Physics on the Fuzzy Onion

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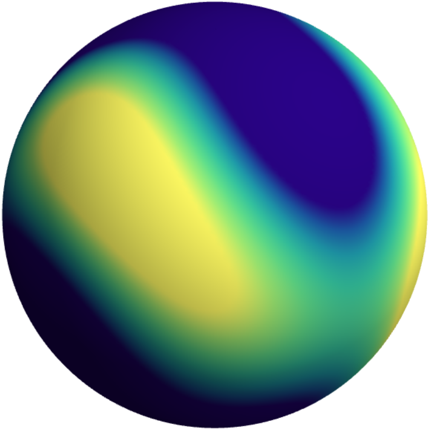
The goal

To construct a simple[1] model of the three-dimensional quantum space.

[1] Easy to work with, versatile and intuitive.

The Fuzzy Sphere



The Fuzzy Sphere

A finite-dimensional representation of $su(2)$ expressed in terms of $N \times N$ Hermitian matrices with a natural cut-off, $l \leq N-1$.

$$[L_i^{(N)}, L_j^{(N)}] = i\varepsilon_{ijk} L_k^{(N)},$$

$$[L_i^{(N)}, [L_i^{(N)}, Y_{lm}^{(N)}]] = l(l+1) Y_{lm}^{(N)}, \quad [L_3^{(N)}, Y_{lm}^{(N)}] = m Y_{lm}^{(N)},$$

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}.$$

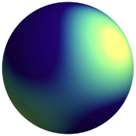
This comes without the mildly annoying ${}^{(N)}$ and usually with physical scales $x_i = \lambda L_i$, $x^2 = r^2$. Note that $N \sim r/\lambda + \ldots$

The Fuzzy Sphere

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}$$

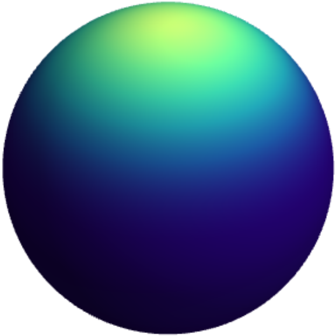
$$\Phi^{(\infty)}(\theta, \phi) = \sum_{l=0}^{\color{red}{N-1}} \sum_{m=-l}^{l} c_{lm}^{(\infty)} Y_{lm}^{(\infty)}$$

$$\begin{pmatrix} -0.182241 & -0.356949 + 0.0169752\,i & 0.0260558 + 0.055678\,i & -0.0418167 - 0.358403\,i \\ -0.356949 - 0.0169752\,i & 0.723061 & -0.266625 - 0.323709\,i & -0.209613 - 0.250825\,i \\ 0.0260558 - 0.055678\,i & -0.266625 + 0.323709\,i & 0.93628 & 0.115833 + 0.0969497\,i \\ -0.0418167 + 0.358403\,i & -0.209613 + 0.250825\,i & 0.115833 - 0.0969497\,i & 0.30945 \end{pmatrix}$$



