A tale of two SUSYs

Stam Nicolis

CNRS–Institut Denis–Poisson(UMR7013)
Université de Tours, Université d'Orléans
Parc Grandmont, 37200 Tours, France

Workshop on the Standard Model and Beyond
Corfu 2024
September 1, 2024

Based on *arXiv:2303.17875*, *arXiv:2404.03959* *and forthcoming.*

Outline

Introduction

Worldline SUSY as an inevitable property of relativistic spinning particles

How worldline supersymmetry can resolve fluctuations

Fluctuations and target space SUSY

The mysteries of the Nicolai map

Conclusions and outlook

What is supersymmetry?

- It is *discretionary*: We can write down supersymmetric theories; but, it seems, that we aren't obliged to do so; or are we, we just haven't realized it yet?
  Within perturbation theory, it *seems* that it is possible. Beyond perturbation theory, much less is known, one way or another.
- It is *inevitable*: We may think that supersymmetry is at our discretion to take into account; it isn't. Just because, historically, it was discovered much later than other possible symmetries doesn't mean anything. History of science, as always, is different from science.

We shall argue that supersymmetry has a bit of both features–but there are many issues that remain to be understood about it.
Two other attributes that will appear are *worldline* and *target space*.

Is relativity at our discretion?

For a long time people didn't know that invariance under global Lorentz transformations wasn't discretionary, but an inevitable property of nature. It took some time, before it became a reflex to write (we use units where $c = 1$ and signature $(+\underbrace{- - \cdots -}_{d-1})$, where d is the dimension of spacetime)

$$S_{\mathrm{NG}}^{(0)}[x] = -m \int d\lambda \sqrt{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$$

for the action of the massive particle, moving in flat spacetime–and still more before realizing that this action is equivalent to the expression

$$S_{\mathrm{P}}^{(0)}[x, e] = \int d\lambda \left\{ \frac{\dot{x}^{\mu}\dot{x}^{\nu}}{2e}\eta_{\mu\nu} + \frac{m^2 e}{2} \right\}$$

which can describe a massive or a massless particle–but that doesn't carry spin.

What happens when spin is included

A relativistic particle is labeled by the two Casimirs of the Poincaré group, its mass and its spin. It took some time before it was noticed (Berezin and Marinov (1976)) that it is possible to describe the property that a particle can carry spin 1/2 by using Grassmann variables, $\psi^I, \{\psi^I, \psi^J\} = 0$, where $I, J = 1, 2, 3$, since the combinations

$$S^I \equiv \frac{1}{2}\varepsilon^{IJK}\psi^J\psi^K$$

satisfy the relations

$$\left[S^I, S^K\right] = \frac{1}{2}\varepsilon^{IJK}S^K$$

The reason the ψ^I are useful is that it is possible to use them to write an action for the free relativistic, massless, particle of spin 1/2 (Brink, Di Vecchia, Howe (1977))

$$S_{\mathrm{P}}[x, e, \psi] = \int d\lambda \left\{ \frac{\dot{x}^\mu \dot{x}^\nu}{2e}\eta_{\mu\nu} + \frac{\mathrm{i}}{2}\psi^\mu\dot{\psi}^\nu\eta_{\mu\nu} \right\}$$

What happens when spin is included

There are additional terms that we can–therefore, must–include, since they are consistent with global Lorentz invariance and local reparametrization invariance of the worldline, viz.

$$S_{\mathrm{P}}^{(1)}[x, e, \psi, \chi] = \int d\lambda\, \mathrm{i}\chi\psi^{\mu}\dot{x}^{\nu}\eta_{\mu\nu}$$

If the particle is massive, we can add the following terms

$$S_{\mathrm{P}}^{(m)}[\psi_*, \chi] = \int d\lambda \left\{ \frac{\mathrm{i}\psi_*}{2}\left(\dot{\psi}_* - m\chi\right) + \frac{m^2 e}{2} \right\}$$

We remark that the fields χ and ψ_* must be Grassmann valued fields and that e and χ are auxiliary fields, that impose constraints–that define the mass and the spin of the particle that moves along the worldline.

