The Strong Coupling Limit of 2-Color 1-Flavour Matrix Model

Quantum Phases, and Towards a Resolution of the Spin Puzzle

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Joint work with

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Based on

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Introduction

Yang-Mills Matrix Model

Strong Coupling Limit, and Numerics

Superselection Sectors, Observables

Results, organized by (B, J)

Baryon Chemical Potential, and a LOFF State

Summary

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2-color 1-flavor QCD

Why 2-color 1-flavor QCD?

- Gauge group is $SU(2) \rightarrow$ Simplest non-Abelian gauge theory.
- computationally less challenging.
- has many interesting features:
  a) baryons (di-quarks and tetra-quarks) are bosonic states,
  b) there are additional global symmetry (Pauli-Gürsey symmetry):
  Fundamental rep of $SU(2)$ is pseudo-real $\Rightarrow$ $U(1)_V$ extended to $SU(2)_B$.
- The fermionic determinant in the path integral has no sign problem. Hence popular in lattice community.

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The 2-color 1-flavor Matrix Model

The 2c-1f matrix model is very easy to describe:

- Building blocks: 2×2 hermitian matrices $M_i(t)$, $i = 1, 2, 3$.
- The space $\mathcal{M}_2 \ni M_i \equiv M_{ia} T_a$ is $M_3(\mathbb{R})$.
- Rotations: $M_i \to R_{ij} M_j$.
- Gauge transformations: $M_i \to g M_i g^\dagger$, $g \in SU(2)$.
- The configuration space $\mathcal{C}_2 = \mathcal{M}_2 / ad\ SU(2)$.
- Complicated topology: $\mathcal{C}_2 \simeq \mathbb{R} \times (S^5 - \mathbb{R}P^2)$.
- Curvature $F_{ij} = -\epsilon_{ijk} M_k - i[M_i, M_j]$. (Narasimhan-Ramadas 1979)
- A natural reduction of $SU(2)$ YM on $S^3 \times \mathbb{R}$ to a matrix model.

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Matrix Model Dynamics

- For dynamics, we need a gauge-invariant Lagrangian.
- The electric $E_i \equiv D_t M_i$ and magnetic $B_i \equiv \frac{1}{2}\epsilon_{ijk} F_{jk}$:

$$E_i = \dot{M}_i - i[M_0, M_i], \qquad B_i = -M_i - \frac{i}{2}\epsilon_{ijk}[M_j, M_k].$$

- M_0: parallel transporter in the temporal direction (set to zero henceforth).
- The matrix model Lagrangian is

$$L_{YM} = \frac{1}{2g^2}\mathrm{Tr}(E_i E_i - B_i B_i) = \frac{1}{2g^2}\mathrm{Tr}(D_t M_i D_t M_i) - V$$

- $V(M)$ has upto quartic terms.
- The matrix model is just a multi-dimensional quartic oscillator.

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Adding a Quark

- Quarks are Grassmann valued matrices $\psi(t)$ transforming as:

$$\text{Fundamental rep. of color:} \qquad \psi \to u(h)\psi, \qquad h \in SU(2)$$

$$\text{spin-}\frac{1}{2}\ \text{rep. of rotations:} \qquad \psi \to D^{1/2}(R)\psi, \qquad R \in SO(3)_{rot}$$

- Dirac quark, $\psi = \begin{pmatrix} b_{\alpha A} \\ d^{\dagger}_{\alpha A} \end{pmatrix}$, $\alpha, A = 1, 2$.
- $b^{\dagger}_{\alpha A}$ and $d^{\dagger}_{\alpha A}$ create quarks and anti-quarks respectively.

anti-commutation algebra: $\{b_{\alpha A}, b^{\dagger}_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB} = \{d_{\alpha A}, d^{\dagger}_{\beta B}\}$

- Quark Hamiltonian: $H_f = gH_{int} + mH_m + \tilde{c}H_c$, with

$$H_{int} = \bar{\psi}\gamma^i M_i \psi,\ H_m = \left(\cos\theta\bar{\psi}\psi + i\sin\theta\bar{\psi}\gamma^5\psi\right),\ H_c = \bar{\psi}\gamma^0\gamma^5\psi$$

