Hopf Algebra Coderivations and Quantum Groups

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Motivations and Plan

Quantum Groups(QG) are a special sort of Hopf algebras that posses a classical limit. In fact, there are two kinds of QG: quantize enveloping algebras (Drinfeld) and matrix (Woronowicz, FRT), which are "mutually dual" each other (FRT).
Differential calculus is a basic tool in NCG. They are natural object for matrix QGs, since in the classical limit one can reconstruct Cartan calculus on Lie group manifolds.Here we argue that for quantized enveloping algebras more natural are codifferential calculi.

PLAN:

- Coalgebra coderivations and First-Order Codifferential Caluli (FOCC)
- Hopf algebra bicocovariant FOCC
- Some examples
- Conclusions and perspectives

Coderivations

Differential calculus on associative unital -algebra $\mathcal{A} = (\mathcal{A}, \mu = \cdot, 1 = 1_{\mathcal{A}})$ is a derivation

$$d : \mathcal{A} \to \Omega \in {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}, \quad \text{such that} \quad d(f \cdot g) = df.g + f.dg$$

We also assume that $\Omega = \mathrm{Span}\{f.dg\} = \mathrm{Span}\{df.g\}$ (FODC). Dually, let $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon = \varepsilon_{\mathcal{C}})$ be a coalgebra. Coderivation (Y. Doi)

$${}^{\mathcal{C}}\mathfrak{M}^{\mathcal{C}} \ni \Upsilon \xrightarrow{\delta} \mathcal{C}, \quad \text{such that} \quad \Delta \circ \delta = (\mathrm{id} \otimes \delta) \circ \Delta_L + (\delta \otimes \mathrm{id}) \circ \Delta_R \,,$$

$$\text{or} \quad \delta(m)_{(1)} \otimes \delta(m)_{(2)} = m_{(-1)} \otimes \delta(m_{<0>}) + \delta(m_{<0>}) \otimes m_{(1)} \,.$$

Restriction of δ to any subbicomodule $\Upsilon_1 \subset \Upsilon$ becomes a coderivation on Υ_1.

The image of any coderivation $\mathrm{Im}\delta$ is a coideal in $\mathcal{C}$

$$\epsilon \circ \delta = 0 \,, \quad \text{i.e.} \quad \mathrm{Im}\delta \subset \mathrm{Ker}\varepsilon \,.$$

Universal coderivations

Similarly to the case of algebras, for each coalgebra there exists a universal coderivation [Doi81]. Firstly, for a coalgebra $(\mathcal{C}, \Delta, \varepsilon)$ one defines a universal bicomodule $\Upsilon_{\mathcal{C}}^{U}$ as a quotient

$$\Upsilon_{\mathcal{C}}^{U} = \mathcal{C} \otimes \mathcal{C} \,/\, \mathrm{Im}\Delta \equiv \mathrm{Coker}\Delta$$

together with $([a_{(1)} \otimes a_{(2)}] = 0)$

$$\begin{aligned} \Delta_L^U[a \otimes b] &:= a_{(1)} \otimes [a_{(2)} \otimes b], \\ \Delta_R^U[a \otimes b] &:= [a \otimes b_{(1)}] \otimes b_{(2)}, \end{aligned}$$

where $[a \otimes b]$ denotes the corresponding equivalence class of a simple tensor $a \otimes b \in \mathcal{C} \otimes \mathcal{C}$ $([\mathrm{Im}\Delta] = 0)$. Then one gets the exact sequence of bicomodules

$$0 \longrightarrow \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\pi} \Upsilon_{\mathcal{C}}^{U} \longrightarrow 0,$$

where $\pi(a \otimes b) = [a \otimes b]$ is a canonical projection. The universal coderivation is defined by

$$\delta^U[a \otimes b] := a\varepsilon(b) - b\varepsilon(a)$$

Universality theorem

Proposition.(Y. Doi) There is one-to-one correspondence between

$$\mathrm{Com}^{(\mathcal{C},\mathcal{C})}(\Upsilon, \Upsilon^U_{\mathcal{C}}) \longleftrightarrow \mathrm{Coder}(\Upsilon, \mathcal{C}) \ni \delta$$

which is an isomorphism of $\mathbb{K}$-spaces.

$$\hat{\delta}(m) \doteq [\delta(m_{<0>}) \otimes m_{(1)}] = -[m_{(-1)} \otimes \delta(m_{<0>})] .$$

