On the Đurđević approach to quantum principal bundles

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Differential geometry dictionary

	Classical differential geometry	**Quantum** differential geometry
observables	smooth manifold M	associative unital algebra A
differential forms	cotangent bundle $\Lambda^\bullet T^*M$	DGA $\Omega^\bullet = \bigoplus_{k\geq 0}\Omega^k$
symmetry	Lie group action $M \times G \to M$	Hopf algebra coaction $\Delta_A\colon A \to A \otimes H$
principal bundle	$\pi\colon P \xrightarrow{\circlearrowleft G} M$ $P \times G \xrightarrow{\cong} P \times_M P$	$B := \{a \in A \mid \Delta_A(a) = a \otimes 1\}$ $\chi\colon A \otimes_B A \xrightarrow{\cong} A \otimes H$

We call $B := A^{\mathrm{co}H} = \{a \in A \mid \Delta_A(a) = a \otimes 1\} \subseteq A$ a **Hopf–Galois extension** or **quantum principal bundle (QPB)** if

$$\chi\colon A \otimes_B A \xrightarrow{\cong} A \otimes H, \qquad \chi(a \otimes_B a') := a\Delta_A(a')$$

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Lost(?) literature

We revisit the quantum principal bundle formalism of **Mićo Đurđević**, in particular

- ĐURĐEVIĆ, M.: *Geometry of Quantum Principal Bundles II - Extended Version*. Rev. Math. Phys. **9**, 5 (1997) 531-607.
- ĐURĐEVIĆ, M.: *Quantum Principal Bundles as Hopf-Galois Extensions*. Preprint arXiv:q-alg/9507022.
- ĐURĐEVIĆ, M.: *Quantum Gauge Transformations and Braided Structure on Quantum Principal Bundles*. Preprint arXiv:q-alg/9605010.

A reworked version (including new non-trivial examples) is

- DEL DONNO, A., LATINI, E. AND TW: *On the Đurđević approach to quantum principal bundles*. Preprint arXiv:2404.07944.

Goals of this talk

Given a QPB $B = A^{\mathrm{co}H} \subseteq A$ make sense of differential calculi

- $\Omega^\bullet(A)$ on the **total space algebra** A
- $\Omega^\bullet(H)$ on the **structure Hopf algebra** H
- $\Omega^\bullet(B)$ on the **base algebra** B
- of **vertical forms** $\mathrm{ver}^\bullet$ and of **horizontal forms** $\mathrm{hor}^\bullet$

such that the Atiyah sequence is exact

$$0 \to \mathrm{hor}^1 \hookrightarrow \Omega^1(A) \xrightarrow{\pi_v} \mathrm{ver}^1 \to 0$$

and we obtain a graded Hopf–Galois extension and braiding

$$\Omega^\bullet(B) = \Omega^\bullet(A)^{\Omega^\bullet(H)} \subseteq \Omega^\bullet(A) \qquad \sigma^\bullet : \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A)$$

Furthermore

- compare to approach of Brzeziński–Majid
- discuss examples!

Structure Hopf algebra

H Hopf algebra, $\Delta(h) = h_1 \otimes h_2$, $\varepsilon\colon H \to \Bbbk$, $S\colon H \to H$

$H^+ := \ker \varepsilon \subseteq H$
$\pi_\varepsilon\colon H \twoheadrightarrow H^+$

Def: A FODC $\Omega^1(H)$ is **left covariant** if ${}_{\Omega^1(H)}\Delta\colon \Omega^1(H) \to H \otimes \Omega^1(H)$ coaction and $\mathrm{d}\colon H \to \Omega^1(H)$ left colinear. We write $\Lambda^1 := {}^{\mathrm{co}H}\Omega^1(H)$.

Theorem (Woronowicz '89)

$$\left\{\begin{matrix} \textit{left covariant} \\ \textit{FODC on } H \end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix} \textit{right } H\textit{-ideal} \\ I \subseteq H^+ \end{matrix}\right\}$$

$\Omega^1 \cong H \otimes \Lambda^1 \cong H \otimes H^+/I$ *via* ***Cartan–Maurer-form*** $\varpi\colon H^+ \twoheadrightarrow \Lambda^1$, $\varpi(h) := S(h_1)\mathrm{d}h_2$
Then $\Omega^1(H)$ *is even bicovariant iff* $\mathrm{Ad}(I) \subseteq I \otimes H$.

