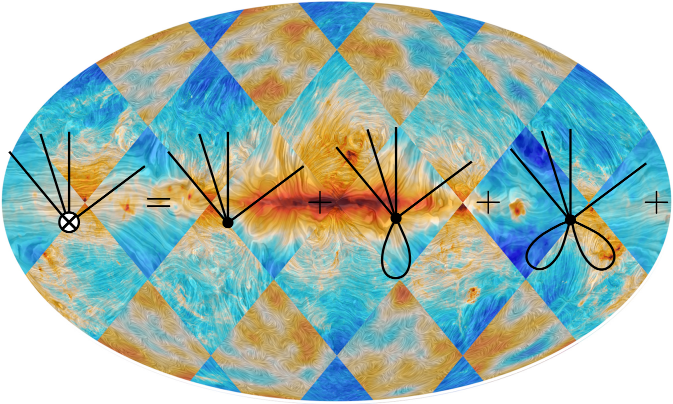
Stochastic dynamics from QFT

Spyros Sypsas
Chulalongkorn U., Thailand

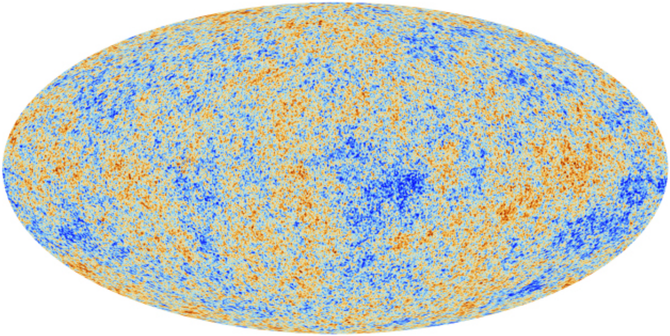


Based on

2309.16474, 2311.17644 with Gonzalo Palma
2406.07610 with Gonzalo Palma, Javier Huenupi and Ellie Hughes

A physical system

The night sky through a strong telescope



Planck Legacy Archive

The night sky through a strong telescope

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In [9]:  pyfits.open('COM_CMB_IQU-smica_1024_R2.02_full.fits').info()

         Filename: COM_CMB_IQU-smica_1024_R2.02_full.fits
         No.    Name         Type      Cards   Dimensions     Format
         0    PRIMARY     PrimaryHDU      16   ()
         1    COMP-MAP    BinTableHDU     61   12582912R x 5C   [E, E, E, B, B]
         2    BEAMTF      BinTableHDU     45   4001R x 2C    [E, E]

In [183]: CMBdata2.info()

         Filename: COM_CMB_IQU-smica-field-Int_2048_R2.01_full.fits
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           1  COMP-MAP      1 BinTableHDU     52   50331648R x 2C   ['E', 'B']
           2  BEAMTF        1 BinTableHDU     41   4001R x 1C    [E]

In [10]: CMBdata[1].data

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                   (-8.9856963e-05,  7.3611503e-08, -6.8458655e-07, 0, 0), ...,
                   (-3.4241329e-04, -1.4987631e-06, -6.5261207e-07, 0, 0),
                   (-3.3521044e-04, -1.5905293e-06, -6.4830010e-07, 0, 0),
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                  dtype=(numpy.record, [('I_STOKES', '>f4'), ('Q_STOKES', '>f4'), ('U_STOKES', '>f4'), ('TMASK', 'u1'),
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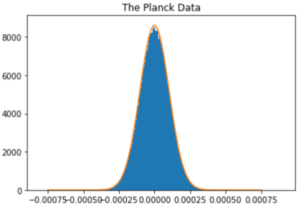
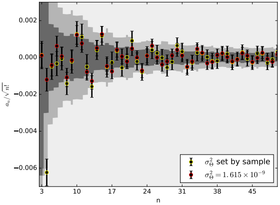


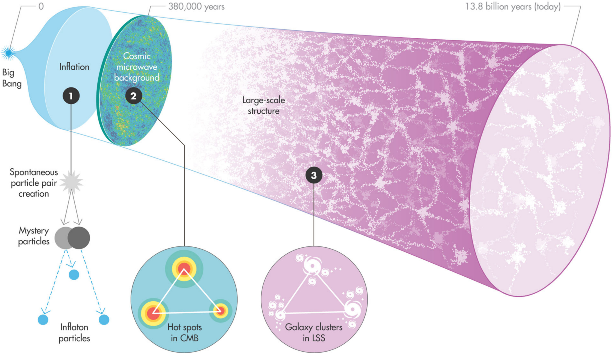


$$\mathrm{P}(\delta T) \sim e^{S[\varphi]}$$

Quadratic action (linear dynamics) $\Leftrightarrow$ Gaussian statistics

Interactions $\Leftrightarrow$ non-Gaussian deviations

CMB temperature fluctuations reflect density fluctuations and small/large scale structure:



