Massless chiral fields in six dimensions

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Workshop on Noncommutative and Generalized Geometry
in String theory, Gauge theory and Related Physical Models

Elusive superconformal theories in $6d$

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  containing a **chiral**, or **self-dual** 2-form;

- $(4,0)$ 'exotic' conformal supergravity [Hull 2000], whose spectrum is

$$\ydiagram{2,2} \oplus (\dots)\,,$$

  containing a **self-dual mixed-symmetric tensor**, argued to play the role of a graviton;

Introduction

Higher spin singletons

These particular mixed-symmetry fields, labelled by a two-row rectangular diagram,

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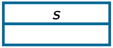

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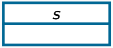

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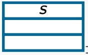
$$F_{\mu_1\dots\mu_s,\nu_1\dots\nu_s,\rho_1\dots\rho_s} = \partial_{\rho_1}\dots\partial_{\rho_s}\varphi_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} + (\dots) \quad \longleftrightarrow$$



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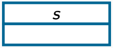

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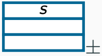

,

and can therefore be either **self-dual or anti-self-dual**, meaning

$$\varepsilon_{\mu\nu\rho}{}^{\alpha\beta\gamma} F_{\ldots\alpha\ldots,\ldots\beta\ldots,\ldots\gamma\ldots} \propto F_{\ldots\mu\ldots,\ldots\nu\ldots,\ldots\rho\ldots}\,,$$

i.e. their Hodge dual in *any column* are proportional to themselves.

Lessons from 4d

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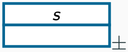

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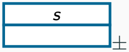

Less indices, but still a mess to deal with.

Lessons from 4d

Fortunately, we can use two-component spinors to simplify things:

$$R_{\mu_1\dots\mu_s,\nu_1\dots\nu_s} \qquad \longleftrightarrow \qquad \psi^{\alpha_1\dots\alpha_{2s}} \quad \text{or} \quad \psi^{\dot{\alpha}_1\dots\dot{\alpha}_{2s}}, \qquad \alpha, \dot{\alpha} = 1, 2$$

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while positive helicities can be described by a symmetric tensor $\psi^{\alpha(2s)}$ which verifies

$$\partial_{\beta\dot\beta}\psi^{\alpha(2s-1)\beta} \approx 0\,.$$

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  and remove additional components by adding an algebraic piece to the gauge symmetry,

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$$\delta_{\xi,\eta}\omega_{\alpha(2s-2)} = \nabla\xi_{\alpha(2s-2)} + e_{\alpha\dot\alpha}\,\eta_{\alpha(2s-3)}{}^{\dot\alpha}\,;$$

- The free action,

$$S[\psi,\omega] = \int_M \psi^{\alpha(2s)}\,H_{\alpha\alpha}\wedge\nabla\omega_{\alpha(2s-2)}\,,\qquad H_{\alpha\alpha} = e_{\alpha\dot\beta}\wedge e_{\alpha}{}^{\dot\beta}\,,$$

  is gauge-invariant thanks to $H_{\alpha\alpha}\wedge e_{\alpha\dot\alpha} = 0$, and produces equivalent equations of motions.

Lessons from 4d

- Promote $\omega_{\alpha(2s-2)}$ and $\psi^{\alpha(2s)}$ to be $\mathfrak{g}$-valued, with $\mathfrak{g}$ a *quadratic* Lie algebra, i.e. endowed with a bilinear symmetric invariant form $\langle -, - \rangle$, and replace $\nabla\omega$ with the 'field strength'

$$F_{\alpha(2s-2)} = \nabla\omega_{\alpha(2s-2)} + \frac{1}{2} \sum_{k+l=s-1} [\omega_{\alpha(2k)}, \omega_{\alpha(2l)}]_{\mathfrak{g}} \,.$$

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- Then, the action

$$S[\Psi, \omega] = \int_M \sum_{s\geq 1} \big\langle \psi^{\alpha(2s)}, H_{\alpha\alpha} \wedge F_{\alpha(2s-2)} \big\rangle \,,$$

  is invariant under

$$\delta_{\xi,\eta}\omega_{\alpha(2s-2)} = \nabla\xi_{\alpha(2s-2)} + \sum_{k+l=s-1} [\omega_{\alpha(2k)}, \xi_{\alpha(2l)}]_{\mathfrak{g}} + e_{\alpha\dot{\alpha}}\, \eta_{\alpha(2s-3)}{}^{\dot{\alpha}} \,,$$

$$\delta_\xi \Psi^{\alpha(2s)} = \sum_{k+l=s} [\Psi^{\alpha(2k)}, \xi^{\alpha(2l)}]_{\mathfrak{g}} \,.$$

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There exists a counterpart of this higher spin extension of self-dual Yang–Mills in $6d$, for **higher spin singletons**.

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In fact, this extends to **arbitrary even dimensions**.