Theorem (Beggs–Majid '20)

The maximal prolongation of a left covariant FODC $\Omega^1(H)$ *is* $\Omega^\bullet(H) \cong H \otimes \Lambda^\bullet$, *where* $\Lambda^\bullet$ *is generated by* Λ^1 *modulo*

$$\wedge \circ (\varpi \circ \pi_\varepsilon \otimes \varpi \circ \pi_\varepsilon)\Delta(I) = 0.$$

Moreover, $\mathrm{d}(\varpi(\pi_\varepsilon(h))) + \varpi(\pi_\varepsilon(h_1)) \wedge \varpi(\pi_\varepsilon(h_2)) = 0$ *(Cartan–Maurer equation)*

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Graded Hopf algebra

Let $\Omega^1(H)$ bicovariant with maximal prolongation $\Omega^\bullet(H)$.

Lemma (Beggs–Majid '20)

$\Delta\colon H \to H \otimes H$ extends to a morphism of DGAs $\Delta^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H) \otimes \Omega^\bullet(H)$.

$\rightsquigarrow$ write $\Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]} \in \bigoplus_{k+\ell=n} \Omega^k(H) \otimes \Omega^\ell(H)$ for $\omega \in \Omega^n(H)$.

Proposition

$\Omega^\bullet(H)$ is a bicovariant DC and a graded Hopf algebra with

$$\Delta^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H) \otimes \Omega^\bullet(H), \qquad \Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]}$$
$$\varepsilon^\bullet\colon \Omega^\bullet(H) \to \Bbbk, \qquad \varepsilon^\bullet(\omega) = 0$$

for all $\omega \in \Omega^\bullet(H)$ with $|\omega| > 0$. The antipode $S^\bullet\colon \Omega^\bullet(H) \to \Omega^\bullet(H)$ is determined by

$$S^\bullet(h^0 \mathrm{d}(h^1) \wedge \ldots \wedge \mathrm{d}(h^k)) = \mathrm{d}(S(h^k)) \wedge \ldots \wedge \mathrm{d}(S(h^1)) S(h^0)$$

for all $h^0 \mathrm{d}(h^1) \wedge \ldots \wedge \mathrm{d}(h^k) \in \Omega^k(H)$.

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Vertical forms and total space forms

From now on let $B = A^{\mathrm{co}H} \subseteq A$ be a QPB and $\Omega^\bullet(H)$ the maximal prolongation of a bicovariant FODC and denote $\Lambda^\bullet = {}^{\mathrm{co}H}\Omega^\bullet(H)$.

Proposition

There is a DC $\mathrm{ver}^\bullet$ on A defined by $\mathrm{ver}^\bullet := A \otimes \Lambda^\bullet$ with wedge product and differential determined by

$$(a \otimes \vartheta) \wedge (a' \otimes \vartheta') := aa'_0 \otimes S(a'_1)\vartheta a'_2 \wedge \vartheta',$$
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*for all $a \otimes \vartheta, a' \otimes \vartheta' \in \mathrm{ver}^\bullet$. We call $\mathrm{ver}^\bullet$ the **vertical forms** on A.*

Definition ([Đurđević '97])

A DC $\Omega^\bullet(A)$ on A is called **complete** if the right H-coaction $\Delta_A \colon A \to A \otimes H$ extends to a morphism

$$\Delta_A^\bullet \colon \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$$

of DGAs. In this case we refer to $\Omega^\bullet(A)$ as the **total space forms**.