- Baryon chemical potential term $\tilde{\mu}H_{\mu} = \frac{1}{2}\tilde{\mu}\bar{\psi}\gamma^0\psi = \tilde{\mu} \times$ Baryon number
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Adding a Quark

- Quarks are Grassmann valued matrices $\psi(t)$ transforming as:

$$\text{Fundamental rep. of color:} \qquad \psi \to u(h)\psi, \qquad h \in SU(2)$$

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Symmetries of The Hamiltonian I

- For $m = 0$: Chiral symmetry $U(1)_A : \psi \to e^{i\theta\gamma^5}\psi$, generated by the chiral charge Q_0. This is broken anomalously.
- Spatial Rotations:

$$L_i = -2\epsilon_{ijk} Tr\left(\Pi_j A_k\right), \quad S_i = \frac{1}{2}\left(b^\dagger_{\alpha A}\sigma^i_{\alpha\beta} b_{\beta A} + d_{\alpha A}\sigma^i_{\alpha\beta} d^\dagger_{\beta A}\right)$$
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Symmetries of The Hamiltonian II

- Pauli-Gürsey Symmetry: $U(1)_B \xrightarrow{\text{extended to}} SU(2)_B$
  Generators of $SU(2)_B \rightarrow B_1, B_2, B_3$

$$[B_i, B_j] = \epsilon_{ijk} B_k, \quad [Q_0, B_{1,2,3}] = 0$$

  Three different cases:
  - $m = 0, \mu = 0 : U(1)_A \xrightarrow{\text{Anomaly}} \mathbb{Z}_2$
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Strong Coupling Regime

- Recale the Hamiltonian as $\Rightarrow M_i \to g^{-\frac{1}{3}} M_i$ and $P_i \to g^{\frac{1}{3}} P_i$

$$H = e_0 \left[Tr \left(P_i P_i + g^{-\frac{4}{3}} M_i M_i + i g^{-\frac{2}{3}} \epsilon_{ijk} [M_i, M_j] M_k - \frac{1}{2} [M_i, M_j]^2 \right) + cH_c + H_{int} + MH_m + \mu H_\mu \right]$$

- $e_0 = g^{2/3}/R$, where R is the radius of S^3.
- Double scaling limit: $g \to \infty$, $R \to \infty$, e_0 =finite $\Rightarrow H$ has well-defined spectra.

Numerical Strategy:

- Hilbert space, $\mathcal{H} = \mathcal{H}_{Fermion} \otimes \mathcal{H}_{Boson}$
- $\mathcal{H}_{Boson}$ is infinite dimensional, $\mathcal{H}_{Fermion}$ is finite.
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- $e_0 = g^{2/3}/R$, where R is the radius of S^3.
- Double scaling limit: $g \rightarrow \infty$, $R \rightarrow \infty$, e_0 =finite $\Rightarrow H$ has well-defined spectra.

Numerical Strategy:

- Hilbert space, $\mathcal{H} = \mathcal{H}_{Fermion} \otimes \mathcal{H}_{Boson}$
- $\mathcal{H}_{Boson}$ is infinite dimensional, $\mathcal{H}_{Fermion}$ is finite.
- Construct colorless trial states, which span $\mathcal{H}_{phys}$.
- Truncate $\mathcal{H}_{phys}$ to a given boson number, and use Rayleigh-Ritz.

General properties of the spectrum

- Colorless energy eigenstates $\Rightarrow$ only even number of quarks.
- Maximum baryon charge is 2 (because $\mathcal{H}_{Fermion}$ is finite).
- Total spin can only be an integer: $J = 0, 1, 2, \cdots$.
- We have *3 types* of states:
  - Mesons: equal number of quarks and anti-quarks (so $B_3 = 0$).
  - Diquarks/Anti-diquarks: Number of quarks and anti-quarks differ by 2 (these have $B_3 = \pm 1$).
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Observables

- Expectation values of observables depend on c and M.
- We compute these in each of the sectors labelled by (B, J).
- We have a variety of interesting observables available for investigation.
  - Ground state energy in each (B, J).
  - Glue spin L and quark spin S.
  - Third and fourth Binder cumulants

$$g_3 = \frac{\det M}{\left(\frac{1}{3} Tr\left(M^T M\right)\right)^{3/2}}, \qquad g_4 = \frac{1}{16}\left[\frac{2 Tr\left(M^T M\right)^2}{\left(\frac{1}{3} Tr\left(M^T M\right)\right)^2} - 9\right].$$