The Fuzzy Sphere

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

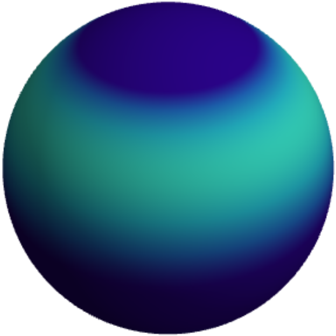


The Fuzzy Sphere

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

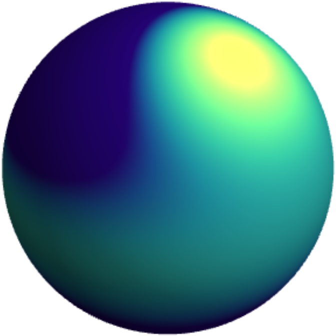
The Fuzzy Sphere

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



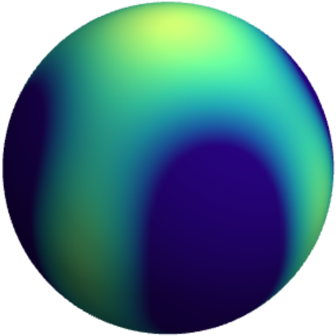
The Fuzzy Sphere

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



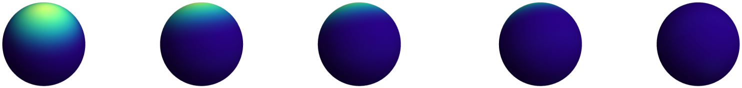
The Fuzzy Sphere

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$



The Fuzzy Sphere

$$\Psi = R \cdot \Psi_{\mathrm{NP}}$$



The Fuzzy Sphere

Also usually one is interested in some kind of field theory on the fuzzy sphere

$$S_N[\Phi^{(N)}] = \frac{4\pi}{N} \mathrm{tr}_N \left(a \; \Phi^{(N)} \mathcal{K}^{(N)} \Phi^{(N)} + b \; (\Phi^{(N)})^2 + c \; (\Phi^{(N)})^4 \right) ,$$

where

$$\mathcal{K}^{(N)} \Phi^{(N)} = [L_i^{(N)}, [L_i^{(N)}, \Phi^{(N)}]] \; .$$

With this, one can compute mean values of observables:

$$\langle \mathcal{O}(\Psi) \rangle = \frac{1}{Z} \int d\Psi e^{-S(\Psi)} \mathcal{O}(\Psi) \; , \; d\Psi = \prod_{N=1}^{M} d\Phi^{(N)} \; .$$

The Fuzzy Sphere

- One can do field theories on it,
- accessible numericaly,
- UV/IR mixing,
- emerges elsewhere (BMN model for example),
- is 2D,
- useful to think in terms of expansion modes,
- aspects of correlations and entanglement,
- nice playground.

Bosonic contruction

A similar starting point

$$[x_i, x_j] = 2\lambda i \varepsilon_{ijk} x_k .$$

We can now invoke an auxiliary Fock space and two sets of c/a bosonic operators:

$$[\mathrm{a}_\alpha, \mathrm{a}^\dagger_\beta] \,=\, \delta_{\alpha\beta}, \quad [\mathrm{a}_\alpha, \mathrm{a}_\beta] \,=\, [\mathrm{a}^\dagger_\alpha, \mathrm{a}^\dagger_\beta] \,=\, 0\,,$$

$$\frac{(\mathrm{a}^\dagger_1)^{n_1}\,(\mathrm{a}^\dagger_2)^{n_2}}{\sqrt{n_1!\,n_2!}}\,|0\rangle \,=\, |n_1, n_2\rangle \quad .$$

Then one can take

$$x_i = \lambda a^\dagger \sigma_i a$$

to satisfy the commutation relation and

$$r = \lambda\left(\mathrm{a}^\dagger \mathrm{a} + 1\right).$$

Bosonic construction of R^3_λ

One can define physical content using this construction; for example take $\Psi = \Psi(x)$ and

$$H_0 \Psi = \frac{1}{2\lambda r}[\mathrm{a}^\dagger_\alpha, [\mathrm{a}_\alpha, \Psi]].$$

This was done thoroughly for the Coulomb problem and the spectrum was found (exactly)

$$E^l_{\lambda\, n} = \frac{\hbar}{m_e \lambda^2}\left(1 - \sqrt{1 + \frac{m_e q \lambda}{\hbar^2 n}}^{\,2}\right)$$

Bosonic construction of R^3_λ

- Important objects defined and tested,
- can be generalized (monopoles, ...),
- preserves many symmetries,
- can lead to phenomenological predictions,
- is 3D, leads to nasty equations sometimes, not easy to solve new problems,
- some ways to move towards relativity.