What happens when spin is included: $\mathcal{N}=1$ worldline SUSY

Now Brink, Di Vecchia and Howe remarked that the action, $S_{\rm P}=S_{\rm P}^{(0)}+S_{\rm P}^{(1)}$, is invariant, up to a total derivative, under the transformations

$$\begin{aligned}\delta_\zeta x^\mu &= A\zeta\psi^\mu\\ \delta_\zeta\psi^\mu &= B\zeta\dot{x}^\mu\end{aligned}$$

where the parameter ζ is a Grassmann variable $\zeta^2=0$ if $A=\mathrm{i}B$; these transformations satisfy the relations

$$[\delta_\zeta,\delta_\eta]=2\mathrm{i}B^2\zeta\eta\frac{d}{d\lambda}$$

which justify calling them supersymmetric; indeed this is $\mathcal{N}=1$ worldline supersymmetry.

It is straightforward to check that the presence of the mass terms can be made consistent with $\mathcal{N}=1$ worldline supersymmetry.

$\mathcal{N} = 1$ worldline SUSY is an inevitable property of relativistic spinning particles

We conclude that the free, relativistic spin $1/2$ particle, in fact, realizes a representation of the $\mathcal{N} = 1$ supersymmetry algebra. The target space of the relativistic spin $1/2$ particle is, thus, eight-dimensional, with d commuting and d anticommuting coordinates–that are mixed through worldline supersymmetry.

Interaction with an external electromagnetic field

Global Lorentz invariance and electromagnetic gauge invariance imply that the interaction of a relativistic spin 1/2 particle with an external electromagnetic field is described by the action

$$S_{\mathrm{int}}[x, e, \psi] = \int d\lambda \, \left\{ q\dot{x}^{\mu} A^{\nu}(x)\eta_{\mu\nu} + q' \psi^{\mu}\psi^{\nu} F^{\rho\sigma}(x)\eta_{\mu\rho}\eta_{\nu\sigma} \right\}$$

A priori the two coupling constants, q and q' are independent; imposing invariance under $\mathcal{N} = 1$ worldline supersymmetry leads to a relation between q and q', viz.

$$q' = \mathrm{i}\frac{q}{2}$$

This shows that $\mathcal{N} = 1$ worldline supersymmetry can be–explicitly–broken, if this relation isn't satisfied.
We note that this incarnation of worldline supersymmetry doesn't seem to require auxiliary fields. The reason is that it is linearly realized.
Also that it describes properties of known particles. For some reason it hasn't received the attention it deserves.

From a particle to a field

Now that we have understood the spinning particle, we can, perhaps, guess how to describe systems of-indefinitely-many spinning particles. Normally we would imagine a field, $\Phi(x^\mu, \psi^\mu)$. Such a field is known-it's called a superfield. However it hasn't been used in this way. It might be interesting to explore why not.

How do fluctuations affect these properties?

Remains to understand what happens in the presence of fluctuations and how does supersymmetry pertain to them. This was addressed by Parisi and Sourlas in 1982.

Supersymmetry and fluctuations

They remarked that the canonical partition function of a non-relativistic quantum particle, moving in one spatial dimension,

$$Z = \int [\mathscr{D}x]\, e^{-\int d\tau \left\{\frac{1}{2}\dot{x}^2 + V(x)\right\}}$$

implies that the scalar potential, $V(x)$, can be written as

$$V(x) = \frac{1}{2}(W'(x))^2$$

up to a constant, since the potential should be bounded from below.

