Signed eigenvalue distributions of complex random tensors and geometric measure of entanglement of multipartite states

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Mainly based on

S. Majumdar, NS, to appear in PTEP, arXiv:2408.01030 [hep-th]
NS, PTEP 2024 (2024) 5, 053A04, arXiv:2404.03385 [hep-th]
M.R. Kloos, NS, Lett.Math.Phys. 114 (2024) 3, 80, arXiv:2403.12427 [hep-th]

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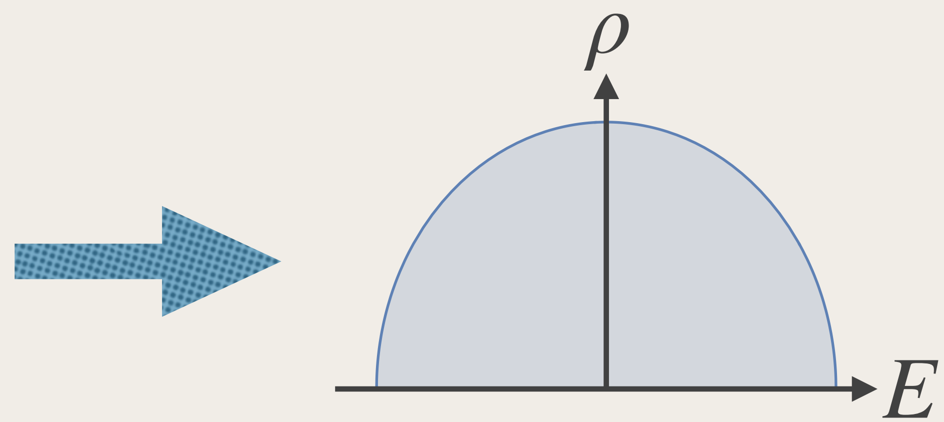
§Introduction

Eigenvalue distributions are important in random matrix models

- Approximate Hamiltonian of atoms (Wigner 1958)

H : Random matrix



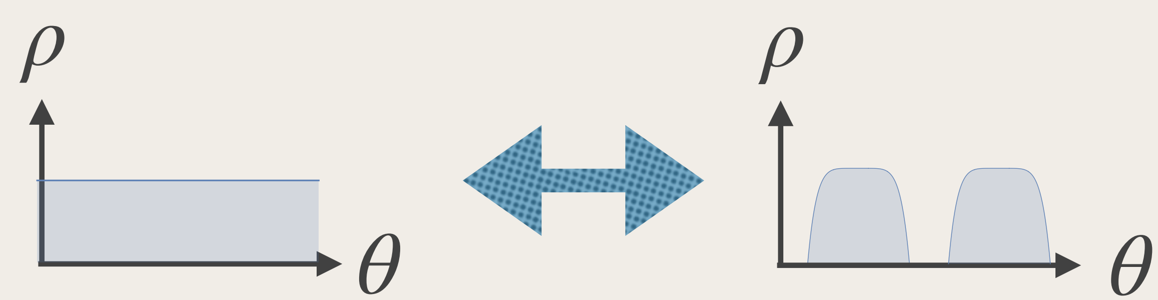
Semi-circle law

- Method of computing matrix models

Brezin-Itzykson-Parisi-Zuber 1978

- Topological transition — Dynamics of QCD

Gross-Witten, Wadia, 1980



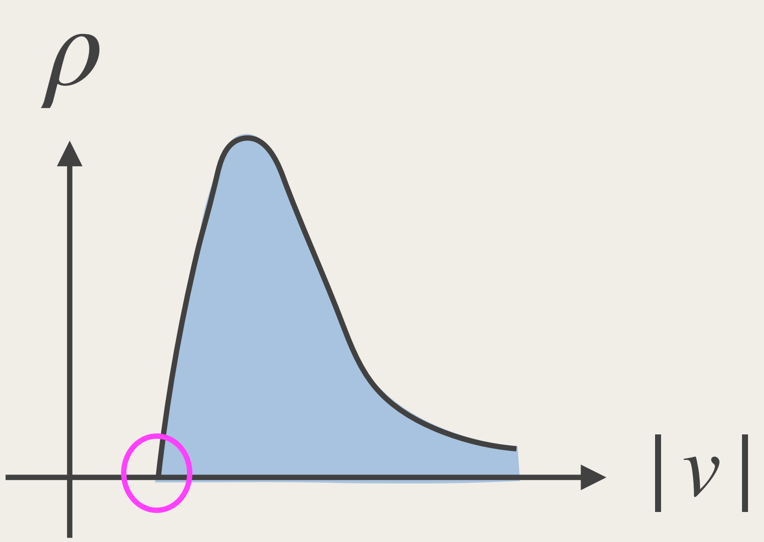
How about tensor eigenvalue distributions ?

Most tensor problems are NP-hard for a tensor. Hillar-Lim 2009

On the other hand, a distribution of tensor eigenvalues/vectors for random tensors can exactly/approximately be computed, as we will do by using quantum field theories.

In $N \to \infty$, the distribution will not depend on a randomly chosen tensor (Thermodynamic limit)

In $N \to \infty$ a sharp edge of the distribution exists, which is important, since it determines the "best" value in applications.



