \quad (n_{15}, m_4) = Q(\tilde{\nu}_5, \tilde{n}_{23})\,, \quad (n_{14}, m_5) = R(-\tilde{\nu}_5, \tilde{n}_{23})\,,$$

where $P, Q, R \in \mathbb{Z}$. This contributes as

$$a_5^{\text{typ}} \simeq \frac{2\pi}{r_{11} t_{12345}} \sum_{\tilde{n}_{23}, \tilde{\nu}_5 \in \mathbb{Z}} \sum_{N>0} \sum_{P,Q,R,m \in \mathbb{Z}} \int_0^\infty dt\, t^{\frac{1}{2}}\, e^{-\pi t \left( N^2 t_{23}^2 L^2 + \frac{m^2}{r_{11}^2} + \left( \frac{P^2}{r_1^2} + \frac{Q^2}{r_4^2} + \frac{R^2}{r_5^2} \right) L^2 \right)} \tag{2}$$

$$\simeq 2\pi \sum_{\tilde{n}_{23}, \tilde{\nu}_5 \in \mathbb{Z}} \sum_{N>0} \sum_{(P,Q,R,m) \neq (0,0,0,0)} \frac{1}{S\, L^2}\, e^{-2\pi N S}\,, \quad L = \sqrt{\tilde{\nu}_5^2 t_{145}^2 + \tilde{n}_{23}^2}\,, \tag{3}$$

$$S = \sqrt{P^2 t_{123}^2 + Q^2 t_{234}^2 + R^2 t_{235}^2 + m^2 \left( \tilde{n}_{23}^2 (r_{11} t_{23})^2 + \tilde{\nu}_5^2 (r_{11} t_{12345})^2 \right)}\,.$$