Two similar models of quantum space

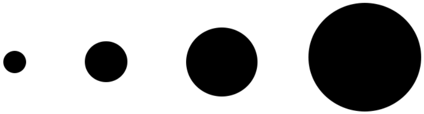
The fuzzy sphere: Convenient to work with, but it is a two-dimensional space.

Bosonic construction of R^3_λ: Is three-dimensional (like our space). Working with it is slightly cumbersome.

The fuzzy onion: a three-dimensional quantum space with matrix realisation that mimics the bosonic construction.

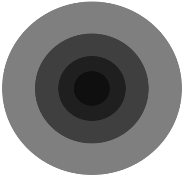
The fuzzy onion

$$\begin{pmatrix} \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



The fuzzy onion

$$\Psi = \begin{pmatrix} \begin{pmatrix} \cdot \end{pmatrix} & & & \\ & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} & & \\ & & \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} & \\ & & & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{pmatrix}$$



The kinetic term

The angular part of the kinetic term is easy to define

$$\mathcal{K}_L \Psi = r^{-2} \begin{pmatrix} \mathcal{K}^{(1)}\Phi^{(1)} & & & & \\ & \mathcal{K}^{(2)}\Phi^{(2)} & & & \\ & & \mathcal{K}^{(3)}\Phi^{(3)} & & \\ & & & \ddots & \\ & & & & \mathcal{K}^{(M)}\Phi^{(M)} \end{pmatrix}$$

Considered before by Wallet, Vitale, Jurić, Poulain and others.

The kinetic term

The radial part of the kinetic term is harder to define. We want to compare objects with different numbers of d.o.f.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \leftrightarrow \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\Phi^{(N)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm}^{(N)} Y_{lm}^{(N)}$$

All expansion coefficients can be matched but the ones corresponding to the highest momentum. Solution: ignore them.

The kinetic term

$$\mathcal{D}:\Phi^{(N+1)}\to\Phi^{(N)}=\sum_{l=0}^{N-1}\sum_{m=-l}^{l}c_{lm}^{(N)}Y_{lm}^{(N)}\ ,\ \ c_{lm}^{(N)}=c_{lm}^{(N+1)}$$

$$\mathcal{U}:\Phi^{(N)}\to\Phi^{(N+1)}=\sum_{l=0}^{N}\sum_{m=-l}^{l}c_{lm}^{(N+1)}Y_{lm}^{(N+1)}\ ,\ \left\{\begin{array}{l} c_{lm}^{(N+1)}=c_{lm}^{(N)} \\ c_{Nm}^{(N+1)}=0 \end{array}\right.$$

$$\partial_r^{(N)}\Phi^{(N)}=\frac{\mathcal{D}\Phi^{(N+1)}-\mathcal{U}\Phi^{(N-1)}}{2\lambda}$$

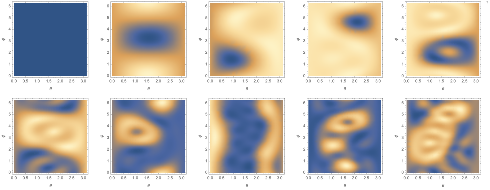
$$\mathcal{K}_R\Psi=\sum_{N,l,m}\frac{(N+1)c_{lm}^{(N+1)}+(N-1)c_{lm}^{(N-1)}-2Nc_{lm}^{(N)}}{N\,\lambda^2}Y_{lm}^{(N)}$$

The fuzzy onion

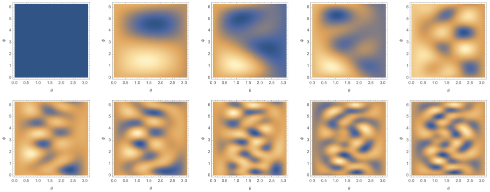
We have the field content Ψ and the kinetic term $\mathcal{K} = \mathcal{K}_L + \mathcal{K}_R$.
Now we can do physics!

