Quantised $\mathfrak{sl}(2)$-differential algebras

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Joint works P. Pandžić (Zagreb)
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I. G-differential algebras

Motivating example

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Then for any $x, y \in \mathfrak{g}$ we have that (as operators on $\Omega^\bullet(M)$)

$$[L_{x_M}, L_{y_M}] = L_{[x,y]_M}, \qquad [L_{x_M}, \iota_{x_M}] = \iota_{x_M}, \qquad [\iota_{x_M}, \mathrm{d}] = L_{x_M}$$

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Hence we have a structure of a Lie superalgebra on the span of d, ι_{x_M} and L_{x_M} for all $x \in \mathfrak{g}$.

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A $\mathfrak{g}$-*differential algebra* is a superalgebra B, equipped with a structure of G-differential space such that $\rho(x) \in \mathrm{Der}\, B$ for all $x \in \widehat{\mathfrak{g}}$.

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- The contractions ι_{e_i} are defined by

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One can show that $H(\Lambda\mathfrak{g}^*, \mathrm{d}) \cong (\Lambda\mathfrak{g}^*)^G \cong H(\mathfrak{g})$.

II. Symmetric and exterior algebra in the braided monoidal categories

Braided monoidal categories

Let C be a monoidal category with the collection of associativity constrains

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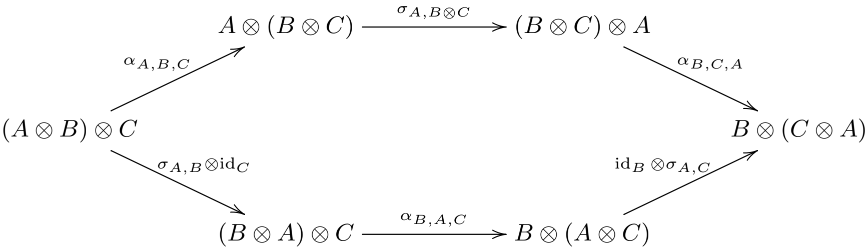
A *braiding* on a monoidal category C is a natural isomophism σ between functors $- \otimes -$ and $- \otimes^{\mathrm{op}} -$ such that the hexagonal diagrams commute,



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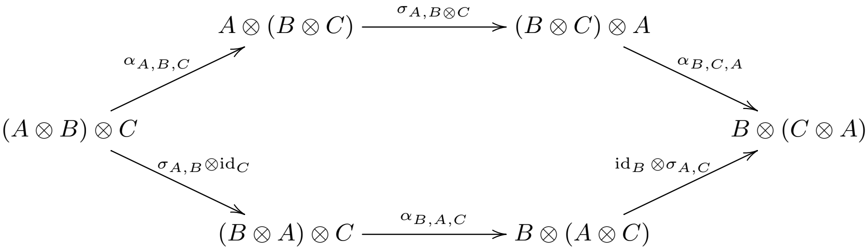
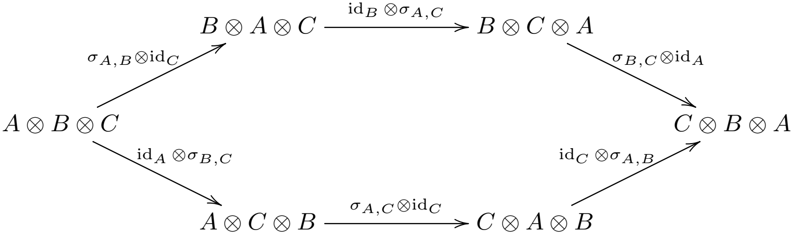


A *braided* monoidal category is a pair consisting of a monoidal category and a braiding.

The Yang–Baxter equation

If C is a strict braided monoidal category with braiding σ then for all $A, B, C \in \mathrm{Obj}(\mathsf{C})$ the braiding satisfies the following Yang–Baxter equation



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Woronowicz'89, K.–Ó Buachalla–Strung'23

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Let C be a braided monoidal category linear over $\mathbb{C}[[\hbar]]$ and assume that the braiding satisfies

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Leads to representations of Cactus groups.

III. Quantised $\mathfrak{sl}(2)$-differential algebras

Drinfel'd–Jimbo Quantum Groups: $\mathfrak{sl}_2$ case

Fix $q \in \mathbb{C}$ such that q is not a root of unity.

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The Hopf algebra structure is given by

$$\Delta(E) = E \otimes K + 1 \otimes E,\ \Delta(F) = F \otimes 1 + K^{-1} \otimes F,\ \Delta(K) = K \otimes K,$$
$$S(K^{\pm 1}) = K^{\mp 1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF,$$
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Note that for $U(\mathfrak{g})$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad x \in \mathfrak{g}.$$

$U_q(\mathfrak{sl}_2)$: Representation Theory

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• If $\mathfrak{sl}_2$ is a dominant weight of $\mathfrak{g}$ then M_λ has a maximal proper submodule I_λ generated by $F^{(\lambda,\alpha^\vee)+1} v_\lambda$ and

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• Such representations are called *type-1 representations*.