We use the short notation

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Proposition

Let $\Omega^\bullet(A)$ be a complete calculus on A. Then

i.) $\Omega^\bullet(A)$ *is a graded right* $\Omega^\bullet(H)$*-comodule algebra.*

ii.) *there is a surjective morphism of DGAs* $\pi_v\colon\Omega^\bullet(A)\to\mathrm{ver}^\bullet$ *extending the identity* $\mathrm{id}\colon A\to A$. *Explicitly,*

$$\pi_v(a^0\mathrm{d}a^1\wedge\ldots\wedge\mathrm{d}a^k)=a^0_0a^1_0\ldots a^k_0\otimes S(a^0_1a^1_1\ldots a^k_1)a^0_2\mathrm{d}a^1_2\wedge\ldots\wedge\mathrm{d}a^k_2$$

for all $a^0,\ldots,a^k\in A$.

iii.) *the vertical forms* $\mathrm{ver}^\bullet$ *are complete, i.e. the right* H*-coaction* $\Delta_A\colon A\to A\otimes H$ *extends to a morphism* $\Delta^\bullet_v\colon\mathrm{ver}^\bullet\to\mathrm{ver}^\bullet\otimes\Omega^\bullet(H)$ *of DGAs. Moreover, the diagram*

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Complete calculi

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iii.) *the vertical forms $\mathrm{ver}^\bullet$ are complete, i.e. the right H-coaction $\Delta_A \colon A \to A \otimes H$ extends to a morphism $\Delta^\bullet_v \colon \mathrm{ver}^\bullet \to \mathrm{ver}^\bullet \otimes \Omega^\bullet(H)$ of DGAs. Moreover, the diagram*

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Horizontal forms and base forms

Definition

For a complete calculus $\Omega^\bullet(A)$ on a QPB A we define the **horizontal forms** as the preimage

$$\mathrm{hor}^\bullet := (\Delta_A^\bullet)^{-1}(\Omega^\bullet(A) \otimes H)$$

of $\Omega^\bullet(A) \otimes H$ under $\Delta_A^\bullet \colon \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$.

- $\mathrm{hor}^\bullet$ is a right H-comodule algebra
- but $\mathrm{hor}^\bullet$ is **not** a DGA!

Definition

Let $\Omega^\bullet(A)$ be a complete calculus on a QPB $B = A^{\mathrm{co}H} \subseteq A$. The corresponding **base forms** are defined as the graded subspace $\Omega^\bullet(B)$ of $\Omega^\bullet(A)$ which is invariant under the right $\Omega^\bullet(H)$-coaction $\Delta_A^\bullet \colon \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$, i.e.

$$\Omega^\bullet(B) := \{\omega \in \Omega^\bullet(A) \mid \Delta_A^\bullet(\omega) = \omega \otimes 1_H\}.$$

- $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ is a DG subalgebra
- but $\Omega^\bullet(B)$ **might not** be generated in degree 0: $B\mathrm{d}B \subsetneq \Omega^1(B)$
- however in all examples we encounter $\Omega^\bullet(B) \subseteq \Omega^\bullet(A)$ equals the pullback calculus

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The Atiyah sequence and Brzeziński–Majid

Theorem

For any complete calculus $\Omega^\bullet(A)$ on a QPB $B = A^{\mathrm{co}H} \subseteq A$ the ***Atiyah sequence***

$$0 \to \mathrm{hor}^1 \hookrightarrow \Omega^1(A) \xrightarrow{\pi_v} \mathrm{ver}^1 \to 0$$

is exact in the category ${}_A\mathcal{M}^H_A$ of right H-covariant A-bimodules.

Definition (Brzeziński–Majid '93)

If $\Omega^1(A)$ is right H-covariant we have a **quantum principal bundle à la Brzeziński–Majid** if the **vertical map**

$$\mathrm{ver}\colon \Omega^1(A) \to A \otimes \Lambda^1, \qquad \mathrm{ver}(a\mathrm{d}_A(a')) = aa'_0 \otimes S(a'_1)\mathrm{d}_H(a'_2)$$

is well-defined & the sequence $0 \to \mathrm{Ad}_A(B)A \hookrightarrow \Omega^1(A) \xrightarrow{\mathrm{ver}} A \otimes \Lambda^1 \to 0$ is exact. We call $\Omega^1_{\mathrm{hor}} := \mathrm{Ad}_A(B)A$ the **horizontal 1-forms à la Brzeziński–Majid**.