Example I: Scalar field theory

$$S[\Psi] = 4\pi\lambda^2 \mathrm{Tr}\ r\left(\ \Psi\mathcal{K}\Psi + b\ \Psi^2 + c\ \Psi^4\right)$$

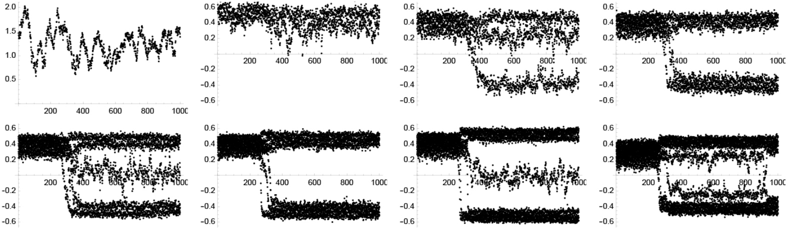


Sample configuraiton of 10 separate fuzzy spheres.



Sample configuraiton of 10 layers of a fuzzy onion.

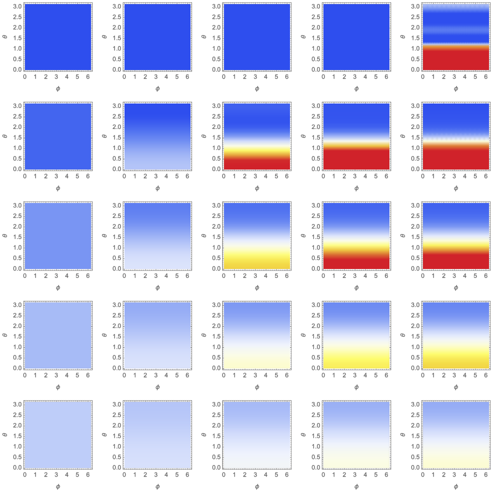
Example I: Scalar field theory



Eigenvalues (HMC trajectories) on 8 fuzzy-onion layers.

Example II: Heat transfer

$$\mathcal{K}\Psi(t) = \alpha\, \partial_t \Psi(t)$$



Example III: Coulomb problem

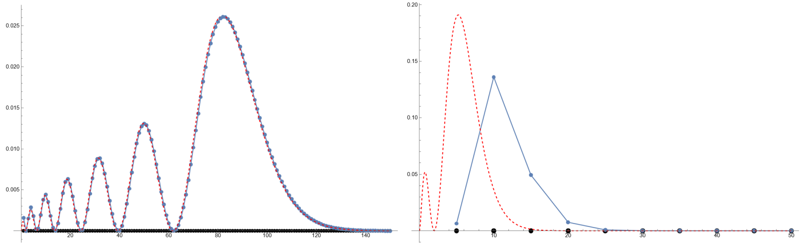
$$H = -\frac{\hbar^2}{2m_e}\mathcal{K} - \frac{q}{r}$$

$$\mathbf{H}\mathcal{C}_{lm} = E\mathcal{C}_{lm}$$

n	1	2	3	4	5	6
E_n	-0.4142	-0.1180	-0.0541	-0.0307	-0.0179	-0.0031
$E^l_{\lambda n}$	-0.4142	-0.1180	-0.0541	-0.0307	-0.0198	-0.0138
E_n^{CQM}	-0.5	-0.125	-0.0556	-0.0313	-0.02	-0.0139

$$E^l_{\lambda\, n} = \frac{\hbar}{m_e\lambda^2}\left(1 - \sqrt{1 + \frac{m_e q \lambda}{\hbar^2 n}^2}\right)$$

Example III: Coulomb problem



- For $\lambda \ll a_0$ and $\lambda N \gg a_0$ we have great agreement with QM.
- For $\lambda \sim a_0$ and $\lambda N \gg a_0$ strong quantum-space effects.
- For $\lambda N \sim a_0$ space is not large enough to capture the physics.