Supersymmetry and fluctuations

This, in turn, implies that the Euclidian action can be written as

$$S[x] = \int d\tau \left\{ \frac{1}{2} \left(\dot{x} + W'(x) \right)^2 \right\}$$

up to total derivatives; which suggests that, if we define the "noise field", $h(\tau)$, by the relation

$$h(\tau) \equiv \dot{x} + W'(x)$$

then, **if** it can be shown that the fluctuations can produce the Jacobian

$$J \equiv \left| \det \frac{\delta h(\tau)}{\delta x(\tau')} \right|$$

we deduce that the canonical partition function can be written as

$$Z = \int [\mathscr{D}h]\, e^{-\int d\tau \frac{1}{2} h(\tau)^2}$$

which is independent of the properties of the potential and may be assigned the value 1 with a suitable choice of units.

Supersymmetry and fluctuations

The reason supersymmetry is relevant here can be understood upon noticing that, by writing

$$\left|\det \frac{\delta h(\tau)}{\delta x(\tau')}\right| = e^{-\mathrm{i}\theta_{\det}} \det\left(\delta(\tau-\tau')\left(\frac{d}{d\tau} + \frac{\partial^2 W}{\partial x(\tau)\partial x(\tau')}\right)\right)$$

the expression involves the determinant of a local operator. It is therefore useful to include it in the Euclidian action using Grassmann valued fields:

$$\det\left(\frac{d}{d\tau} + \frac{\partial^2 W}{\partial x(\tau)\partial x(\tau')}\right) = \int [\mathscr{D}\psi][\mathscr{D}\chi]\, e^{\int d\tau\, \psi(\tau)\left(\frac{d}{d\tau} + \frac{\partial^2 W}{\partial x(\tau)^2}\right)\chi(\tau)}$$

leading to the expression for the classical action

$$S[x,\psi,\chi] = \int d\tau \left\{\frac{1}{2}\left(\dot{x} + W'(x)\right)^2 - \psi\left(\dot{\chi} + W''(x)\chi\right)\right\}$$

Enter the Nicolai map

The map

$$h(\tau) \equiv \dot{x} + W'(x)$$

is known as the Nicolai map, cf. Nicolai 1980.

It maps the field $x(\tau)$ to the field $h(\tau)$. While $x(\tau)$ is an interacting field–it has a non-trivial potential–$h(\tau)$ is a Gaussian field–but it has ultra–local 2-point function.

Were it possible to solve the differential equation it defines exactly, it would be possible to express $x(\tau)$ in terms of $h(\tau)$ thus providing a new way for describing interacting fields in terms of free fields.

The Nicolai map and fermions

Another property of the Nicolai map (which was in fact Nicolai's motivation for introducing it) is that it allows taking account of fermions in terms of their superpartners, since

$$\det \frac{\delta h(\tau)}{\delta x(\tau')} = \int [\mathscr{D}\psi][\mathscr{D}\chi]\, e^{\int d\tau\, d\tau' \psi(\tau) \frac{\delta h(\tau)}{\delta x(\tau')} \chi(\tau')}$$

Therefore whether the anticommuting fields, ψ and χ, are worldline or target space fermions depends on the properties of the operator

$$\frac{\delta h(\tau)}{\delta x(\tau')}$$

Supersymmetry and fluctuations: $\mathcal{N} = 2$ worldline SUSY

By introducing an auxiliary field, $F(\tau)$, we may write the classical action in the form

$$S[x, \psi, \chi, F] = \int d\tau \left\{ \frac{1}{2}\dot{x}^2 - \frac{1}{2}F^2 + FW'(x) - \psi\left(\dot{\chi} + W''\chi\right) \right\}$$

and show that it is invariant under *two* supersymmetric transformations, *linear* in the fields. These realize $\mathcal{N} = 2$ worldline SUSY.