- Ground state energy of spin glass
- Largest eigenvalue
- Best rank-one decomposition of tensor
- Geometric measure of entanglement of random multipartite states

§Geometric measure of entanglement of multipartite states

- **Bipartite state** $|\psi\rangle = M_{ab}|a\rangle_A|b\rangle_B$ $\quad |a\rangle_A \in H_A \quad |b\rangle_B \in H_B$

Entanglement entropy

$$S = -\mathrm{Tr}_A(\Omega_A \log \Omega_A) = -\mathrm{Tr}_B(\Omega_B \log \Omega_B)$$

$$\Omega_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$$
$$\Omega_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$$

- **Tripartite state**

$$|\psi\rangle = C_{abc}|a\rangle_A|b\rangle_B|c\rangle_C \qquad |a\rangle_A \in H_A \quad |b\rangle_B \in H_B \quad |c\rangle_C \in H_C$$

Generally,

$$-\mathrm{Tr}_A(\Omega_A \log \Omega_A) \neq -\mathrm{Tr}_B(\Omega_B \log \Omega_B)$$

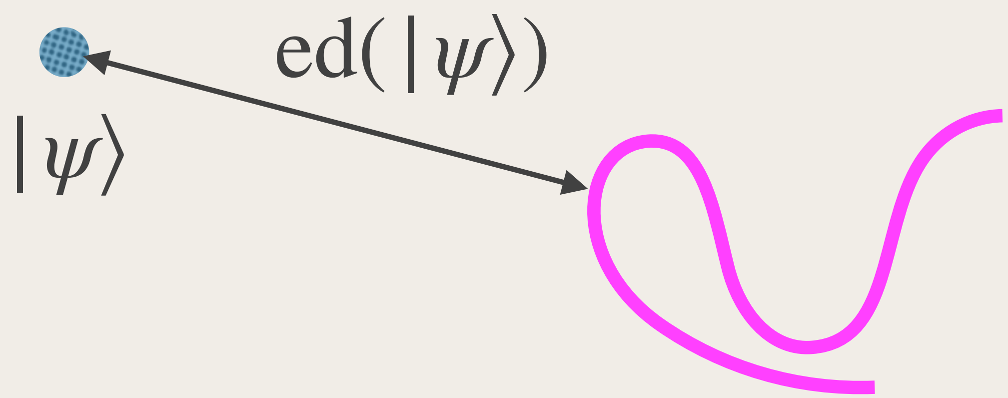
$$\Omega_A = \mathrm{Tr}_{BC}(|\psi\rangle\langle\psi|)$$
$$\Omega_B = \mathrm{Tr}_{AC}(|\psi\rangle\langle\psi|)$$

How can we measure entanglement of multipartite states ?

Define entanglement by minimum distance from separable states

Shimony 1995, Barnum-Linden 2001, Wei-Goldbart 2003

$$\mathrm{ed}(|\psi\rangle) = \min_{\psi_{A,B,C}} \Big|\, |\psi\rangle - |\psi_A\rangle_A \otimes |\psi_B\rangle_B \otimes |\psi_C\rangle_C \Big|$$



Orbit of separable states

$$|\psi_A\rangle_A \otimes |\psi_B\rangle_B \otimes |\psi_C\rangle_C$$

Representation in tensor $|\psi\rangle = C_{abc}|a\rangle_A \otimes |b\rangle_B \otimes |c\rangle_C$

$$|\psi_A\rangle_A = v_a^{(A)}|a\rangle_A$$

$$|\psi_B\rangle_B = v_b^{(B)}|b\rangle_B$$

$$|\psi_C\rangle_C = v_c^{(C)}|c\rangle_C$$

$$\frac{\partial\, \mathrm{ed}(|\psi\rangle)}{\partial v_{a,b,c}^{(A,B,C)}} = 0 \quad \Rightarrow \quad \begin{cases} C^*_{abc} v_b^{(B)} v_c^{(C)} = v_a^{(A)*} \\ C^*_{abc} v_a^{(A)} v_c^{(C)} = v_b^{(B)*} \\ C^*_{abc} v_a^{(A)} v_b^{(B)} = v_c^{(C)*} \end{cases}$$

A system of eigenvector equations

Eigenvector of smallest $|v| = |v_i|$ determines $\mathrm{ed}(|\psi\rangle) \rightarrow$ The edge

§Tensor eigenvalues/vectors

Qi, Lim, 2005 Cartwright-Sturmfels 2013

- **Tensor eigenpair equation**

  Ex. Symmetric order-three tensor

  $$C_{abc} w_b w_c = \zeta\, w_a \qquad |w| = \sqrt{w_a w_a} = 1$$

  ζ : eigenvalue, w : eigenvector

  Because of non-linearity one can absorb ζ into w , unless $\zeta = 0$.

  ↓

- **Tensor eigenvector equation**

  $$C_{abc} v_b v_c = v_a \qquad (\zeta = |v|^{-1})$$

This talk uses tensor eigenvector equation, since it is simpler to handle and equivalent, as $|v| = \infty$ is ignorable in most applications.