Smearing the potential

$$\mathcal{S}\phi^{(n)} = \frac{\phi^{(n)} + \sum_i \alpha_i \left(\mathcal{U}^i \phi^{(n-i)} + \mathcal{D}^i \phi^{(n+i)}\right)}{1 + \sum_i \alpha_i},$$

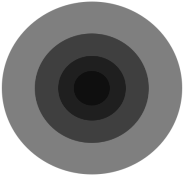
For example with $\alpha_1 = \frac{1}{2}, \alpha_{2+} = 0$:

$$\mathcal{S}\phi^{(n)} = \frac{\phi^{(n)} + \frac{1}{2}\mathcal{D}\phi^{(n+1)} + \frac{1}{2}\mathcal{U}\phi^{(n-1)}}{2},$$

$$V(\Psi) = \sum_{j=1}^{N_m} V\left(\mathcal{S}\phi^{(j)}\right).$$

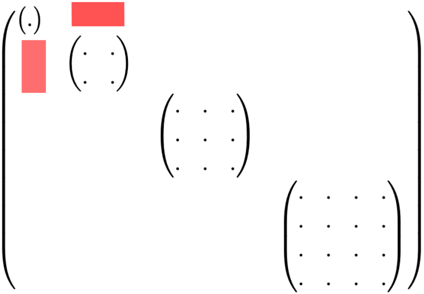
Considering string states [2]

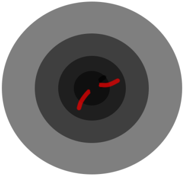
$$\Psi = \begin{pmatrix} (\cdot) & & & \\ & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} & & \\ & & \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} & \\ & & & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \end{pmatrix}$$



[2] Steinacker, Tekel 2203.02376

Consider string states [3]

$\Psi =$ 



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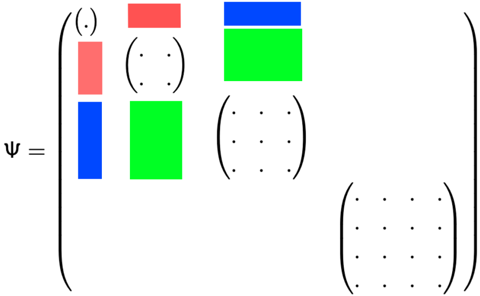
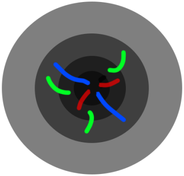
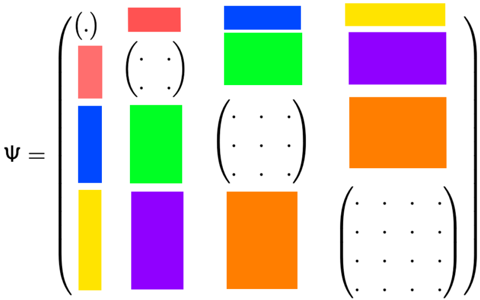
Consider string states [4]

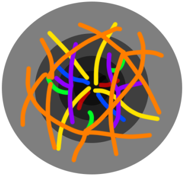




[4]Steinacker, Tekel 2203.02376

Consider string states [5]





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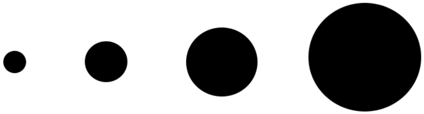
Model of expanding toy universe

$$\begin{pmatrix}\cdot\end{pmatrix}, \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}, \begin{pmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{pmatrix}, \begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}$$



Model of expanding toy universe

As the universe expands:

1. the number of Planck cells increases,
2. the size of Planck cells increases.

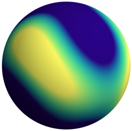
Model of expanding toy universe

As the universe expands:

1. the number of Planck cells increases,
2. the size of Planck cells increases,
3. space is not made of a (finite) number of Planck cells (in such a simple way).

Outlook

- Proper field theory study.
- String state formalism.
- Connection to other theories
- Classical physics/astrophysical aplications.



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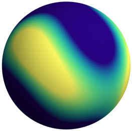

Thank you for your attention!