Serre–Swan Theorem for Graded Vector Bundles

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Motivation, hypothesis

Serre–Swan theorem

Fundamental relation of geometry and algebra:
Vector bundles over M correspond (almost one-to-one) to finitely generated projective modules over the algebra of functions on M.

- Serre (1955) - for algebraic vector bundles over affine varieties;
- Swan (1962) - (continuous) vector bundles over Hausdorff topological spaces;
- Nestruev (2003) - **The category of smooth vector bundles over a smooth manifold M and the category of finitely generated projective modules over $C^\infty(M)$ are equivalent.**

Theorem (**Graded Serre–Swan**)

The category of $\mathbb{Z}$-graded vector bundles over a $\mathbb{Z}$-graded manifold $\mathcal{M}$ and the category of finitely generated projective graded modules over $\mathcal{C}^\infty_{\mathcal{M}}(M)$ are equivalent.

Graded always means $\mathbb{Z}$-graded.

Projective graded modules

Definition (Graded A-module)

Let A be a graded commutative associative algebra. By a **graded A-module P (over $\mathbb{R}$)**, we mean a graded real vector space P together with a degree zero linear map $\triangleright : A \otimes_{\mathbb{R}} P \to P$, such that

$$(a \cdot b) \triangleright p = a \triangleright (b \triangleright p), \;\; 1 \triangleright p = p,$$

where we write simply $a \triangleright p = \triangleright(a \otimes p)$.

Example

Let K be graded vector space. Let $A[K] := A \otimes_{\mathbb{R}} K$ and set

$$a \triangleright (b \otimes k) := (a \cdot b) \otimes k, \;\; \forall a, b \in A, \;\; \forall k \in K.$$

Definition (Free graded A-modules)

We say that a graded A-module P is **free**, if it is isomorphic to $A[K]$.

Definition (**Projective graded A-modules**)

We say that a graded A-module P is **projective**, if there is a free graded A-module F and some graded A-module Q, such that

$$F = P \oplus Q.$$

Definition (**Finitely generated A-modules**)

We say that P is a **finitely generated A-module**, if there is a finite collection $\{p_i\}_{i=1}^k \subseteq P$, such that every $p \in P$ can be written as $p = a^i \triangleright p_i$ for some (not necessarily unique) $a^i \in A$.

Remark

- Every free graded A-module is projective;
- We say that A has an **invariant graded rank property**, if $A[K] \cong A[K']$ implies $K \cong K'$. We suppose this is the case.
- A free graded A-module P is finitely generated, iff $P \cong A[K]$ for a finite dimensional K;
- A projective graded A-module is finitely generated, iff F can be chosen to be finitely generated.

Sheaves

Definition

Let M be a given topological space. By a **presheaf of graded algebras**, we mean a functor $\mathcal{A} : \mathbf{Op}(M)^{\mathrm{op}} \to \mathbf{gcAs}$, where

1. $\mathbf{Op}(M)$ is the category of open subsets of M, there is an arrow $i^U_V : V \to U$ if $V \subseteq U$;
2. **gcAs** is a category of graded commutative associative unital graded algebras.

Remark

Explicitly, presheaf $\mathcal{A}$ consists of the following data:

1. For each $U \in \mathbf{Op}(M)$, one has $\mathcal{A}(U) \in \mathbf{gcAs}$;
2. For each $V \subseteq U$, one has a restriction algebra morphism $\mathcal{A}^U_V : \mathcal{A}(U) \to \mathcal{A}(V)$. One writes $a|_V := \mathcal{A}^U_V(a)$ for $a \in \mathcal{A}(U)$.
3. For any $W \subseteq V \subseteq U$, one has

$$\mathcal{A}^V_W \circ \mathcal{A}^U_V = \mathcal{A}^U_W, \;\; \mathcal{A}^U_U = \mathbb{1}_{\mathcal{A}(U)}.$$

Example (Constant presheaf)

Let $A \in \mathbf{gcAs}$ be fixed. For each $U \in \mathbf{Op}(M)$, let $\mathcal{A}(U) := A$. For each $V \subseteq U$, let $\mathcal{A}^U_V := \mathbb{1}_A$.