§ Complex eigenvector problems

S. Majumdar, NS, to appear in PTEP, arXiv:2408.01030 [hep-th]
NS, PTEP 2024 (2024) 5, 053A04, arXiv:2404.03385 [hep-th]

We compute the distributions of eigenvectors of complex order-three random tensors with symmetric or independent indices.

- Symmetric indices case

$$C_{abc} = C_{\sigma_a \sigma_b \sigma_c},\ v_a \in \mathbb{C} \ \ (\sigma: \text{arbitrary perms. of } a, b, c)$$

$$C^*_{abc} v_b v_c = v^*_a \quad : \text{Eigenvector equation}$$

Corresponds to $|\psi\rangle = C_{abc} |a\rangle |b\rangle |c\rangle$

- Independent indices case

$$C^*_{abc} v_b^{(B)} v_c^{(C)} = v_a^{(A)*}$$
$$C^*_{abc} v_a^{(A)} v_c^{(C)} = v_b^{(B)*}$$
$$C^*_{abc} v_a^{(A)} v_b^{(B)} = v_c^{(C)*}$$

$C_{abc}, v_a^{(A)}, v_b^{(b)}, v_c^{(C)} \in \mathbb{C}$

: A system of eigenvector eqs.

Corresponds to $|\psi\rangle = C_{abc} |a\rangle_A |b\rangle_B |c\rangle_C$

§ Field theoretical method

cf. A. Crisanti, L. Leuzzi, and T. Rizzo, Eur. Phys. J. B 36, 129-136 (2003)

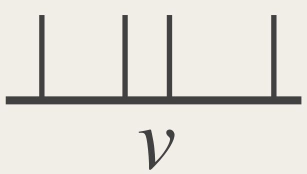
General form of the problem

Number of d.o.f. of v

$$f_i(v, C) = 0 \quad : \text{linear in } C \qquad i = 1,2,\cdots,\#v$$

Distribution of solutions v^α ($\alpha = 1,2,\cdots,\#\text{sol}$) for a C

$$\rho(v, C) = \sum_{\alpha=1}^{\#\text{sol}} \delta^{\#v}(v - v^\alpha) = |\det M(v, C)| \prod_{i=1}^{\#v} \delta^{\#v}\big(f_i(v, C)\big)$$

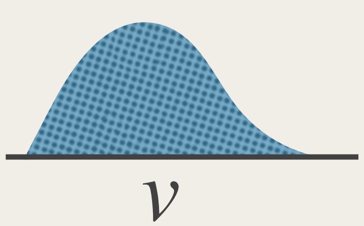


: NP-hard

Jacobian $M_{ij}(v, C) = \dfrac{\partial f_i(v, C)}{\partial v_j}$

Distribution of v for a Gaussian ensemble of C

$$\rho(v) = \frac{1}{\mathcal{N}} \int_{\mathbb{C}^{\#C}} dC\, e^{-\alpha\, C^*_{abc} C_{abc}}\, \rho(v, C) \qquad \alpha \in \mathbb{R}^+$$



: Computable

Rewrite $\rho(v) = \int dC\, e^{-\alpha C^*_{abc} C_{abc}} \,|\det M(v, C)|\, \prod_{i=1}^{\#v} \delta(f_i(v, C))$

$$\prod_{i=1}^{\#v} \delta(f_i(v, C)) = \frac{1}{(2\pi)^{\#v}} \int d^{\#v}\lambda\, e^{i\,\lambda_j f_j(v,C)} \longleftarrow \int d\lambda\, e^{i\lambda x} = 2\pi\delta(x)$$

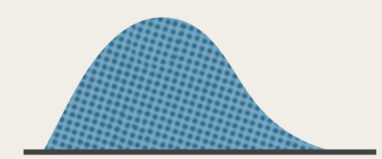
The determinant factor can be treated in two different ways:

- **Genuine distribution** Harder to compute

$$|\det M| = \lim_{\epsilon \to +0} \frac{\det(M^2 + \epsilon I)}{\sqrt{\det(M^2 + \epsilon I)}}$$

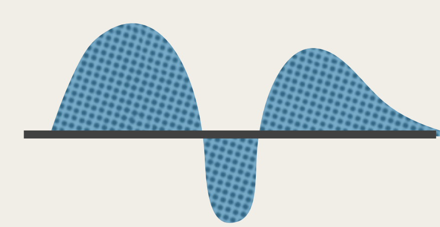
$\longrightarrow$ Fermions (numerator)

$\longrightarrow$ Bosons (denominator)



- **Signed distribution** Exactly computable, closed-forms sometimes

$$|\det M| \longrightarrow \det M = \int d\bar{\psi} d\psi\, e^{\bar{\psi} M \psi}$$: Fermions only