Supersymmetry and fluctuations: Target space SUSY

It is possible to show that this resolution of the degrees of freedom, that can describe the fluctuations–of relativistic spinning fields–can be captured by *target space* supersymmetry. The idea of Parisi and Sourlas consists in showing that the Euclidian action for $N_{\mathrm{f}} = 2$ scalars, in $d = 2$ spacetime dimensions,

$$S^{(0)}[\{\phi_A\}] = \int d^2x \left\{ \frac{1}{2}\delta^{\mu\nu}\delta^{AB}\partial_\mu\phi_A\partial_\nu\phi_B \right\}$$

can be written as

$$\int d^2x \left\{ \frac{1}{2}\delta^{AB}\left(\sigma^\mu\partial_\mu\phi_A\right)\left(\sigma^\nu\partial_\nu\phi_B\right) \right\}$$

where the σ^μ generate the Clifford algebra

$$\left\{\sigma^\mu_{AC}, \sigma^\nu_{CB}\right\} = 2\delta^{\mu\nu}\delta_{AB}$$

Supersymmetry and fluctuations: Target space SUSY

The canonical partition function is given by the expression

$$Z = \int [\mathscr{D}\phi_A] \, e^{-S^{(0)}[\{\phi_A\}]}$$

Parisi and Sourlas remark that the classical action can be written as

$$S^{(0)}[\{\phi_A\}] = \int d^2x \, \frac{1}{2}\delta^{AB} h_A(x) h_B(x)$$

where

$$h_A(x) \equiv [\sigma^\mu]_{AB}\partial_\mu\phi_B$$

What is crucial here is that it is possible to show that

$$[\sigma^\mu]_{AB}\partial_\mu\phi_B[\sigma^\nu]_{BC}\partial_\nu\phi_C = \delta^{AC}\delta^{\mu\nu}\partial_\mu\phi_A\partial_\nu\phi_C$$

up to total derivatives.

Supersymmetry and fluctuations: Target space SUSY

The idea, now, is to show that the fluctuations can be described through the appearance of the term

$$J = \left|\det\frac{\delta h_A(x)}{\delta\phi_B(x')}\right| = e^{-\mathrm{i}\theta_{\rm det}}\det\frac{\delta h_A(x)}{\delta\phi_B(x')} = \det\left(\delta(x-x')\right)e^{-\mathrm{i}\theta_{\rm det}}\det\left(\sigma^\mu_{AB}\partial_\mu\right)$$

which implies that the–canonical–partition function, when the fluctuations can be resolved, is given by the expression

$$Z = \int[\mathscr{D}\phi_A]\left|\det\frac{\delta h_A(x)}{\delta\phi_B(x')}\right|e^{-\int d^2x\,\frac{\delta^{AB}}{2}h_A(x)h_B(x)} = \int[\mathscr{D}h_A(x)]\,e^{-\int d^2x\,\frac{\delta^{AB}}{2}h_A(x)h_B(x)} = 1 =$$
$$\det\left(\delta(x-x')\right)\int[\mathscr{D}\phi_A]\,e^{-\mathrm{i}\theta_{\rm det}}\det\left(\sigma^\mu_{AB}\partial_\mu\right)\,e^{-S^{(0)}[\{\phi_A\}]} =$$
$$\det\left(\delta(x-x')\right)\int[\mathscr{D}\phi_A]\,[\mathscr{D}\psi_A][\mathscr{D}\chi_A]\,e^{-\mathrm{i}\theta_{\rm det}}\times$$
$$e^{-\int d^2x\left(\frac{1}{2}\delta^{AB}(\sigma^\mu\partial_\mu\phi_A)(\sigma^\nu\partial_\nu\phi_B)-\psi_A\sigma^\mu_{AB}\partial_\mu\chi_B\right)}$$

Supersymmetry and fluctuations: Target space SUSY

We recognize that, this time, the anticommuting fields, ψ_A and χ_B, can be identified with target space, not worldline, fermions and the classical, free, action is invariant under $\mathcal{N} = 2$ *target space* SUSY.

There are several conceptual issues that remain to be fully clarified here.