The left adjoin action of $U_q(\mathfrak{sl}_2)$ on itself is defined by

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Denote by $\mathfrak{sl}_q(2)$ the span of

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$$\mathrm{ad}_E\, v_2 = 0, \qquad \mathrm{ad}_K\, v_2 = q^2 v_2, \qquad \mathrm{ad}_F\, v_2 = -v_0,$$
$$\mathrm{ad}_E\, v_0 = -(q + q^{-1})v_2, \qquad \mathrm{ad}_K\, v_0 = v_0, \qquad \mathrm{ad}_F\, v_0 = (q + q^{-1})v_{-2}$$
$$\mathrm{ad}_E\, v_{-2} = v_0, \qquad \mathrm{ad}_K\, v_{-2} = q^{-2}v_{-2}, \qquad \mathrm{ad}_F\, v_{-2} = 0.$$

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- $\sigma_{\mathcal{R}} \circ \sigma_{\mathcal{R}} \neq \text{id}$: the category is not symmetric!

Quantum exterior algebras

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- Eigenvalues of $\sigma_{\mathcal{R}}$ on $V_{2\pi} \otimes V_{2\pi}$:

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• For any $V \in \mathrm{Rep}_1(U_q(\mathfrak{g}))$, let us denote

$$S^2_q V := \{x \in V \otimes V \mid \tilde{\sigma}_{\mathcal{R}}(x) = x\}, \quad \Lambda^2_q V := \{x \in V \otimes V \mid \tilde{\sigma}_{\mathcal{R}}(x) = -x\}.$$

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- the *BZ quantum exterior algebra* $\Lambda_q(V)$ of V to be

$$\Lambda_q(V) := \mathcal{T}(V)/\langle S^2_q V\rangle.$$

For $U_q(\mathfrak{sl}_2)$, the algebra $\Lambda_q V_{2\pi}$ has the classical dimension.

$$v_2 \wedge v_2 = 0, \qquad v_{-2} \wedge v_{-2} = 0,$$

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The $U_q(\mathfrak{sl}_2)$-module $V_{2\pi}$ admits a nondegenerate invariant bilinear form given by

$$\langle v_2, v_{-2} \rangle = c, \quad \langle v_0, v_0 \rangle = q^{-3}(1+q^2)c, \quad \langle v_{-2}, v_2 \rangle = cq^{-2},$$

where $c \in \mathbb{C}[q, q^{-1}]$.

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Note that $\langle \cdot, \cdot \rangle$ is symmetric with respect to $\tilde{\sigma}_{\mathcal{R}}$.

Definition

Let $\mathrm{Cl}_q(V_{2\pi}, \tilde{\sigma}_{\mathcal{R}}, \langle\cdot,\cdot\rangle) := T(V_{2\pi})/I$, where the corresponding two-sided ideal I is generated by

$$x \otimes y + \tilde{\sigma}_{\mathcal{R}}(x \otimes y) - 2\langle x, y\rangle 1 \qquad \text{for all } x, y \in V_{2\pi}, \tag{2}$$

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In what follows we refer to $\mathrm{Cl}_q(V_{2\pi}, \tilde{\sigma}_{\mathcal{R}}, \langle\cdot,\cdot\rangle)$ as the q-deformed Clifford algebra of $\mathfrak{sl}_2$ and denote it by $\mathrm{Cl}_q(\mathfrak{sl}_2)$.

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- Dirac operators K.–Pandžić'25

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v_2v_2 &= 0, & v_{-2}v_{-2} &= 0,\\
v_0v_2 &= -q^{-2}v_2v_0, & v_{-2}v_0 &= -q^{-2}v_0v_{-2},\\
v_0v_0 &= \frac{(1-q^4)}{q^3}v_2v_{-2} + \frac{q^2+1}{q}c1, & v_{-2}v_2 &= -v_2v_{-2} + \frac{q^2+1}{q^2}c1,
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where $c \in \mathbb{C}[q, q^{-1}]$.

- Dirac operators K.–Pandžić'25
- (quasi-)Poisson geometry Alekseev–K.'25

Lemma

The algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is of the PBW type.

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- Dirac operators K.–Pandžić'25
- (quasi-)Poisson geometry Alekseev–K.'25
- q-deformed Clifford analysis K.–Lávička–Souček WIP

Quantised $\mathfrak{sl}(2)$-differential spaces and

Definition

A supervector space W is called a quantised $\mathfrak{sl}_2$-differential space if it is equipped with

1. Lie derivatives $L_x \in \mathrm{End}(W)$ for $x \in U_q(\mathfrak{sl}_2)$ which define a $U_q(\mathfrak{sl}_2)$-module structure on W;
2. a $U_q(\mathfrak{sl}_2)$-equivariant action $\iota\colon \bigwedge_q V_{2\pi} \otimes W \to W$ of $\bigwedge_q V_{2\pi}$;
3. a $U_q(\mathfrak{sl}_2)$-equivariant differential $\mathrm{d}_W\colon W \to W$;
4. such that they satisfy Cartan's magic formula

$$L_x = \iota_x \circ \mathrm{d}_W + \mathrm{d}_W \circ \iota_x \qquad \text{for } x \in \mathfrak{sl}_q(2).$$

A morphism between two quantised $\mathfrak{sl}_2$-differential spaces is a morphism in the category of $U_q(\mathfrak{sl}_2)$-modules which intertwines contractions and differentials (and also Lie derivatives).