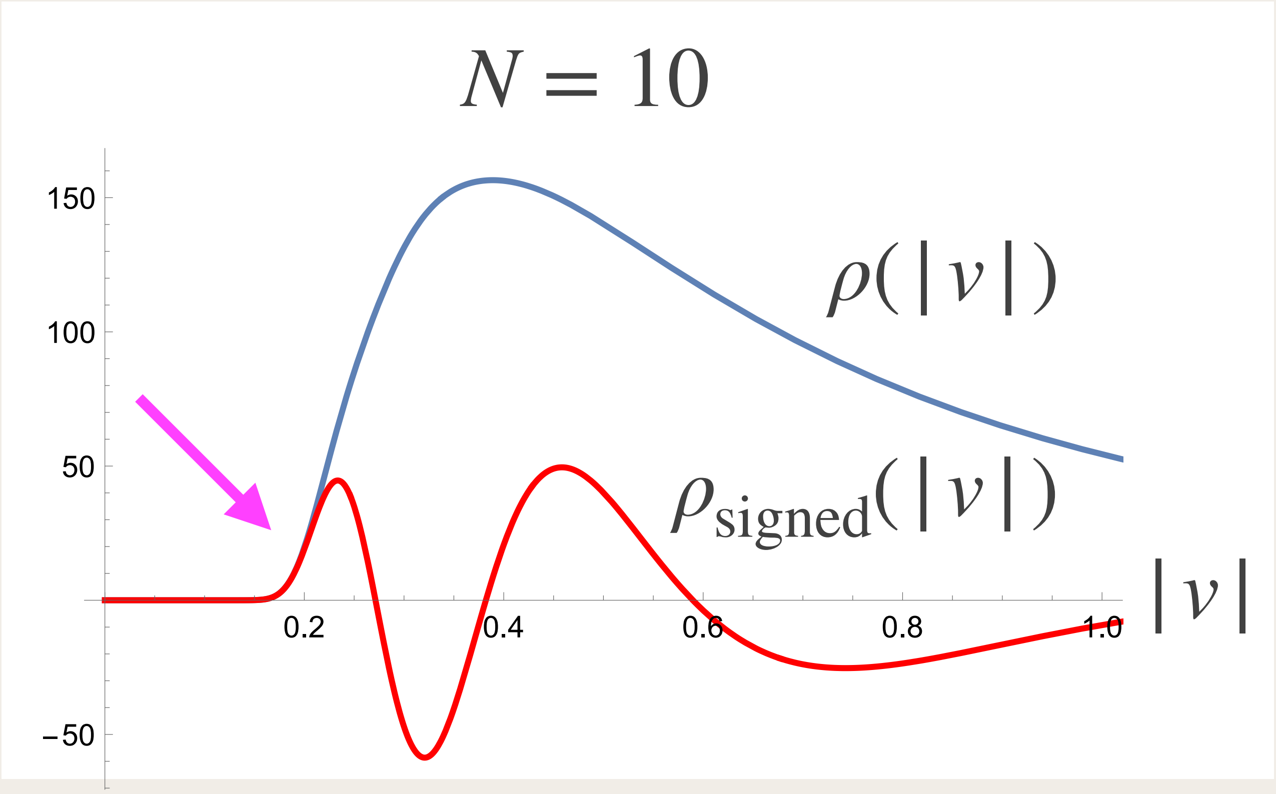


Ignoring the positivity

The location of the edge can be derived from the signed distribution

M.R. Kloos, NS, Lett.Math.Phys. 114 (2024) 3, 80, arXiv:2403.12427 [hep-th]

Ex. Real eigenvector distribution of real symmetric random tensor



The genuine and signed distributions are intimately related and have a common edge in the large N limit.

$$\rho(|v|) \sim e^{N\,\mathrm{Re}[h(|v|)]}$$

$$\rho_{\text{sigend}}(|v|) \sim \mathrm{Re}[e^{N h(|v|)}]$$

$h(|v|) < 0$ for $|v| < |v|_{\text{edge}}$

$h(|v|_{\text{edge}}) = 0$

$h(|v|) > 0$ for $|v|_{\text{edge}} < |v| \leq |v|_c$

$h(|v|)$: complex for $|v|_c < |v|$

$$\rho_{\text{signed}}(v) = \frac{1}{(2\pi)^{\# v} \mathcal{N}} \int dC d\lambda d\bar{\psi} d\psi \, e^{S}$$

$$S = -\alpha \, C^{*}_{abc} C_{abc} + i \, \lambda_j f_j(v, C) + \bar{\psi}_i \frac{\partial f_i(v, C)}{\partial v_j} \psi_j$$

Since $f_i(v, C)$ is linear in C_{abc}, integration over C, λ is a Gaussian integration and can explicitly be performed.

Then we see that the signed distribution is given by a partition function of a four-fermi theory:

$$\rho_{\text{signed}}(v) = \mathcal{N}' \int d\bar{\psi} d\psi \, e^{S_{ff}}$$

S_{ff} : A fermionic action with four-fermi interactions

The four-fermi actions

- **Symmetric indices case**

$$S_{ff} = \bar{\psi}\cdot\psi + \bar{\varphi}\cdot\varphi + \frac{2|v|^2}{3\alpha}\left(\bar{\psi}\cdot\varphi\,\bar{\varphi}\cdot\psi - \bar{\psi}\cdot\psi\,\bar{\varphi}\cdot\varphi\right) + \text{parallel to } v, v^*$$