The last two expressions differ by $[\Delta(\delta(m))] = 0$. Therefore, $\hat{\delta}(\Upsilon) \subset \Upsilon^U_{\mathcal{C}}$ is a subbicomodule. Moreover,

$$\delta = \delta^U \circ \hat{\delta}$$

. **Definition** First order codifferential calculus (FOCC) over a coalgebra $\mathcal{C}$ is a pair (Υ, δ), where $\Upsilon \in {}^{\mathcal{C}}\mathfrak{M}^{\mathcal{C}}$ and $\delta : \Upsilon \to \mathcal{C}$ is a coderivation, such that the corresponding bicomodule homomorphism $\hat{\delta}$ is injective, i.e. $\mathrm{Ker}\hat{\delta} = 0$.

Corollary. This allows to classify FOCC classifying subbicomodules of $\Upsilon^U_{\mathcal{C}}$ (in fact, up to a coalgebra automorphism $\phi : \mathcal{C} \to \mathcal{C}$).

Morphisms of coderivations

Let $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a coalgebras morphism and $\Psi : \Upsilon_1 \to \Upsilon_2$ the corresponding bicomoduls morphism. Then the following diagram commutes (${}_L\Delta_R = (\Delta_L \otimes \mathrm{id}) \circ \Delta_R$):

$$
\begin{array}{ccccccccc}
\mathbb{K} & \xleftarrow{\varepsilon_1} & \mathcal{C}_1 & \xleftarrow{\delta_1^U} & \Upsilon^U_{\mathcal{C}_1} & \xleftarrow{\widehat{\delta_1}} & \Upsilon_1 & \xrightarrow{(\phi \otimes \mathrm{id} \otimes \phi) \circ {}_L\Delta^1_R} & \mathcal{C}_2 \otimes \Upsilon_1 \otimes \mathcal{C}_2 \\
 & \nwarrow^{\varepsilon_2} & \downarrow \phi & & \downarrow [\phi \otimes \phi] & & \downarrow \Psi & & \downarrow \mathrm{id} \otimes \Psi \otimes \mathrm{id} \\
 & & \mathcal{C}_2 & \xleftarrow{\delta_2^U} & \Upsilon^U_{\mathcal{C}_2} & \xleftarrow{\widehat{\delta_2}} & \Upsilon_2 & \xrightarrow{{}_L\Delta^2_R} & \mathcal{C}_2 \otimes \Upsilon_2 \otimes \mathcal{C}_2
\end{array}
$$

- If ϕ is injective, i.e. $\mathcal{C}_1 \equiv \mathrm{Im}\phi$ is a subcoalgebra of $\mathcal{C}_2$ then $[\phi \otimes \phi]$ is also injective, and $\Upsilon^U_{\mathcal{C}_1} \subset \Upsilon^U_{\mathcal{C}_2}$.
- If ϕ is surjective then its kernel is a coideal in $\mathcal{C}_1$ and $\mathcal{C}_2 \equiv \mathcal{C}_1/\mathrm{Ker}\phi$. Then $[\phi \otimes \phi]$ is also surjective and $\Upsilon^U_{\mathcal{C}_2} \equiv \Upsilon^U_{\mathcal{C}_1}/\mathrm{Ker}[\phi \otimes \phi]$.
- If ϕ is an automorpismm then $[\phi \otimes \phi]$ is an automorphism.

Some remarks on $\dim \mathcal{C} = N < \infty$ case

Then $\dim \Upsilon^U_{\mathcal{C}} = N(N-1)$ and $\dim \ker \delta^U = (N-1)^2$. Taking the dual $\mathcal{C}^*$ one gets algebra with the multiplication given by the transposition map $\Delta^*(\alpha \otimes \beta)(a) \equiv \alpha \star \beta(a) \doteq \alpha(a_{(1)})\beta(a_{(2)})$ (convolution product), ε serves as the algebra unit. There is a dual pairing $< \alpha, a > \doteq \alpha(a)$ given by the evaluation map. It extends to the pairing between $\ker \Delta^*$ and $\Upsilon^U_{\mathcal{C}}$:

$$< \sum \alpha^i \otimes \beta^i, [a \otimes b] > \doteq \sum \alpha^i(a)\beta^i(b) \quad \text{for} \quad \sum \alpha^i \star \beta^i = 0$$

Remembering that $\ker \Delta^*$ is a bimodule of universal differential one-forms, one finds

$$< d^U\alpha, [a \otimes b] > = < \alpha, \delta^U([a \otimes b]) > = \varepsilon(b)\alpha(a) - \varepsilon(a)\alpha(b) \, .$$

where $d^U\alpha = \alpha \otimes \epsilon - \epsilon \otimes \alpha$.