- QPB à la Brzeziński–Majid $\Rightarrow \Omega^1(A)$ is first order complete.
- $\Omega^1(A)$ first order complete $\Rightarrow \Omega^1(A)$ is QPB à la Brzeziński–Majid iff $\Omega^1_{\mathrm{hor}} = \mathrm{hor}^1$.

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Graded Hopf–Galois extension

Given a complete calculus $\Omega^\bullet(A)$ we define

$$\begin{aligned}\chi^\bullet\colon \Omega^\bullet(A\otimes_B A) &\to \Omega^\bullet(A)\otimes\Omega^\bullet(H)\\ \omega\otimes_{\Omega^\bullet(B)}\eta &\mapsto \omega\Delta_A^\bullet(\eta)=\omega\wedge\eta_{[0]}\otimes\eta_{[1]}\end{aligned}$$

Theorem

$\Omega^\bullet(A)$ complete calculus on QPB $B=A^{\mathrm{co}H}\subseteq A$. Then

$$\Omega^\bullet(B)=\Omega^\bullet(A)^{\Omega^\bullet(H)}\subseteq\Omega^\bullet(A)$$

is a graded Hopf–Galois extension.

The inverse of $\chi^\bullet$ is $\omega\otimes\theta\mapsto\omega\wedge\tau^\bullet(\theta)$ with well-defined graded map

$$\tau^\bullet\colon\Omega^\bullet(H)\to\Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)$$

extending the translation map $\tau\colon H\to A\otimes_B A$, $\tau(h)=h^{\langle 1\rangle}\otimes_B h^{\langle 2\rangle}$. On 1-forms:

$$\begin{aligned}\tau^1\colon\Omega^1(H)&\to\Omega^\bullet(A)\otimes_{\Omega^\bullet(B)}\Omega^\bullet(A)\\ h\mathrm{d}g&\mapsto\mathrm{d}(g^{\langle 1\rangle})h^{\langle 1\rangle}\otimes_{\Omega^\bullet(B)}h^{\langle 2\rangle}g^{\langle 2\rangle}+g^{\langle 1\rangle}h^{\langle 1\rangle}\otimes_{\Omega^\bullet(B)}h^{\langle 2\rangle}\mathrm{d}(g^{\langle 2\rangle})\end{aligned}$$

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Canonical braiding

Pull back the tensor product multiplication on $A \otimes H$ to $A \otimes_B A$:

$$\begin{array}{ccc} (A \otimes H) \otimes_B (A \otimes H) & \xrightarrow{m_{A\otimes H}} & A \otimes H \\ {\scriptstyle \chi\otimes_B\chi}\uparrow & & \downarrow{\scriptstyle \chi^{-1}} \\ (A \otimes_B A) \otimes_B (A \otimes_B A) & \xrightarrow{m_{A\otimes_B A}} & A \otimes_B A \end{array}$$

Namely $m_{A\otimes_B A}((a \otimes_B a') \otimes_B (c \otimes_B c')) = a\sigma(a' \otimes_B c)c'$ with

$$\sigma(a \otimes_B a') := a_0 a' \tau(a_1) = a_0 a' (a_1)^{\langle 1\rangle} \otimes_B (a_1)^{\langle 2\rangle}$$

to which we colloquially refer to as the **Đurđević braiding**. Then

- $\chi\colon A \otimes_B A \to A \otimes H$ and the translation map $\tau\colon H \to A \otimes_B A$ are algebra morphisms
- $\sigma\colon A \otimes_B A \to A \otimes_B A$ is an isomorphism in ${}_B\mathcal{M}_B$ satisfying

$$(\sigma \otimes_B \mathrm{id}) \circ (\mathrm{id} \otimes_B \sigma) \circ (\sigma \otimes_B \mathrm{id}) = (\mathrm{id} \otimes_B \sigma) \circ (\sigma \otimes_B \mathrm{id}) \circ (\mathrm{id} \otimes_B \sigma).$$

- A is braided-commutative with respect to σ, i.e.

$$m_A \circ \sigma = m_A,$$

  where m_A denotes the multiplication $A \otimes_B A \to A$.