- **Independent indices case**

$$S_{ff} = \sum_{i=1}^{3}\left(\bar{\psi}_i\cdot\psi_i + \bar{\varphi}_i\cdot\varphi_i\right) + \frac{|v|^2}{\alpha}\sum_{i<j}^{3}\left(\bar{\psi}_i\varphi_j + \bar{\psi}_j\varphi_i\right)\cdot\left(\bar{\varphi}_i\psi_j + \bar{\varphi}_j\psi_i\right)$$
$$+\text{parallel to } v, v^*$$

The partition function of these four-fermi theories can exactly be computed by using the following type of manipulations:

$$e^{g\,\bar{\psi}\cdot\psi\,\bar{\varphi}\cdot\varphi} = e^{g\frac{\partial}{\partial k_1}\frac{\partial}{\partial k_2}} e^{k_1\,\bar{\psi}\cdot\psi + k_2\,\bar{\varphi}\cdot\varphi}\Big|_{k_1=k_2=0}$$

Exact closed-form expressions are given in terms of generating functions.

- **Symmetric indices case**

$$\rho_{\text{signed}}(|v|^2) = -3^N \alpha^N |v|^{-2N-2} e^{-\frac{\alpha}{|v|^2}} (1+g\,l)^{-2} \exp\left(\frac{l}{1+g\,l}\right)\Bigg|_{l^{N-1}}$$

$$g = 2|v|^2/(3\alpha)$$

Taking the l^{N-1}-th order

- **Independent indices case**

$$\rho_{\text{signed}}(|v|^2) = -\alpha |v|^{-4} e^{-\frac{\alpha}{|v|^2}} (1-t_2+2t_3)^{-2} \exp\left(\frac{t_1-2t_2+3t_3}{g(1-t_2+2t_3)}\right)\Bigg|_{\prod_{i=1}^{3} l_i^{N_i-1}}$$