$\dim \mathcal{C} = \infty$

$$0 \longrightarrow \mathcal{C} \xrightarrow{\ \Delta\ } \mathcal{C} \otimes \mathcal{C} \xrightarrow{\ \pi\ } \Upsilon^{U}_{\mathcal{C}} \longrightarrow 0$$

Dualizing the above exact sequence of bicomodeles, one gets, in general, a commutative diagram with exact rows of being bimodule morphisms

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{C}^* & \xleftarrow{\ \Delta^*\ } & (\mathcal{C} \otimes \mathcal{C})^* & \xleftarrow{\ \pi *\ } & (\Upsilon^{U}_{\mathcal{C}})^* & \longleftarrow & 0 \\
 & & & \nwarrow^{\mu} & \uparrow \cup & & \cup \uparrow & \swarrow & \\
 & & & & \mathcal{C}^* \otimes \mathcal{C}^* & \xleftarrow{\ \supset\ } & \ker \mu & &
\end{array}$$

where $\Phi \in (\mathcal{C} \otimes \mathcal{C})^*$ are bilinear forms on the vector space $\mathcal{C}$, and $(\Upsilon^{U}_{\mathcal{C}})^*$ consist of the forms vanishing on all coproducts, i.e. $\Phi(\Delta(a)) = 0$ for arbitrary $a \in \mathcal{C}$. The vertical arrows are injective maps that become identities in the finite-dimensional case. Hoever, by choosing appropriate subalgebra $\mathcal{A} \subset \mathcal{C}^*$ being in dual pair with $\mathcal{C}$, one can recover previous result.

Bicovariant bimodules

Let $H = (H, \Delta, \mu = \cdot, \varepsilon, 1, S)$ be a Hopf algebra.

Definition. Bicovariant bimodule (a.k.a. Hopf bimodule) over Hopf algebra H is an object $(M, \triangleright, \triangleleft, \Delta_L, \Delta_R) \in {}^H_H\mathfrak{M}^H_H$, such that:

- $(M, \triangleright, \triangleleft) \in {}_H\mathfrak{M}_H$ is a bimodule, $(M, \Delta_L, \Delta_R) \in {}^H\mathfrak{M}^H$ is a bicomodule, and the following compatibility conditions are satisfied

$$\Delta_L(a \triangleright m \triangleleft b) = (a_{(1)} \cdot m_{(-1)} \cdot b_{(1)}) \otimes (a_{(2)} \triangleright m_{<o>} \triangleleft b_{(2)}),$$

and $(1 \triangleright m = m \triangleleft 1 = m)$

$$\Delta_R(a \triangleright m \triangleleft b) = (a_{(1)} \triangleright m_{<o>} \triangleleft b_{(1)}) \otimes (a_{(2)} \cdot m_{(1)} \cdot b_{(2)}).$$

The structure theorem for such objects indicates (e.g. Schauenburg '93) that $M \cong V_L \otimes H \cong H \otimes V_R$, where $(V_L, \Delta_L, \triangleright) \in {}^H_H\mathfrak{YD}$, $(V_R, \Delta_R, \triangleleft) \in \mathfrak{YD}^H_H$

$$\Delta_L(a \triangleright u) = a_{(1)} v_{(-1)} S(a_{(3)}) \otimes a_{(2)} \triangleright v_{<o>}$$

$$\Delta_R(v \triangleleft a) = v_{<0>} \triangleleft a_{(2)} \otimes S(a_{(1)}) v_{(1)} a_{(3)}$$

The case of universal bicomodule

Theorem. For any Hopf algebra the universal bicomodule $\Upsilon^U_H \cong \bar{H}_L \otimes H \cong H \otimes \bar{H}_R$ is bicovariant, where $\bar{H} \doteq H/\{1\mathbb{K}\} \ni \bar{a} = \overline{a + \lambda 1}, a \in H$.
Here, $\bar{H}_L = (\bar{H}, \Delta_L, \triangleright) \in {}^H_H\mathfrak{YD}$

$$\Delta_L(\bar{a}) = a_{(1)} \otimes \overline{a_{(2)}}, \quad x \triangleright \bar{a} = \overline{x_{(1)} a S(x_{(2)})},$$

and $\bar{H}_R = (\bar{H}, \Delta_R, \triangleleft) \in {}^H_H\mathfrak{YD}$

$$\Delta_R(\bar{a}) = \overline{a_{(1)}} \otimes a_{(2)}, \quad \bar{a} \triangleleft x = \overline{S(x_{(1)}) a x_{(2)}}$$

are two canonical Yetter-Drinfeld structures inherited from H.