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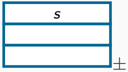
d	$\mathbb{K}$	$SL(2,\mathbb{K})$-tensors	$\widetilde{SO}(1, d-1)$-irrep
3	$\mathbb{R}$	$\alpha = 1, 2$ $\varphi_{\alpha(2s)}$	
4	$\mathbb{C}$	$\alpha, \dot{\alpha} = 1, 2$ $\psi_{\alpha(2s)}$ or $\psi_{\dot{\alpha}(2s)}$	
6	$\mathbb{H}$	$A = 1, 2, 3, 4$ $\Psi_{A(2s)}$ or $\Psi^{A(2s)}$	

$SL(2,\mathbb{H}) \cong SU^*(4)$ formulation in 6d

Singletons of spin s and positive chirality can be described *either* by a gauge potential

$$\partial_{A,B}\varphi_{A(2s-1)}{}^{B} \approx 0\,, \qquad \delta_\xi \varphi_{A(2s-1)}{}^{B} = \partial^{B,C}\xi_{A(2s-1),C}\,,$$

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One subtlety with respect to the 4*d* case: the gauge symmetry is **reducible**, since

$$\mathring{\xi}_{A(2s-1),B} = \partial_{A,B}\zeta_{A(2s-2)} \qquad \Longrightarrow \qquad \delta_{\mathring{\xi}}\varphi_{A(2s-1)}{}^{B} = 0\,,$$

for any $\zeta_{A(2s-2)}$.

Free theory

Remark that one can embed the gauge potential into a 2-form

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where ∇ is torsion-free and has constant curvature. Concretely,

$$\nabla^2 \varphi_{\dots A \dots}{}^{\dots B \dots} = \sum -H_A{}^C\, \varphi_{\dots C \dots}{}^{\dots B \dots} + H_C{}^B\, \varphi_{\dots A \dots}{}^{\dots C \dots}\,,$$

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upon normalising the curvature. The self-dual 3-forms $H_{AA} = H_A{}^B \wedge e_{B,A}$ satisfy the identities

$$H_{AA} \wedge e_{A,B} = 0 \qquad \Longrightarrow \qquad H_{AA} \wedge H_A{}^B = 0\,,$$

ensures that the action

$$S[\Psi, \Phi] = \int_M \Psi^{A(2s)} H_{AA} \wedge \nabla \varpi_{A(2s-2)}\,,$$

is gauge invariant.

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which contains more than the potential $\Phi_{A(2s-1)}{}^B$, but can be removed by adding an **algebraic piece** to the gauge transformations

$$\delta_{\xi,\eta}\omega_{A(2s-2)} = \nabla\xi_{A(2s-2)} + e_{A,B} \wedge \eta_{A(2s-3)}{}^B\,,$$

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where both $\xi_{A(2s-2)}$ and $\eta_{A(2s-3)}{}^B$ are 1-forms. The previous action, with ϖ replaced by $\omega_{A(2s-2)}$,

$$S[\Psi,\omega] = \int_M \Psi^{A(2s)} H_{AA}\wedge\nabla\omega_{A(2s-2)}\,.$$

is gauge-invariant **thanks to the identity**

$$H_{AA}\wedge e_{A,B} = 0\,.$$

'Yang–Mills-type' interaction

For convenience, let us introduce generating fields

$$\Omega^2_M \otimes \mathfrak{g} \otimes \mathbb{C}[y]^{\mathbb{Z}_2} \ni \omega := \sum_{s\geq 1} \frac{1}{(2s-2)!}\,\omega_{A(2s-2)}y^{A(2s-2)}\,,$$

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and the pairing

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so that the sum of the previous free action for $s \geq 1$ reads

$$S[\Psi, \omega] = \int_M p\big(\Psi, H \wedge \omega\big)\,, \qquad H := \tfrac{1}{2}\, H_{AA}\, y^A y^A\,.$$

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Now we can introduce a Yang–Mills gauge field $A \in \Omega^1_M \otimes \mathfrak{g}$ which is a 1-form valued in a Lie algebra $\mathfrak{g}$, equipped with a symmetric bilinear **invariant** form denoted $\langle -, - \rangle : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$.

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$$S_{\min}[\Psi, \omega; A] = \int_M p \circ \langle \Psi, H \wedge D\omega \rangle\,, \qquad D := \nabla + [A, -]_{\mathfrak{g}}\,,$$

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$$S_{\min}[\Psi, \omega; A] = \int_M p \circ \langle \Psi, H \wedge D\omega \rangle , \qquad D := \nabla + [A, -]_{\mathfrak{g}} ,$$

is **almost gauge-invariant** under

$$\delta_{\epsilon,\xi,\eta}\omega = D\xi + \sigma_+\eta - [F, \sigma_-^\dagger D\eta]_{\mathfrak{g}} + [\omega, \epsilon]_{\mathfrak{g}} , \quad \delta_\epsilon A = D\epsilon , \quad \delta_\epsilon \Psi = [\Psi, \epsilon]_{\mathfrak{g}} ,$$

where

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$$S_{\min}[\Psi, \omega; A] = \int_M p \circ \langle \Psi, H \wedge D\omega \rangle \,, \qquad D := \nabla + [A, -]_{\mathfrak{g}} \,,$$

is **almost gauge-invariant** under

$$\delta_{\epsilon,\xi,\eta}\omega = D\xi + \sigma_+\eta - [F, \sigma_-^\dagger D\eta]_{\mathfrak{g}} + [\omega, \epsilon]_{\mathfrak{g}} \,, \quad \delta_\epsilon A = D\epsilon \,, \quad \delta_\epsilon \Psi = [\Psi, \epsilon]_{\mathfrak{g}} \,,$$

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- $\sigma_-^\dagger := -\frac{2}{N_y(N_{\bar{y}}+3)}\, y^A \bar{\partial}^B\, e^\mu_{A,B}\, \frac{\partial}{\partial(\mathrm{d}x^\mu)}$ with N_y and $N_{\bar{y}}$ the number operators for the variables y ad $\bar{y}$.