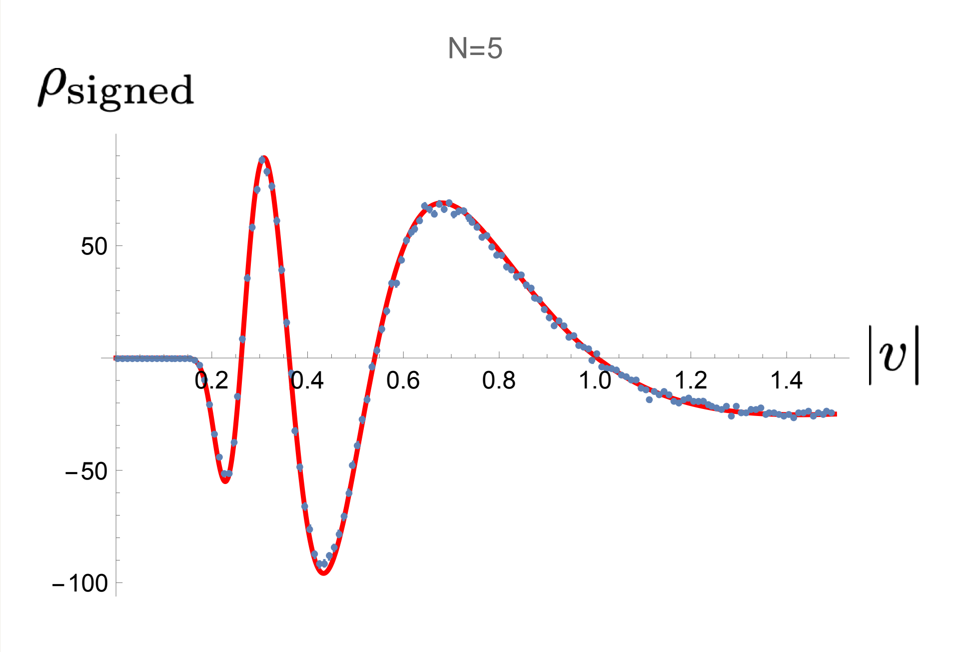
$$g = |v|^2/\alpha$$

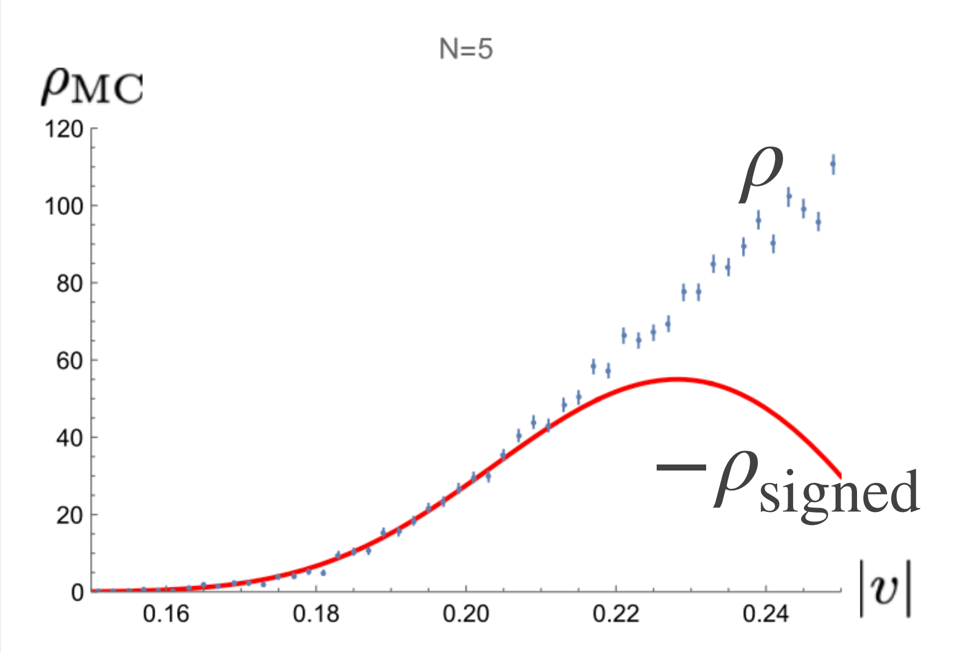
N_i : dimension of i-th index

$$t_1 = l_1 + l_2 + l_3$$
$$t_2 = l_1 l_2 + l_2 l_3 + l_3 l_2$$
$$t_3 = l_1 l_2 l_3$$

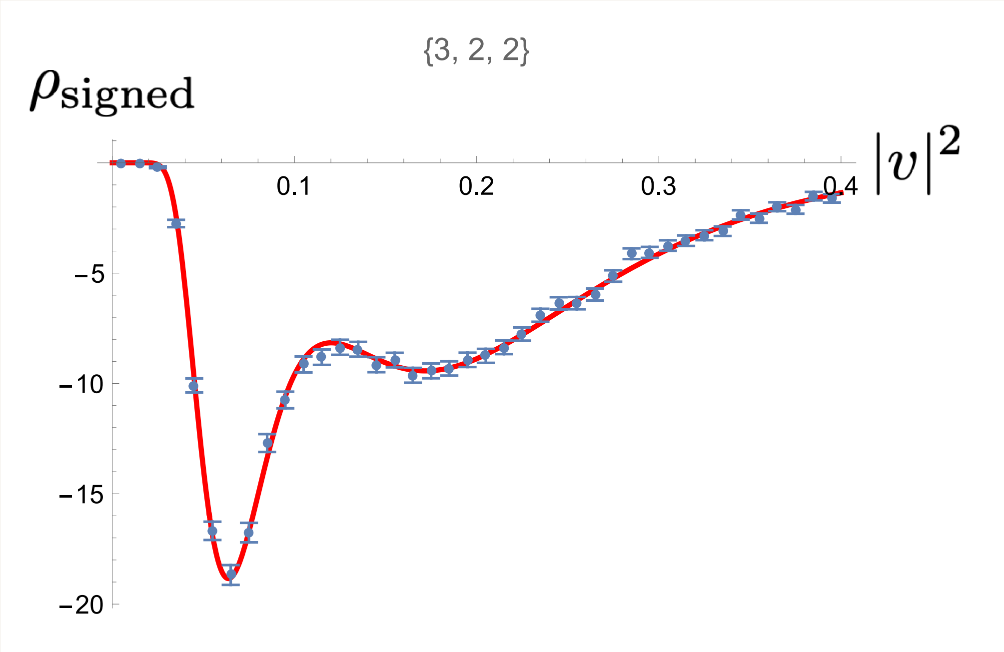
§ Checked with Monte Carlo simulations

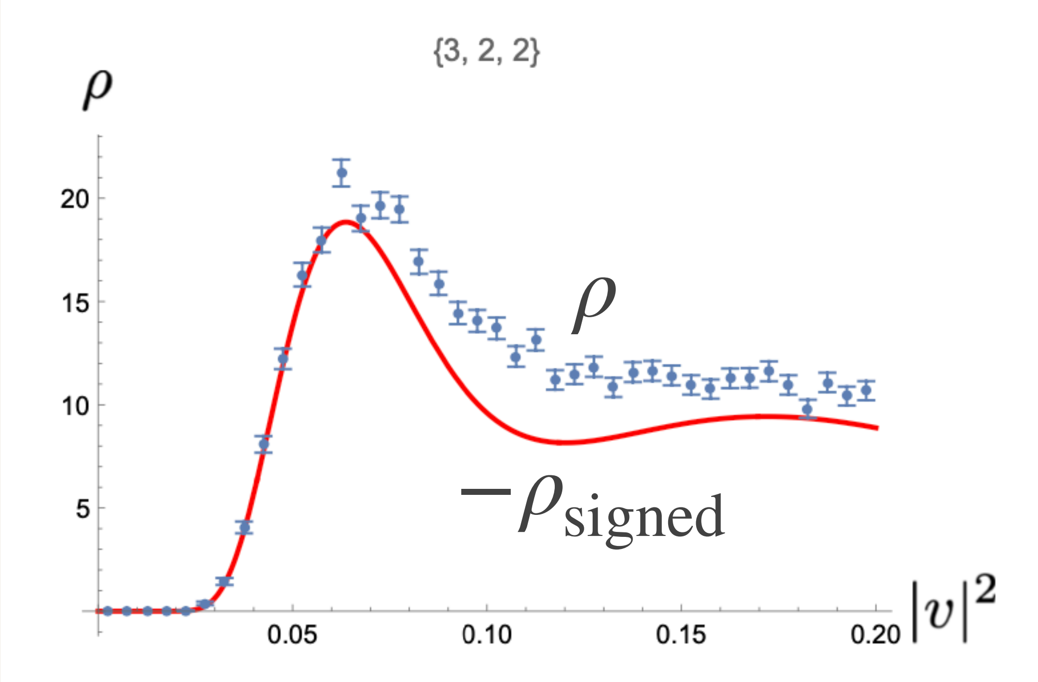
Symmetric indices case $N = 5$





Independent indices case $(N_1, N_2, N_3) = (3,2,2)$





The asymptotic form in the large N limit can be extracted from the exact closed-form expression.