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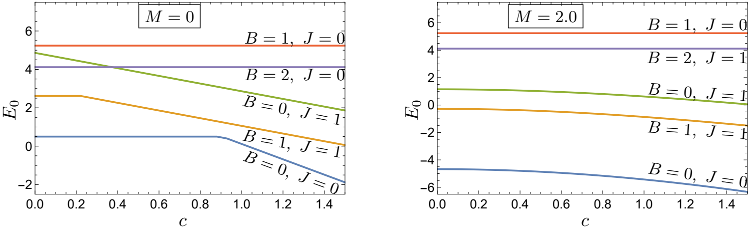
Results

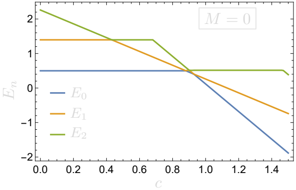
- Low-lying eigenvalues as a function of c from each sector (at $\mu = 0$).

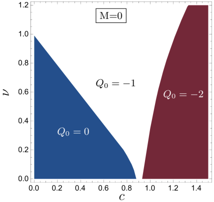


- Ground state is unique and belongs to $B = 0, J = 0$ sector.

Phases of $(B, J) = (0, 0)$ sector and their properties

- Level crossing in the (B,J)=(0,0) is rather special $\Rightarrow$ Triple crossing.
- Plot of ν $(= g^{-2/3})$ vs c shows three distinct phases. For $g \to \infty$ or $\nu \to 0$ two transition lines merge at the triple point.





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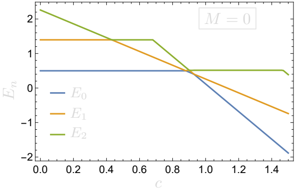

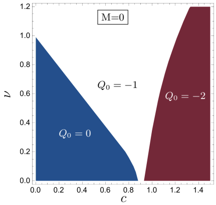

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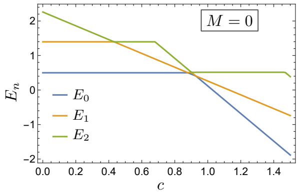
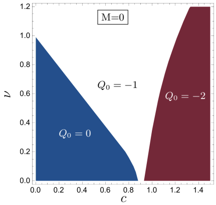
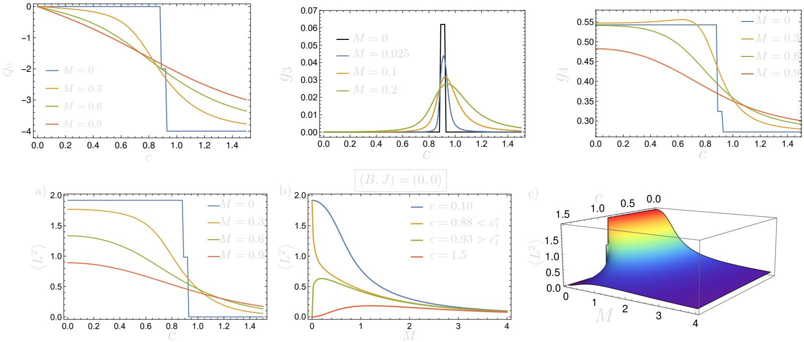




$(B, J) = (0, 0)$ sector and its properties

Critical point $(c, M) \approx (0.928, 0)$. Q_0, g_3 and g_4 are singular.



$$\lim_{M \to 0} \langle L^2 \rangle \simeq \begin{cases} 1.9 & \text{for } c < c_0^* \quad \text{(so mostly spin 1 glue)} \\ 0 & \text{for } c > c_0^* \quad \text{(so mostly spin 0 glue)} \end{cases}$$

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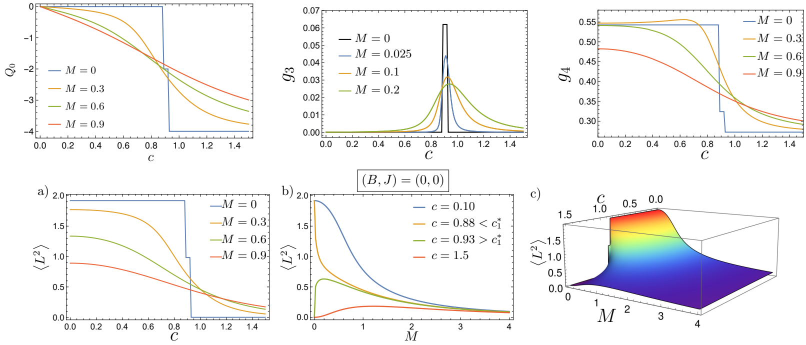