Definition

Let $\mathcal{A}$ be a presheaf of graded algebras. We say that $\mathcal{A}$ is a **sheaf of graded algebras**, if for any $U \in \mathbf{Op}(M)$ and any its open cover $\{U_\alpha\}_{\alpha \in I}$, one has

1. The **locality property**: if $a, b \in \mathcal{A}(U)$ satisfy $a|_{U_\alpha} = b|_{U_\alpha}$ for all $\alpha \in I$, then $a = b$.
2. The **gluing poperty**: for any collection $\{a_\alpha\}_{\alpha \in I}$ of the same degree, such that $a_\alpha|_{U_\alpha \cap U_\beta} = a_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in I$, there exists $a \in \mathcal{A}(U)$ with $a|_{U_\alpha} = a_\alpha$ for all $\alpha \in I$.

Remark

Not every presheaf is a sheaf, e.g. constant presheaf. There is a universal **sheafification** procedure, making each presheaf $\mathcal{A}$ into a sheaf $\mathcal{A}^{\mathsf{sff}}$.

Example (**Constant sheaf**)

Let $A \in \mathbf{gcAs}$ be fixed. For each $k \in \mathbb{Z}$ and $U \in \mathbf{Op}(M)$, let

$$\mathcal{A}(U)_k := \{ f : M \to A_k \mid f \text{ locally constant} \}$$

Graded algebra structure by "pointwise multiplication" and restrictions are restrictions.

Definition

- $\mathcal{A}$ a given sheaf of graded algebras.
- $\mathcal{F} : \mathbf{Op}(M)^{\mathrm{op}} \to \mathbf{gVect}$ a sheaf of graded vector spaces.

We say that $\mathcal{F}$ is a **sheaf of graded $\mathcal{A}$-modules**, if

1. For each $U \in \mathbf{Op}(M)$, $\mathcal{F}(U)$ is a graded $\mathcal{A}(U)$-module;
2. Restrictions are compatible with the structure, that is $(a \triangleright f)|_V = a|_V \triangleright f|_V$.

Example

Let $K \in \mathbf{gVect}$ be finite-dimensional. For each $U \in \mathbf{Op}(M)$, define $\mathcal{F}(U) := \mathcal{A}(U)[K] \equiv \mathcal{A}(U) \otimes_{\mathbb{R}} K$ with obvious restrictions. This makes $\mathcal{F}$ into a sheaf of graded $\mathcal{A}$-modules.

Graded manifolds and vector bundles

Definition (Graded manifold)

A graded manifold $\mathcal{M}$ consists of the following data:

1. second countable Hausdorff topological space M;
2. (certain) sheaf $\mathcal{C}^\infty_{\mathcal{M}}$ of graded commutative associative algebras;
3. atlas $\mathcal{A}$ making $\mathcal{C}^\infty_{\mathcal{M}}$ locally isomorphic to a certain "model sheaf". It also makes M into a smooth manifold.

Example (The model space)

- Let $M = \mathbb{R}^n$ with coordinates $(x^1, \dots, x^n)$
- Suppose we have "purely graded coordinate functions" $(\xi_1, \dots, \xi_m)$, each of them assigned a **degree** $|\xi_\mu| \in \mathbb{Z} - \{0\}$, such that

$$\xi_\mu \xi_\nu = (-1)^{|\xi_\mu||\xi_\nu|} \xi_\nu \xi_\mu.$$

- For each $U \in \mathbf{Op}(M)$, we declare $\mathcal{C}^\infty_{\mathcal{M}}(U)$ to be the graded algebra of formal power series in ξ's with coefficients in $\mathcal{C}^\infty_{\mathbb{R}^n}(U)$.

Definition (**Graded vector bundles**)

A **graded vector bundle** $\mathcal{E}$ **over a graded manifold** $\mathcal{M}$ is a locally freely and finitely generated sheaf $\Gamma_{\mathcal{E}}$ (on M) of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-modules of a constant graded rank.

Remark (**Local frames**)

Conditions on $\Gamma_{\mathcal{E}}$ are equivalent to the following: For each $m \in M$, there exists $U \in \mathbf{Op}_m(M)$ and $\{\Phi_\lambda\}_{\lambda=1}^{r} \subseteq \Gamma_{\mathcal{E}}(U)$, such that

- $|\Phi_\lambda| = |\vartheta_\lambda|$, where $(\vartheta_\lambda)_{\lambda=1}^{r}$ is some fixed total basis of some fixed graded vector space K;
- For each $V \in \mathbf{Op}(U)$, $\{\Phi_\lambda|_V\}_{\lambda=1}^{r}$ freely generates $\Gamma_{\mathcal{E}}(V)$.

$\{\Phi_\lambda\}_{\lambda=1}^{r}$ is called the **local frame for** $\mathcal{E}$ **over** U.