Notice that $\bar{H}_R \subset \Upsilon^U_H$ by the identification $[1 \otimes a] \mapsto \bar{a}$, which turns out to be an isomorphism of right-right Yetter-Drineld moduls, hence $\delta^U(\bar{a}) = \varepsilon(a)1 - a$. Similarly, the identification $[a \otimes 1] \mapsto \bar{a}$ provides a left-left YD embedding $\bar{H}_L \subset \Upsilon^U_H$.

Definition Let $\Upsilon \in {}_H^H\mathfrak{M}_H^H$ then a coderivation (Υ, δ) over H is called bicovariant if

$$\delta(a \triangleright m \triangleleft b) = a\,\delta(m)\,b$$

Proposition 1. The universial FOCC $\delta^U : \Upsilon_H^U \to H$ is bicovariant.
2. Let $\Upsilon \in {}_H^H\mathfrak{M}_H^H$ then a coderivation (Υ, δ) over H is bicovarian iff $\hat{\delta}(\Upsilon)$ is bicovariant subbicomodule in Υ_H^U
3. Classification of bicovariabt FOCC can be reduced to the classification of left-left YD subcomodules in $\bar{H}_L$ up to Hopf algebra automorphisms.[1]

More exactly, we have to find non-isomorphic YD sumbmodules $U \subset \bar{H}$, i.e.e subspaces satatisfying the following conditions

$$\mathrm{Ad}_{\bar{H}}^L U \subset U\,, \quad \Delta_L(U) \subset H \otimes U$$

and

$$\mathrm{Ad}_{\bar{H}}^R U \subset U\,, \quad \Delta_R(U) \subset U \otimes H$$

[1]Assuming bijective antipod.

Some remarks on $\dim H = N < \infty$ case

Then $\dim \Upsilon_H^U = N(N-1)$ and $\dim \ker \delta^U = (N-1)^2$. Taking the dual $H^* = (H^*, \mu^*, \Delta^*, \epsilon^*, 1^* = \varepsilon_H)$ one gets a dual Hopf algebra with dual pairing $< \alpha, a > \doteq \alpha(a)$ given by the evaluation map. All dual maps are obtained by the transposition.
$\Delta^*(\alpha \otimes \beta)(a) \equiv \alpha \star \beta(a) \doteq \alpha(a_{(1)})\beta(a_{(2)})$ (convolution product),
$\epsilon^*(\alpha) = \alpha(1_H)$,
The coproduct $\alpha(ab) = \mu^*(\alpha)(a \otimes b) = \alpha_{(1)}(a)\alpha_{(2)}(b)$. There is the pairing between $\ker \Delta^*$ and $\Upsilon_{\mathcal{C}}^U$:

$$< \sum \alpha^i \otimes \beta^i, [a \otimes b] > \doteq \sum \alpha^i(a)\beta^i(b) \quad \text{for} \quad \sum \alpha^i \star \beta^i = 0$$

As before, one finds

$$< d^U \alpha, [a \otimes b] > = < \alpha, \delta^U([a \otimes b]) > = \varepsilon(b)\alpha(a) - \varepsilon(a)\alpha(b) .$$

where $d^U \alpha = \alpha \otimes \epsilon - \epsilon \otimes \alpha$.

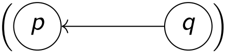
There is also induced pairing between $\ker \epsilon^*$ and $\bar{H}$:
$\alpha(\bar{a}) = \alpha(\overline{a + \lambda 1_h})$, since $\alpha(1_H) = 0$.
Now $\ker \Delta^* \cong (\ker \epsilon^*)_L \otimes H^* \cong (\ker \epsilon^*)_R$ is a bicovariant bimodule dual to Υ^U_H, where $(\ker \epsilon^*)_L \in {}^{H^*}_{H^*}\mathfrak{YD}$ with the YD structure dual to that of $\bar{H}_L \in {}^H_H\mathfrak{YD}$.
More exactly, $((\ker \epsilon^*)_L, \Xi_L, \gg)$ is defined as ($\alpha \in \ker \epsilon^*$)