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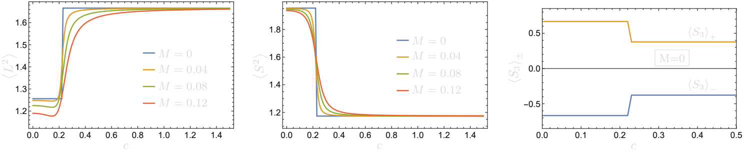
$(B, J) = (1, 1)$ sector

- We can also look at the spin division for the (B,J)=(1,1) sector.
- Here the QPT occurs at $(c, M) =\approx (0.22, 0)$
- Glue (L) and quark (S) spin contribution:



- For $c < c_1^*$, the quark contributes significantly, and it is opposite for $c > c_1^*$.
- More information is there in $\langle S_3 \rangle_\pm$:

$$\text{For } M = 0 : \ \langle S_3 \rangle_\pm = \begin{cases} \pm 0.67 & \text{for } c < c_1^* \\ \pm 0.33 & \text{for } c > c_1^* \end{cases}$$

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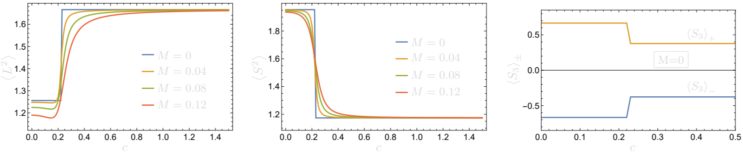

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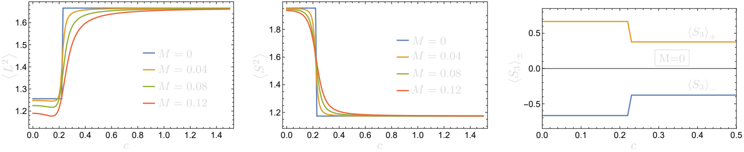

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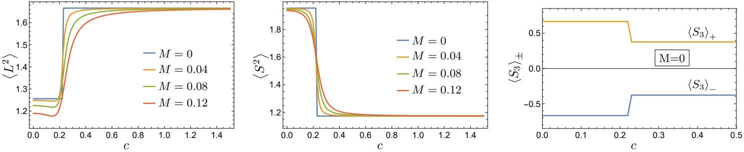

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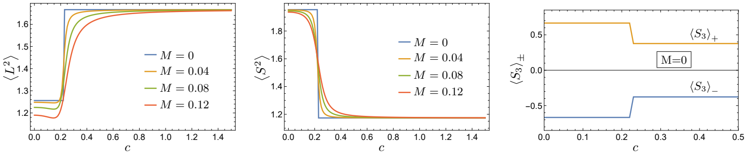

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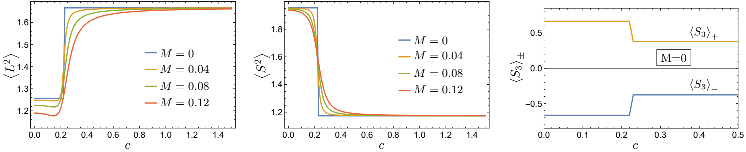

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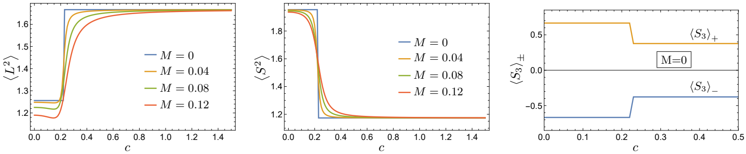

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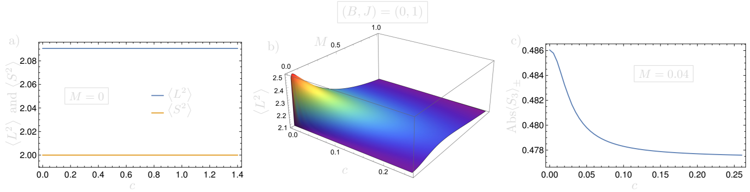