$\rightsquigarrow$ extends to braiding $\sigma^\bullet\colon \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes_{\Omega^\bullet(B)} \Omega^\bullet(A)$

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Example: The noncommutative 2-torus

Let $A := \mathcal{O}_\theta(\mathbb{T}^2) := \mathbb{C}[u, u^{-1}, v, v^{-1}]/\langle vu - e^{i\theta} uv\rangle$ for $\theta \in \mathbb{R}$.

It is a right $H := \mathcal{O}(U(1))$-comodule algebra $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}$ and a faithfully flat Hopf–Galois extension, with coinvarint subalgebra $B := A^{\mathrm{co}H} = \mathrm{span}_\mathbb{C}\{(uv)^{\pm k}\}$ and cleaving map $j \colon H \to A$, $\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^k \\ v^k \end{pmatrix}$ for $k \geq 0$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{\mathrm{d}u, \mathrm{d}v\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{\mathrm{d}u \wedge \mathrm{d}v\}}$ and $\Omega^\bullet(H) = H \oplus \underbrace{\Omega^1(H)}_{=t^{\pm k}\mathrm{d}t}$.

Proposition (DelDonno-Latini-TW '24)

$\Omega^\bullet(A)$ is a complete calculus on the noncommutative 2-torus and $\Omega^\bullet(B)$ is the usual pullback calculus.

$$\Omega^1(A) \to \underbrace{\Omega^1(A) \otimes H}_{\Delta_{\Omega^1(A)}} \oplus \underbrace{A \otimes \Omega^1(H)}_{\mathrm{ver}}$$

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$$\Omega^2(A)\to\underbrace{\Omega^2(A)\otimes H}_{\Delta_{\Omega^2(A)}}\oplus\underbrace{\Omega^1(A)\otimes\Omega^1(H)}_{\mathrm{ver}^{1,1}\colon\, \mathrm{d}u\wedge\mathrm{d}v\mapsto-\mathrm{d}(uv)\otimes t^{-1}\mathrm{d}t}$$

Already known in the literature. Well-defined according to our calculations.

Example: The noncommutative 2-torus

Let $A := \mathcal{O}_\theta(\mathbb{T}^2) := \mathbb{C}[u, u^{-1}, v, v^{-1}]/\langle vu - e^{i\theta} uv\rangle$ for $\theta \in \mathbb{R}$.

It is a right $H := \mathcal{O}(U(1))$-comodule algebra $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}$ and a faithfully flat Hopf–Galois extension, with coinvarint subalgebra $B := A^{\mathrm{co}H} = \mathrm{span}_{\mathbb{C}}\{(uv)^{\pm k}\}$ and cleaving map $j \colon H \to A$, $\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^k \\ v^k \end{pmatrix}$ for $k \geq 0$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{\mathrm{d}u,\mathrm{d}v\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{\mathrm{d}u\wedge\mathrm{d}v\}}$ and $\Omega^\bullet(H) = H \oplus \underbrace{\Omega^1(H)}_{=t^{\pm k}\mathrm{d}t}$.

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Already known in the literature. Well-defined according to our calculations.

Example: $\mathcal{O}_q(\mathrm{SU}_2)$ and Podleś sphere

Let $A := \mathcal{O}_q(\mathrm{SU}_2)$ with $H := \mathcal{O}(U(1))$ coaction $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \mapsto \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$.

This is known to be a faithfully flat Hopf–Galois extension with coinvariant subalgebra the Podleś sphere $B := A^{\mathrm{co}H}$.

Define $\Omega^\bullet(A) = A \oplus \underbrace{\Omega^1(A)}_{=\mathrm{span}_A\{e^\pm, e^0\}} \oplus \underbrace{\Omega^2(A)}_{=\mathrm{span}_A\{e^\pm \wedge e^0, e^+ \wedge e^-\}} \oplus \underbrace{\Omega^3(A)}_{=\mathrm{span}_A\{e^+ \wedge e^- \wedge e^0\}}$ and

$\Omega^\bullet(H) = H \oplus \Omega^1(H)$ with $\mathrm{d}t \cdot t = q^2 t\mathrm{d}t$.

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Thank you for your attention!