$$\rho_{\mathrm{signed}}(|v|^2) \sim \mathrm{Re}[e^{N h(|v|)}]$$

Symmetric indices case

$$h(|v|) = \log(2) - \frac{\alpha}{|\tilde{v}|^2} - \log \tilde{l} + \frac{\tilde{l}}{\tilde{g}(1+\tilde{l})} \qquad |\tilde{v}| = \sqrt{N}|v|$$

$$\tilde{l} = \frac{1 - 2\tilde{g} - \sqrt{1 - 4\tilde{g}}}{2\tilde{g}} \qquad \tilde{g} = \frac{2|\tilde{v}|^2}{3\alpha}$$

The edge is numerically determined by $h(|v|_{\mathrm{edge}}) = 0$

$$|v|_{\mathrm{edge}} = 0.603501\sqrt{\frac{\alpha}{N}} \qquad \left(|v|_c = \sqrt{\frac{3\alpha}{8N}} \right)$$

Independent indices case

$$h = -1 - \log \tilde{g} - \frac{1}{\tilde{g}} - \sum_{i=1}^{3} n_i \log n_i + s_{\text{eff}}$$

$$n_i = N_i/(N_1 + N_2 + N_3) \qquad \tilde{g} = (N_1 + N_2 + N_3)\,|\,v\,|^2/\alpha$$

$$s_{\text{eff}} = 2\sum_{i=1}^{3} n_i Q_i - \tilde{g}\sum_{i\neq j}^{3} n_i n_j Q_i Q_j - 2\sum_{i=1}^{3} n_i \log Q_i$$

$$Q_i = \frac{\sqrt{1 + 4q^2 n_i \tilde{g}} - 1}{2q n_i \tilde{g}}$$

where q is the solution to $1 + 2q - \sum_{i=1}^{3}\sqrt{1 + 4q^2 n_i \tilde{g}} = 0$

For $N_i = N$

$$|\,v\,|_{\text{edge}} = 0.348431\sqrt{\frac{\alpha}{N}} \qquad \left(|\,v\,|_c = \sqrt{\frac{\alpha}{8N}}\right)$$

§ Agreement with a pervious numerical study

K. Fitter, C. Lancien, I. Nechita, “Estimating the entanglement of random multipartite quantum states,” [arXiv:2209.11754 [quant-ph]]

Symmetric indices case $\left(|C|_{\text{inj}} = \max_{|w|=1} C_{abc} w_a w_b w_c\right)$

$$|C|_{\text{inj}} = 1/|v|_{\text{edge}} = 2.34335 \quad (\alpha = N/2)$$

FLN result = 2.356248 Error~0.5%

Independent indices case $\left(|C|_{\text{inj}} = \max_{|w^i|=1} C_{abc} w_a^1 w_b^2 w_c^3\right)$

$$|C|_{\text{inj}} = 1/|v|_{\text{edge}} = 4.0588 \quad (\alpha = N/2)$$

FLN result = 4.143529 Error~2%

The numbers are coincident, since the errors are smaller than 4%, which is of the established case (real case).

Summary

As in matrix models, tensor eigenvalue/vector distributions will become important in various applications.

The quantum field theoretical method is a powerful practical method of computing them.

In particular signed distributions are the easiest but useful, and can be computed by four-fermi theory.

We have computed the signed eigenvalue/vector distributions of complex random tensors, and have derived the asymptote of the geometric measure of quantum entanglement analytically for the first time. (cf. Dartois, McKenna, arXiv:2404.03627)

Future prospects

Studying tensor eigenvalue/vector distributions is rather a new subject. We expect more results to come in the near future.

Thank you !