Supersymmetry and fluctuations: Target space SUSY–interactions

To describe interactions, we need a superpotential. If we set

$$h_A(x) = [\sigma^\mu]_{AB}\partial_\mu\phi_B + \frac{\partial W}{\partial\phi_A}$$

we deduce the scalar potential $V(\{\phi_A\})$:

$$V(\{\phi_A\}) = \frac{1}{2}\delta^{AB}\frac{\partial W}{\partial\phi_A}\frac{\partial W}{\partial\phi_B}$$

We would like to understand under what conditions this is invariant under SO(2) flavor transformations.
However there's an additional term, that appears, namely,

$$\sigma^\mu_{AB}\frac{\partial\phi_B}{\partial x^\mu}\frac{\partial W}{\partial\phi_B}$$

We'd like to understand, under what conditions this term is a total derivative.

A worked example: $\mathcal{N} = 2, d = 2$ Wess-Zumino

An interesting case is provided by the equations

$$\frac{\partial W}{\partial \phi_1} = g(\phi_1^2 - \phi_2^2)$$
$$\frac{\partial W}{\partial \phi_2} = 2gs\phi_1\phi_2$$

with $s = \pm 1$, which was studied by Parisi and Sourlas (1982).
We remark that

$$\frac{\partial^2 W}{\partial \phi_1^2} + \frac{\partial^2 W}{\partial \phi_2^2} = 2g(s+1)\phi_1$$

A worked example: $\mathcal{N} = 2, d = 2$ Wess-Zumino

but that the crossterm

$$[\sigma^\mu]_{AB}\partial_\mu\phi^A \frac{\partial W}{\partial \phi_B} =$$
$$\partial_x\left(\phi_1^2\phi_2 - \frac{\phi_2^3}{3}\right) + \partial_y\left(\frac{\phi_1^3}{3} - \phi_1\phi_2^2\right) +$$
$$2g\phi_1\phi_2(s-1)\left(\partial_x\phi_1 - \partial_y\phi_2\right)$$

A worked example: $\mathcal{N} = 2, d = 2$ Wess-Zumino

This implies that the superpotential is *either* a holomorphic function of the scalars, *or* it leads to a scalar potential that respects SO(2) coordinate invariance (in Euclidian signature)–i.e. Lorentz invariance in Lorentzian signature. Obviously we care about the latter more than about the former.

Target space SUSY and fluctuations beyond two dimensions

The attempt by Parisi and Sourlas to generalize their approach beyond $d > 2$ spacetime dimensions hits obstacles.

- For $d \not\equiv 2 \mod 8$ (in particular for $d = 3$ and $d = 4$) the generators of the Clifford algebra don't have a Majorana representation. This can be easily remedied, by doubling the degrees of freedom:

$$h_A(x) = \sigma^\mu_{AB} \partial_\mu \phi_B + \frac{\partial W}{\partial \phi_A}$$
$$h_A(x)^\dagger = \sigma^\mu_{BA} \partial_\mu \phi_B^\dagger + \left(\frac{\partial W}{\partial \phi_A}\right)^\dagger$$

since $[\sigma^\mu_{AB}]^\dagger = \sigma^\mu_{BA}$.

Target space SUSY and fluctuations beyond two dimensions

- However there is another problem, namely, whether the crossterms

$$\sigma^{\mu}_{AB}\partial_{\mu}\phi_{B}\left(\frac{\partial W}{\partial\phi_{A}}\right)^{\dagger}+\sigma^{\mu}_{BA}\partial_{\mu}\phi^{\dagger}_{B}\frac{\partial W}{\partial\phi_{A}}$$

  are total derivatives;
- but, even more significantly, whether the crossterms, encountered upon expanding the term

$$\sigma^{\mu}_{AB}\partial_{\mu}\phi_{B}\sigma^{\nu}_{BC}\partial_{\nu}\phi^{\dagger}_{C}$$

  are total derivatives.