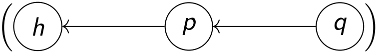
$$\beta \gg \alpha = \beta \star \alpha \,, \quad \Xi_L(\alpha) = \alpha_{(1)} \gg S^*(\alpha_{(3)}) \otimes \alpha_{(2)}$$

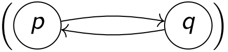
If $\alpha(1_H) = 0$ then
$\alpha(\overline{a_{(1)} b S(a_{(2)})}) = \alpha(a_{(1)} b S(a_{(2)}) = \alpha_{(1)}(a_{(1)}) \alpha_{(2)}(b) \alpha_{(3)}(S(a_{(2)})) =$
$\alpha_{(1)}(a_{(1)}) S^*(\alpha_{(3)})(a_{(2)}) \alpha_{(2)}(b) =< \alpha_{(1)} \star S^*(\alpha_{(3)}) \otimes \alpha_{(2)}, a \otimes \bar{b} >$.

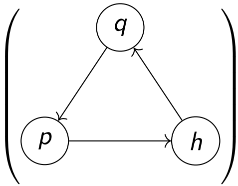
Set coalgebra $\mathbb{K}(X) = \oplus_{x\in X}\mathbb{K}_x$

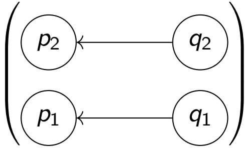
$$\Delta(p) = p \otimes p\,, \qquad \delta^U[p\otimes q] = p - q$$

$\{[p\otimes q]\} \quad \rightarrow$  , (1)

$\{[h\otimes p],[p\otimes q]\} \quad \rightarrow$  , (2)

$\{[p\otimes q],[q\otimes p]\} \quad \rightarrow$  , (3)

$\{[h\otimes p],[p\otimes q],[q\otimes h]\} \quad \rightarrow$  , (4)

$\{[p_1\otimes q_1],[p_1\otimes q_2]\} \quad \rightarrow$  . (5)

Coalgebra generated by a vector space: $\mathcal{C}_V = \mathbb{K} \oplus V$

$$\Delta(v) = v \otimes 1 + 1 \otimes v\,, \quad \delta^U[v \otimes 1] = -\delta^U[1 \otimes v] = v\,, \text{for} \;\; v \in V$$

There is a class of subbicomodules generated by bilinear forms $\Upsilon_\omega = \mathrm{gen}\{\omega^{ij}[v_i \otimes v_j] \,|\, v_i \,\text{basis in}\, V\}$.
Automorphisms of $\mathcal{C}_V$ are generated by automorphisms of V. Thus, the equivalent forms: $\omega' = A\omega A^t$ provide isomorphic bicomodules.

Quantize enveloping Hopf algebra $U_q(\mathfrak{sl}(2))$

$$KK^{-1} = K^{-1}K = 1, \qquad KE = q^2 EK,$$
$$[E,F] = \frac{K - K^{-1}}{q - q^{-1}}, \qquad KF = q^{-2} FK.$$

$$\begin{array}{ccc} \Delta(K) = K \otimes K & \varepsilon(K) = 1 & S(K) = K^{-1} \\ \Delta(E) = E \otimes K + 1 \otimes E & \varepsilon(E) = 0 & S(E) = -EK^{-1} \\ \Delta(F) = F \otimes 1 + K^{-1} \otimes F & \varepsilon(F) = 0 & S(F) = -KF \end{array}$$

$$\mathcal{T}^n \quad = \quad \{\overline{FK^n}\} \quad \text{for} \quad n \in \mathbb{Z}/\{0\}.$$

For $n < 0$ and F, the dimension is infinite.

Lowest dimensional bicovariant $U_q(\mathfrak{sl}(2))$ FOCC

The lowest-dimensional bicovariant FOCC have four elements:

$$\upsilon_{00} = \overline{K}, \qquad \upsilon_{10} = \overline{E},$$
$$\upsilon_{01} = \overline{FK}, \qquad \upsilon_{11} = \overline{EF - q^2 FE}.$$

the second lowest-dimensional bicovariant FOCC is nine dimensional.