Σας ευχαριστώ

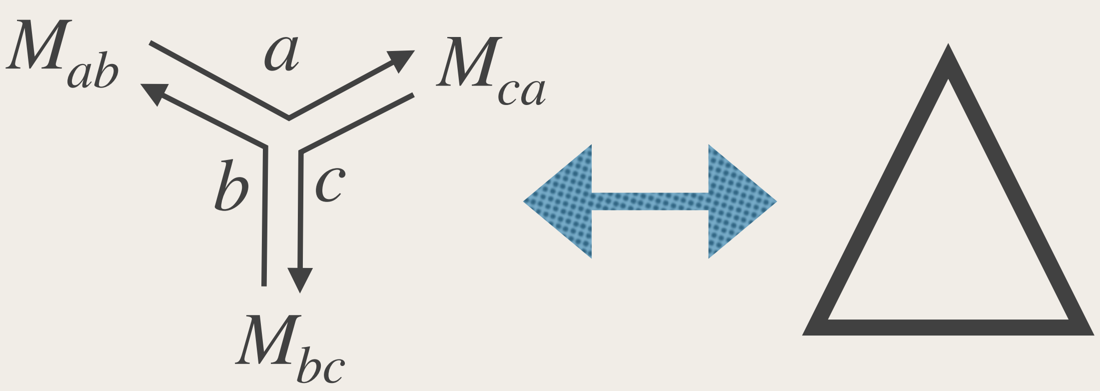
Random tensor models

- **Discretized model of quantum gravity of dim ≥ 3**

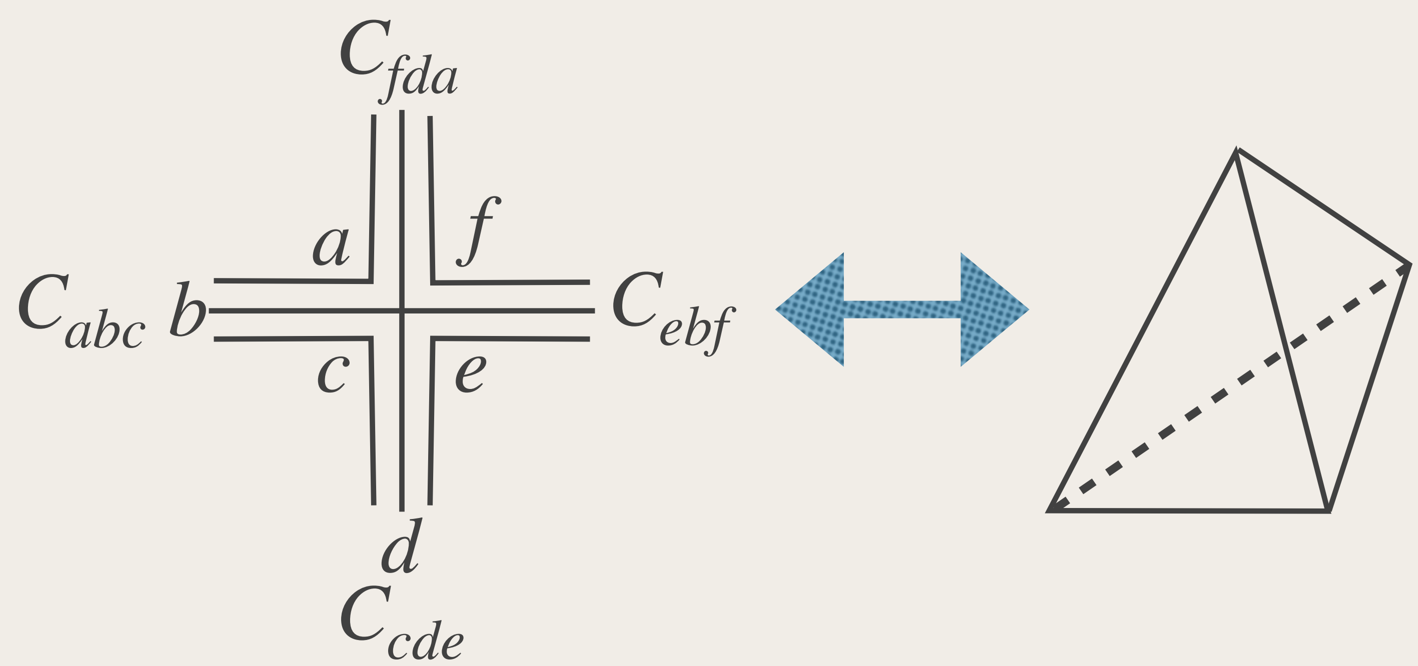
  Ambjorn-Durhuus-Jonsson, NS, Godfrey-Gross 1990

  Extension of matrix models for discretized 2-dim QG

2-dim



3-dim



- **Colored tensor model** $1/N$ expansion Gurau 2011

There are recent applications to some new subjects:

- **Glasses** Spherical p-spin model of spin glass
- **AdS/CFT correspondence** Gurau-Witten model
- **Data analysis**

$$D_{abc} = C^0_{abc} + C_{abc}$$

Target signal: constant tensor (C^0_{abc})

Noise: Random tensor (C_{abc})

$$C_{abc} w_b w_c = \zeta w_a$$: Tensor eigen problem

$$C_{abc} = \sum_{r=1}^{R} \phi^r_a \phi^r_b \phi^r_c$$: Tensor rank decomposition

- **Quantum information theory**

$$|\Psi\rangle = C_{abc} |a\rangle_1 |b\rangle_2 |c\rangle_3$$ Random multipartite states

Most tensor problems are NP-hard for a tensor. Hillar-Lim 2009

On the other hand, the distribution of a quantity (like tensor eigenvalues) for an ensemble of tensors can exactly/approximately computed, as we will do.

Thermodynamic limit is expected in the large-N limit, where the distribution of a quantity (like tensor eigenvalues) does not depend on a tensor in the ensemble.

(Rigorously proven for a particular case. Subag 2017)

So, random tensor models provide an interesting angle to more easily approach these NP-hard problems