Target space SUSY and fluctuations beyond two dimensions

These crossterms are the "leftovers" between the terms generated by the Nicolai map and the terms imposed by the supersymmetry algebra. If they aren't total derivatives, their presence can be probed by the anomalies in the identities that describe how supersymmetry is realized.

In the conventional approach to supersymmetric theories such terms never appear in the first place, that's why they haven't been noticed to date. The reason they appear in this approach is the Nicolai map, that provides a map between the noise fields and the dynamical fields–assuming, however, that the superpartners describe the fluctuations of each. This need not be the case, as the relativistic spinning particle shows.

What does the Nicolai map actually imply?

It provides the hint that supersymmetry is a more general property of physical systems. How this map can be defined for gauge theories is, still, an open problem. Despite recent efforts, it hasn't been possible to define it, yet, in as explicit aa form as for Wess-Zumino models.
On the lattice, on the other hand, it has been possible to define the so-called "trivializing map",
cf. M. Lüscher (2010),

$$[dV_\mu] = [dU_\mu]\, e^{-S[U]}$$

This map transforms the non–uniform measure over the gauge group into a uniform measure. On a compact manifold–which is the case of the Lie groups of relevance to particle physics–the uniform distribution has the "universality" property" that the Gaussian has for non–compact manifolds. However, curiously, the relation with the Nicolai map hasn't been established..

Conclusions and outlook

- Worldline SUSY is an inevitable property of relativistic spinning particles. Its effects can be probed by their coupling to external electromagnetic fields. What is not that obvious is how to adapt it for relativistic spinning fields.
- Worldline SUSY can, also, provide a way for resolving the degrees of freedom, with which dynamical degrees of freedom are in equilibrium.
- However, as the example of the relativistic spinning particle illustrates, the degrees of freedom that are related by worldline SUSY need not resolve the other's fluctuations.

Conclusions and outlook

- Target space SUSY can be, also, understood as providing the resolution of fluctuations–however its realization, for spacetime dimensions $d > 2$, is, typically, anomalous. This anomaly can be probed by the correlation functions of the Nicolai map, without needing to solve for it. The challenge is to translate the identities satisfied by the Nicolai map–and their anomalies–into signals that can be probed by real experiments. Perhaps in polarized structure functions?

Conclusions and outlook

- Understanding the meaning of the Nicolai map for (super)Yang-Mills theories (target space SUSY) remains to be fully spelled out.
  Cf. the recent work by Nicolai *et al.* (2021) and by Lechtenfeld (2024)
  and it is interesting to compare it to the work by de Alfaro, Fubini, Furlan and Veneziano (1985)

Conclusions and outlook for the two SUSYs

- So we see that there are two kinds of SUSY: The SUSY–where the superpartners have distinct physical significance, already, at the classical level: For the relativistic spinning particle the superpartners are position and spin. This is, indeed, the "conventional" approach to SUSY.
- Then there is the SUSY, where the superpartners possess another kind of physical significance, namely, where they resolve the degrees of freedom that define the bath of fluctuations. This turns out to be the case for the *non-relativistic* particle, in equilibrium with thermal or with quantum fluctuations.

Conclusions and outlook for the two SUSYs

- So for the relativistic spinning particle, if it is in equilibrium with a bath of fluctuations, the fluctuations of the position are encoded by the superpartners of the position–that are **not** the components of the spin!; and the fluctuations of the spin are encoded by the superpartners of the spin–that are **not** the components of the position! So there are *three* supersymmetries at work here that are, however, definitely, related.

Conclusions and outlook for the two SUSYs

In any event the Nicolai map is useful for probing supersymmetric anomalies, in a new way. In the picture of Parisi and Sourlas supersymmetry describes the symmetry between the dynamical degrees of freedom and the degrees of freedom that can serve as fluctuations; what this implies for field theories in four spacetime dimensions remains to be understood.

Conclusions and outlook for the two SUSYs

There is, still, much more to learn about SUSY!