Example (**Tangent bundle**)

By declaring $\Gamma_{T\mathcal{M}} = \mathfrak{X}_{\mathcal{M}}$, $\mathfrak{X}_{\mathcal{M}}$ is a sheaf of vector fields (graded derivations of $\mathcal{C}^{\infty}_{\mathcal{M}}$), we define the **tangent bundle** $T\mathcal{M}$ **of** $\mathcal{M}$. Local frame = cordinate vector fields.

$\Gamma_{\mathcal{E}}(M)$ is finitely generated projective

Statement 1: $\Gamma_{\mathcal{E}}(M)$ is a finitely generated graded $\mathcal{C}^\infty_{\mathcal{M}}(M)$-module.

Proof (sketch): There is *finite* open cover $\{U_i\}_{i=1}^k$ of M with a local frame $\{\Phi_\lambda^{(i)}\}_{\lambda=1}^r$ for $\mathcal{E}$ over U_i. Let $\{\rho_i\}_{i=1}^k \subseteq \mathcal{C}^\infty_{\mathcal{M}}(M)$ be a partition of unity. Then the following collection generates $\Gamma_{\mathcal{E}}(M)$:

$$\{\{\rho_i \cdot \Phi_\lambda^{(i)}\}_{\lambda=1}^r\}_{i=1}^k \subseteq \Gamma_{\mathcal{E}}(M)$$

Statement 2: $\Gamma_{\mathcal{E}}(M)$ is a projective graded $\mathcal{C}^\infty_{\mathcal{M}}(M)$-module.

Proof (sketch): Let $\{\Phi_i\}_{i=1}^k \subseteq \Gamma_{\mathcal{E}}(M)$ be the finite generating set. Let $\mathcal{E}' = \mathcal{M} \times K$ be the trivial vector bundle, where $K = \mathbb{R}\{\Phi_i\}_{i=1}^k$. $\Gamma_{\mathcal{E}'}(M)$ is free and one constructs an epimorphism $F : \Gamma_{\mathcal{E}'}(M) \to \Gamma_{\mathcal{E}}(M)$. Short exact sequences of *graded vector bundles* split, so $\Gamma_{\mathcal{E}'}(M) \cong \Gamma_{\mathcal{E}}(M) \oplus \ker(F)$.

The converse statement

The issue: Graded vector bundles are not determined by their fibers.

Step 1: For any sheaf $\mathcal{F}$ of graded $\mathcal{C}^\infty_\mathcal{M}(M)$-modules and any finitely generated graded submodule $P \subseteq \mathcal{F}(M)$, there is a unique sheaf $\mathcal{P}$ of $\mathcal{C}^\infty_\mathcal{M}$-submodules, such that $\mathcal{P}(M) = P$.

Proof (sketch): For each $U \in \mathbf{Op}(M)$, the submodule $\mathcal{P}(U) \subseteq \mathcal{F}(U)$ is defined by the property:

$$\psi \in \mathcal{P}(U) \Leftrightarrow (\forall m \in U)(\exists V \in \mathbf{Op}_m(U))(\exists \psi' \in P)(\psi|_V = \psi'|_V).$$

$\mathcal{P}$ always forms a sheaf of $\mathcal{C}^\infty_\mathcal{M}$-submodules, such that $P \subseteq \mathcal{P}(M)$.
The converse inclusion requires P to be closed under "locally finite sums", i.e. sums of possibly infinite collections of elements of P, whose supports form a locally finite set (and hence the sums are well-defined). Finitely generated P have this property.
One can also show that $\mathcal{P}(U)$ is finitely generated for any $U \in \mathbf{Op}(M)$.

Step 2: If P is a finitely generated projective $\mathcal{C}^{\infty}_{\mathcal{M}}(M)$-module, there exists a trivial vector bundle $\mathcal{E} = \mathcal{M} \times K$ and its sheaves $\mathcal{P}, \mathcal{Q}$ of graded $\mathcal{C}^{\infty}_{\mathcal{M}}$-submodules, such that $\Gamma_{\mathcal{E}} = \mathcal{P} \oplus \mathcal{Q}$, and $P \cong \mathcal{P}(M)$.