Definition

An algebra A is called a quantised $\mathfrak{sl}_2$-differential algebra if it is a quantised $\mathfrak{sl}_2$-differential space such that

1. the Lie derivatives satisfy
$$L_x(ab) = \sum (L_{x_{(1)}}a)(L_{x_{(2)}}b) \qquad \text{for } a, b \in A,\ x \in U_q(\mathfrak{sl}_2),$$
in other words, A is an algebra in the monoidal category of $U_q(\mathfrak{sl}_2)$-modules;
2. the differential d_A satisfies the (graded) Leibniz rule.

A morphism between two quantised $\mathfrak{sl}_2$-differential algebras is an algebra morphism in the category of $U_q(\mathfrak{sl}_2)$-modules which intertwines contractions and differentials (and also Lie derivatives).

Example: $\mathrm{Cl}_q(\mathfrak{sl}_2)$

For $x, y \in \mathrm{Cl}_q(\mathfrak{sl}_2)$ homogeneous with respect to parity set

$$[x, y]_{\tilde{\sigma}} := \left(m_{\mathrm{Cl}_q} - (-1)^{p(x)p(y)} m_{\mathrm{Cl}_q} \circ \tilde{\sigma} \right) (x \otimes y),$$

where m_{Cl_q} denotes the multiplication map in $\mathrm{Cl}_q(\mathfrak{sl}_2)$.

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Define a linear map $\beta_q \colon \mathfrak{sl}_q(2) \to \mathrm{Cl}_q(\mathfrak{sl}_2)$ by

$$\beta_q(X) = -\frac{1}{c} v_2 v_0, \; \beta_q(Y) = -\frac{1}{c} v_0 v_{-2}, \; \beta_q(Z) = \frac{1+q^2}{q}\left(\frac{1}{c} v_2 v_{-2} - 1\right).$$

Proposition

For $\omega \in \mathrm{Cl}_q(\mathfrak{sl}_2)$*,* $x \in \mathfrak{sl}_q(2) = \mathrm{Span}(v_2, v_0, v_{-2})$

$$L_x \omega = [\beta_q(X), \omega]_{\tilde{\sigma}}, \qquad \iota_x \omega = \frac{1}{2}[x, \omega]_{\tilde{\sigma}}, \qquad \mathrm{d}_{\mathrm{Cl}} \omega = [\gamma_q, \omega]_{\tilde{\sigma}},$$

where $\gamma_q = -\frac{1}{2c^2}(c v_0 + v_2 v_0 v_{-2})$*.*

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For $x, y \in \mathfrak{sl}_q(2) = \mathrm{Span}(v_2, v_0, v_{-2})$ we have that

$$[\beta_q(x), \beta_q(y)]_{\tilde{\sigma}} = \beta_q(\mathrm{ad}_x\, y)$$

Example: $\bigwedge_q V_{2\pi}$

- Since $\bigwedge_q V_{2\pi}$ is the associated graded to $\mathrm{Cl}_q(\mathfrak{sl}_2)$, it is a quantised $\mathfrak{sl}(2)$-deformed algebra too.

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- For $x, y, z \in \mathfrak{sl}_q(2)$

$$\iota_x \iota_y \mathrm{d}_{\mathrm{Cl}_q} z = \langle \mathrm{ad}_x\, y, z \rangle$$

Quantised Chevalley–Elenberg complex

(Joint works in progress with E. Boffo and T. Weber)

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so

$$\langle v_i^*, v_j \rangle = \delta_{i,j}$$

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Lemma

Let V be a $U_q(\mathfrak{sl}_2)$-module. Set $C_q(\mathfrak{g}, V) = V \otimes \bigwedge_q \mathfrak{sl}_q(2)$ and

$$\mathrm{d}_{\mathrm{CE}}(w\otimes\omega) = \sum_i (L\otimes m_{\wedge_q})\circ(\mathrm{id}\otimes\sigma_{\mathcal{R}}\otimes\mathrm{id})(v_i\otimes v_i^*\otimes w\otimes\omega) + \frac{q(q^2-1)^2}{q^2+1}(C_q w)\otimes \mathrm{d}_{\wedge_q}(\omega),$$

where $L\colon \mathfrak{sl}_q(2)\otimes V \to V$ denotes the action map. Then $\mathrm{d}_{\mathrm{CE}}^2 = 0$.

Thank you!