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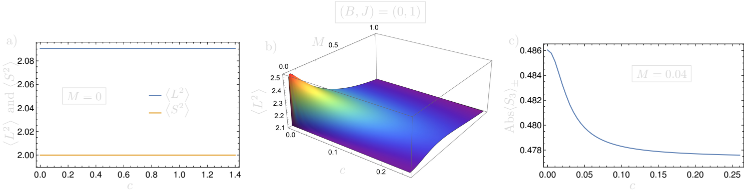

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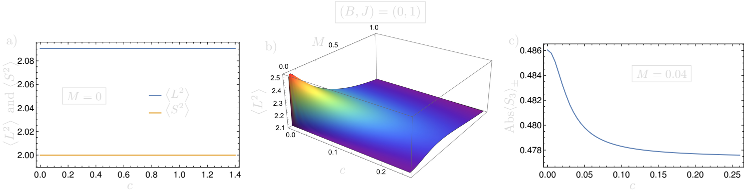

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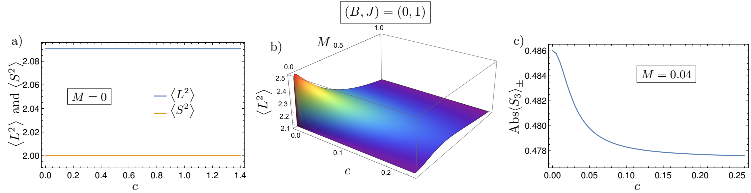

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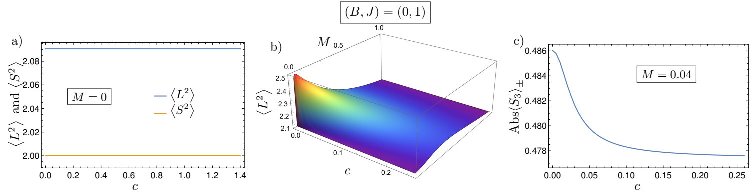

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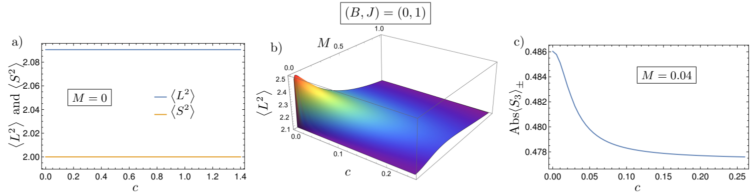

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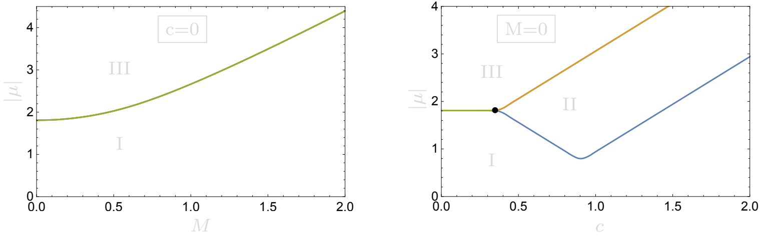
- $$E(\mu) = E(\mu = 0) + \mu B_3$$

- Degeneracy between mesons, di-quarks and tetra-quarks is lifted.
- New phases emerge:



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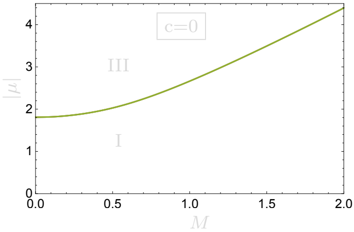

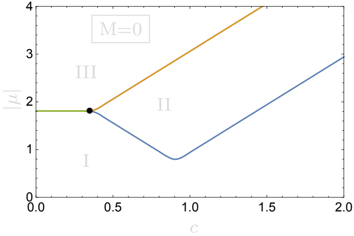

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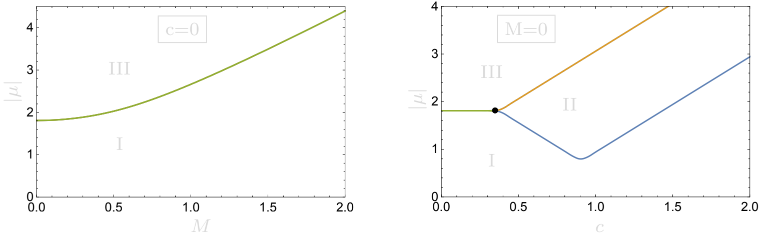