Proof (sketch): We have $F = P \oplus Q$ for F free and finitely generated. But $F \cong \mathcal{C}^{\infty}_{\mathcal{M}}(M)[K] \equiv \Gamma_{\mathcal{E}}(M)$ for $\mathcal{E} = \mathcal{M} \times K$. Hence we can assume

$$\Gamma_{\mathcal{E}}(M) = P \oplus Q.$$

$Q \cong \Gamma_{\mathcal{E}}(M)/P$ is also finitely generated. By Step 1, there are $\mathcal{P}, \mathcal{Q} \subseteq \Gamma_{\mathcal{E}}$ with $P = \mathcal{P}(M)$ and $Q = \mathcal{Q}(M)$. Using partitions uf unity, one shows

$$\Gamma_{\mathcal{E}}(U) = \mathcal{P}(U) + \mathcal{Q}(U).$$

Since $\mathcal{P} \cap \mathcal{Q}$ is a sheaf of submodules having the property $(\mathcal{P} \cap \mathcal{Q})(M) = P \cap Q = 0$, we have $\mathcal{P} \cap \mathcal{Q} = 0$, so the sum is direct.

Step 3: Let $\mathcal{E}$ be any graded vector bundle. Suppose M is connected. Let $\mathcal{P}, \mathcal{Q} \subseteq \Gamma_{\mathcal{E}}$ be two sheaves of $\mathcal{C}^{\infty}_{\mathcal{M}}$-submodules, such that

$$\Gamma_{\mathcal{E}} = \mathcal{P} \oplus \mathcal{Q}.$$

Then both $\mathcal{P}$ and $\mathcal{Q}$ are sheaves of sections of subbundles of $\mathcal{E}$, hence sheaves of sections of graded vector bundles.

Proof (sketch): For each $m \in M$, there is a finite-dimensional graded vector space $\mathcal{E}_m$ called the **fiber of $\mathcal{E}$ at** m, defined as a quotient

$$\mathcal{E}_m = \Gamma_{\mathcal{E}}(M)/(\mathcal{J}^m_{\mathcal{M}}(M) \triangleright \Gamma_{\mathcal{E}}(M)),$$

where $\mathcal{J}^m_{\mathcal{M}}(M) = \{f \in \mathcal{C}^{\infty}_{\mathcal{M}}(M) \mid f(m) = 0\}$. By $\psi \mapsto \psi|_m$ we denote the quotient map. One can then define the subspace

$$\mathcal{P}_{(m)} := \{\psi|_m \mid \psi \in \mathcal{P}(M)\} \subseteq \mathcal{E}_m.$$

$\mathcal{Q}_{(m)}$ is defined analogously. The assumptions ensure that

$$\mathcal{E}_m = \mathcal{P}_{(m)} \oplus \mathcal{Q}_{(m)}.$$

Now comes the hard bit. One has to show the following two facts:

- The graded dimension of $\mathcal{P}_{(m)}$ is constant in $m \in M$.
- The total basis of $\mathcal{E}_m$ adapted to the decomposition can be extended to a local frame for $\mathcal{E}$ over U adapted to the decomposition $\mathcal{P} \oplus \mathcal{Q}$.

This can be used to construct local frames for $\mathcal{P}$ and $\mathcal{Q}$.

Theorem

To any finitely generated projective graded $\mathcal{C}^\infty_{\mathcal{M}}(M)$-module P, there exists a graded vector bundle $\mathcal{F}$ over $\mathcal{M}$, such that $P \cong \Gamma_{\mathcal{F}}(M)$.

Proof: By Step 2, we can construct a trivial vector bundle $\mathcal{E}$ and a sheaf of submodules $\mathcal{P} \subseteq \Gamma_{\mathcal{E}}$ satisfying $\mathcal{P}(M) \cong P$.
By Step 3, we have $\mathcal{P} = \Gamma_{\mathcal{F}}$ for a graded vector bundle $\mathcal{F}$. Rather tautologically, one has $P \cong \Gamma_{\mathcal{F}}(M)$.

Theorem (graded Serre-Swan theorem)

The functor $\mathcal{E} \mapsto \Gamma_{\mathcal{E}}(M)$ is fully faithful and essentially surjective functor from the category of graded vector bundles over $\mathcal{M}$ to the category of finitely generated projective graded $\mathcal{C}^\infty_{\mathcal{M}}(M)$-modules.

- We assume that M is connected. Otherwise we have to allow *locally* constant graded ranks.
- The proof works flawlessly for ordinary manifolds, supermanifolds, $\mathbb{Z}_2^n$-manifolds, etc.
- Morye (2009) proved Serre–Swan for a huge class of locally ringed spaces $(X, \mathcal{O}_X)$, where "vector bundles" are locally free sheaves of $\mathcal{O}_X$-modules of a bounded rank.
- I claimed for two years that Serre–Swan does not work. Counterexample involves carefully constructed complicated arguments starting from $\tau : T\mathcal{M} \to T\mathcal{M}$ having the property $\tau^2 = 1$, which is "easy to see". Except τ has no such property.

Thank you for your attention!