$$\upsilon_{00} = \overline{K^2}, \qquad \upsilon_{10} = \overline{EK}, \qquad \upsilon_{20} = \overline{E^2},$$
$$\upsilon_{01} = \overline{FK^2}, \qquad \upsilon_{11} = \overline{(EF - q^4 FE)K}, \qquad \upsilon_{12} = \overline{E^2 F - q^4 FE^2},$$
$$\upsilon_{02} = \overline{F^2 K^2}, \qquad \upsilon_{21} = \overline{(EF^2 - q^4 F^2 E)K}, \qquad \upsilon_{22} = \,.$$

$$\upsilon_{22} = \overline{E^2 F^2 - (q^2 + q^4) EF^2 E + q^6 F^2 E^2}$$

κ-Poincaré Hopf algebra

Let's introduce κ-Poincaré Hopf algebra by a system of generators $H = \mathrm{gen}\{\Pi_0,\ \Pi_0^{-1},\ P_j,\ N_j,\ M_j | j = 1, 2, 3\}$ (AB, A. Pachol)

$$
\begin{array}{lll}
[\mathcal{P}_i, \Pi_0] = 0, & [\mathcal{P}_j, \mathcal{P}_k] = 0, & [M_j, M_k] = i\epsilon_{jkl} M_l, \\
[M_j, \Pi_0] = 0, & [M_j, \mathcal{P}_k] = i\epsilon_{jkl}\mathcal{P}_l, & [N_j, M_k] = i\epsilon_{jkl} N_l, \\
[N_j, \Pi_0] = \frac{i}{\kappa}\mathcal{P}_j, & [N_j, \mathcal{P}_k] = -i\delta_{jk}\mathcal{P}_0, & [N_j, N_k] = -i\epsilon_{jkl} M_l.
\end{array}
$$

$$
\begin{aligned}
\Delta\Pi_0 &= \Pi_0 \otimes \Pi_0, \\
\Delta\mathcal{P}_j &= \mathcal{P}_j \otimes \Pi_0 + 1 \otimes \mathcal{P}_j, \\
\Delta M_j &= M_j \otimes 1 + 1 \otimes M_j, \\
\Delta N_j &= N_j \otimes 1 + \Pi_0^{-1} \otimes N_j - \frac{1}{\kappa}\epsilon_{jkl}\mathcal{P}_k \Pi_0^{-1} \otimes M_l.
\end{aligned}
$$

$$
\mathcal{P}_0 \doteq \frac{\kappa}{2}\left(\Pi_0 - \Pi_0^{-1}(1 - \frac{1}{\kappa^2}\vec{\mathcal{P}}^2)\right).
$$

Lowest dimensional bicovariant FOCC

It is 5-dimensional with its right-free module basis [2]:
$\Upsilon = \langle \upsilon_C, \upsilon_0, \upsilon_j \rangle \subset \bar{H}_L$.

$$\delta(\upsilon_0) = \Pi_0 - 1, \qquad \delta(\upsilon_j) = \mathcal{P}_j, \qquad \delta(\upsilon_C) = \kappa^2(\Pi_0 + \Pi_0^{-1} - 2) - \overrightarrow{\mathcal{P}}^2\Pi_0^{-1}.$$

$$\Delta_L(\upsilon_C) = \Pi_0^{-1} \otimes \upsilon_C + 2(\kappa\mathcal{P}_0 - \overrightarrow{\mathcal{P}}^2\Pi_0^{-1}) \otimes \upsilon_0 - 2\mathcal{P}_k\Pi_0^{-1} \otimes \upsilon_k,$$
$$\Delta_L(\upsilon_0) = \Pi_0 \otimes \upsilon_0, \qquad \Delta_L(\upsilon_j) = \mathcal{P}_j \otimes \upsilon_0 + 1 \otimes \upsilon_j.$$

$$[N_j, \upsilon_0] = -\tfrac{i}{\kappa}\upsilon_j, \qquad [M_j, \upsilon_k] = i\epsilon_{jkl}\upsilon_l,$$
$$[N_j, \upsilon_k] = i\delta_{jk}(\tfrac{1}{2\kappa}\upsilon_C - \kappa\upsilon_0).$$

! Corresponds to known results by A. Sitarz'95.

[2] $\upsilon_0 = \bar{\Pi}_0, \upsilon_i = \bar{P}_i, \upsilon_C = \bar{C}_\kappa,\ C_\kappa = \kappa^2(\Pi_0 + \Pi_0^{-1} - 2) - P_iP^i\Pi_0^{-1}$.

Conclusions and open problems

- Bicovariant codifferential culculi on quantized enveloping algebras are dual counterpart of Woronowicz bicovariant differential calculi on matrix quantum groups
- Higher-order bicovariant codifferential calculus
- coConnection
- CoVectorfields .

THANK YOU !