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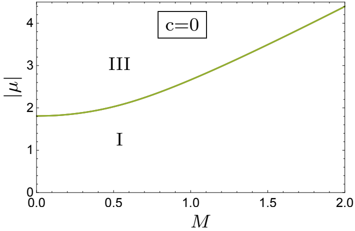

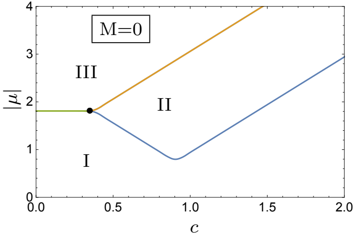

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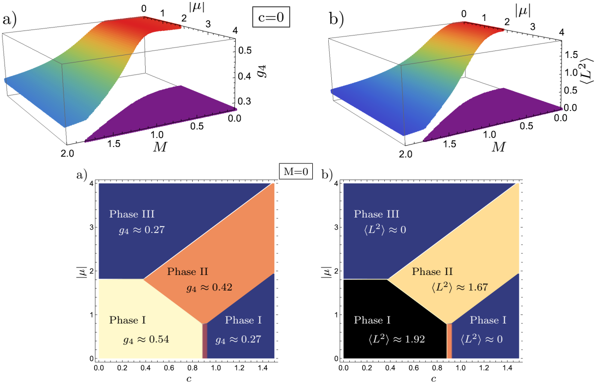

- When $c = 0$, $g_4 \simeq 0.3$ and $\langle L^2 \rangle = 0$ are both constants in phase III, but non-trivial functions of M in phase I.
- In phase II, the ground state is a spin triplet $\Rightarrow$ rotational symmetry is broken (analog of LOFF).

Summary

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THANK YOU

Predictions of QCD Matrix Model

Masses of π, ρ, Λ^{-}, Δ are used as inputs.

Particle	Spin	Isospin	Matrix Model Mass (MeV)	Observed Mass (MeV)	% Error
$K^{\pm}/K^{0}/\bar{K}^{0}$	0	1/2	395.95	495.65	-20.1 %
η	0	0	138	547.86	-75 %
η'	0	0	653.85	957.78	-31.7 %
$K^{*\pm}/K^{*0}/\bar{K}^{*0}$	1	1/2	1030.99	893.65	+15.3 %
ω	1	0	775.03	782.65	-0.9 %
ϕ	1	0	1287.87	1019.46	+26.4 %

Table: Comparison of the meson masses

Particle	Spin	Isospin	Matrix Model Mass (MeV)	Observed Mass (MeV)	% Error
p/N	1/2	1/2	935.06	938.91	-0.3 %
$\Xi^{0/-}$	1/2	1/2	1448.94	1314.86	+10.2 %
Λ^{0}	1/2	0	1190.01	1115.61	+6.6 %
$\Sigma^{*(\pm/0)}$	3/2	1	1488.99	1384.6	+7.4 %
$\Xi^{*0/-}$	3/2	1/2	1745.94	1533.4	+15.5 %
Ω^{-}	3/2	0	2002.89	1672.45	+19.8 %

Table: Comparison of the baryon masses

(M. Pandey and S. Vaidya, Phys. Rev. D **101**, no.11, 114020 (2020).)

Glueball states J^C	Physical masses from matrix model (MeV)	Physical masses from lattice QCD (MeV)
0^+	$1757.08^\dagger$	1580 - 1840
2^+	$2257.08^\dagger$	2240 - 2540
0^+	2681.45	2405 - 2715
0^{*+}	3180.82	2360 - 2980
1^-	3235.41	2810 - 3150
2^+	3054.97	2850 - 3230
0^{*+}	3568.02	3400 - 3880
1^-	3535.66	3600 - 4060
2^{*+}	3435.75	3660 - 4120
2^-	4050.14	3765 - 4255

$^\dagger \equiv$ (input)

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