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which, being proportional to the field strength F, suggests to add a **BF-term**

$$S_{\text{BF}}[A, B] = \mathrm{g} \int_M \big\langle B, F \big\rangle\,, \quad \delta_{\epsilon,\xi} B = [B, \epsilon]_{\mathfrak{g}} - \tfrac{1}{\mathrm{g}}\, p\big([\Psi, \xi - D\sigma_-^\dagger \eta]_{\mathfrak{g}} \wedge H\big)\,,$$

where the gauge transformations of the field $B \in \Omega^4_M \otimes \mathfrak{g}$ are adjusted so as to compensate the previous variation.

Pause: why the weird modification?

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where the dots vanish on 0-forms, so that one can easily verify that $\delta_{\mathring{\xi},\mathring{\eta}}\omega = 0$, even in presence of $F \neq 0$.

Short summary

Free formulation

$$S[\Psi,\omega \qquad] = \int_M p\big(\Psi, H \wedge \nabla\omega\big)\,,$$

with gauge symmetries

$$\delta_{\xi,\eta}\omega = \nabla\xi + \sigma_+\eta\,,$$

reducible for

$$\mathring{\xi} = \nabla\zeta\,, \qquad \mathring{\eta} = \sigma_-\zeta\,.$$

Short summary

Interacting formulation

$$S[\Psi, \omega; A, B] = \int_M p \circ \langle \Psi, H \wedge D\omega \rangle + \mathrm{g} \langle B, F \rangle \,,$$

with gauge symmetries

$$\delta_{\xi,\eta}\omega = D\xi + \sigma_+\eta - [F, \sigma_-^\dagger]_{\mathfrak{g}} + [\omega, \epsilon]_{\mathfrak{g}} \,,$$

$$\delta_\epsilon \Psi = [\Psi, \epsilon]_{\mathfrak{g}} \,, \qquad \delta_{\epsilon,\xi,\eta} B = [B, \epsilon]_{\mathfrak{g}} - \tfrac{1}{\mathrm{g}}\, p\big([\Psi, \xi - D\sigma_-^\dagger \eta]_{\mathfrak{g}} \wedge H\big) \,,$$

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Higher spin version

(*i*) Extend A and B to

$$\Omega^1_M \otimes \mathfrak{g} \otimes \mathbb{C}[y] \ni \mathcal{A} = \sum_{s\geq 1} \tfrac{1}{(2s-2)!}\, A_{A(2s-2)}\, y^{A(2s-2)}\,,$$

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(ii) Define $\bullet : \mathbb{C}[\bar{y}] \otimes \mathbb{C}[y]$ via

$$p(\psi, f\cdot g) = p(\psi \bullet f, g)\,,$$

for $\psi \in \mathbb{C}[\bar{y}]$ and $f, g \in \mathbb{C}[y]$, and

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(iii) Now we have

$$p \circ \big\langle \psi, [f, g]_{\mathfrak{g}} \big\rangle = p \circ \big\langle [\psi \overset{\bullet}{,} f]_{\mathfrak{g}}, g \big\rangle\,,$$

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Higher spin version

The same properties / mechanisms as before ensure that

$$S[\Psi,\omega;\mathcal{A},\mathcal{B}] = \int_M p\circ\left\langle\Psi, H\wedge\mathcal{D}\omega\right\rangle + \mathrm{g}\, p\circ\left\langle\mathcal{B},\mathcal{F}\right\rangle,$$

is invariant under the gauge symmetries

$$\delta_{\xi,\eta}\omega = \mathcal{D}\xi + \sigma_+\eta - [\mathcal{F},\sigma_-^\dagger]_{\mathfrak{g}} + [\omega,\epsilon]_{\mathfrak{g}}\,,$$

$$\delta_\epsilon\Psi = [\Psi \overset{\bullet}{,} \epsilon]_{\mathfrak{g}}\,, \qquad \delta_{\epsilon,\xi,\eta}\mathcal{B} = [\mathcal{B}\overset{\bullet}{,}\epsilon]_{\mathfrak{g}} - \tfrac{1}{\mathrm{g}}\,[\Psi\overset{\bullet}{,}(\xi - D\sigma_-^\dagger\eta)\wedge H]_{\mathfrak{g}}\,,$$

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(iii) The existence of a commutative algebra whose decomposition under the Lorentz algebra corresponds to **tower of singletons of all integer spin**.

Outlook

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Thanks for your attention!