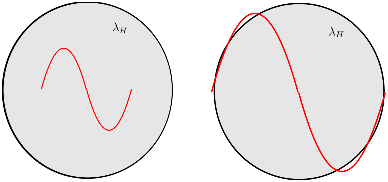
$$P(\varphi) \to P(\delta T) \to P(\delta_g)\ ,\ P(\delta_{pbh})$$

The problem

Probe scalar in the Poincare patch of (rigid) de Sitter space ($H =$ const):

$$\varphi,\, V(\varphi) \quad \Big| \quad ds^2 = a^2(\tau)(-dt^2 + dx^2),\;\; a = (H\tau)^{-1}.$$

Wavelengths stretch $\lambda = \ell a$, wavevectors shrink $p = k/a$.



- ⋆ Q: What is the statistics (correlators/PDF) of long modes? (random process in an open system)
- ⋆ Methods: QFT (Feynman graphs) = Stochastic (Langevin eq)

$\mathcal{V}(\varphi)$ on dS

$$S = \int \mathrm{d}^3 x \, \mathrm{d}\tau \, a^2 \left[\frac{1}{2}\dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{2} - a^2 \mathcal{V}(\varphi) \right]$$

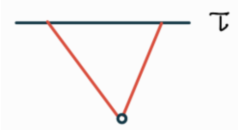
$$\mathcal{V}(\varphi) = \sum_{n=2}^{\infty} \frac{\lambda_n^{\mathrm{bare}}}{n!} \varphi^n$$

We will be interested in single-point statistics (i.e. histograms marginalised over $\boldsymbol{x}$)

$$\langle \varphi^n \rangle = \langle \varphi(\boldsymbol{x}_1) \cdots \varphi(\boldsymbol{x}_n) \rangle \Big|_{\boldsymbol{x}_1 = \cdots = \boldsymbol{x}_n} \propto \lambda_n^{\mathrm{obs}}$$

UV/IR structure of correlators

The simplest correlator is the (free theory) one-point variance:



$$\sigma^2 = H^2 \int_0^\infty \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{1}{k^3} \left(1 + (k\tau)^2\right)$$

or, in physical momentum $p \equiv Hk\tau$

$$\sigma^2 = \left(\frac{H}{2\pi}\right)^2 \int_0^\infty \frac{\mathrm{d}p}{p} \left(1 + \frac{p^2}{H^2}\right)$$

UV divergence: $\propto p^2$

IR divergence: $\propto \ln p$

Cure for external legs: restrict to long modes

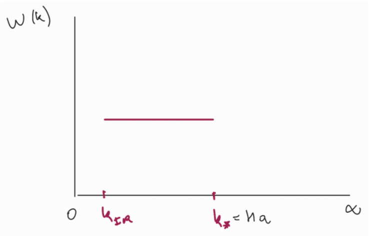
Long-mode statistics

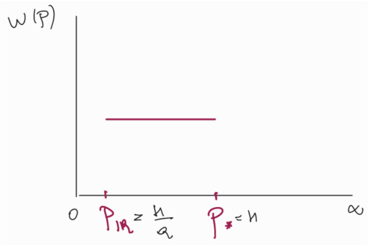
Long-short split: $\varphi = \varphi_S + \varphi_L$:

$$\varphi_L(x,t) \equiv \hat{\boldsymbol{L}}\{\varphi(x,t)\} = \int_k e^{ikx} W(k)\tilde{\varphi}_k(t)$$

$$W(k) = \theta(k_*(t) - k) \times \theta(k - k_*(t_{\rm i})) \,||\, W(p) = \theta(H - p) \times \theta(p - k_*(t_{\rm i})/a)$$

Cut off external legs at $k_*(t_{\rm i}) \equiv k_{\rm IR}$. This is our choice!





Secular structure of correlators

Now

$$\sigma_L^2(t) = \left(\frac{H}{2\pi}\right)^2 \int_{H/a}^{H} \frac{\mathrm{d}p}{p} = \frac{H^3 t}{4\pi^2}$$

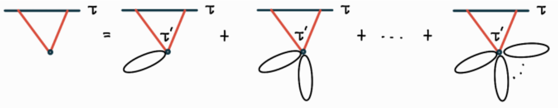
We regularized but we now have a secular behaviour.

Linear modes independent $\rightarrow$ variance additive $\rightarrow$ blows up asymptotically (secular).

Cure: resummation (e.g. stochastic formalism)

Daisy loops

Do loops change the secular growth?



$$\mathcal{V}(\varphi) = \lambda_4^{\mathrm{bare}}\varphi^4 + \lambda_6^{\mathrm{bare}}\varphi^6 + \cdots\,, \quad (\text{e.g. axion } \mathcal{V} \propto 1 - \cos\varphi/f)$$

Recall that loops are virtual processes; they control the validity.

Cure for internal legs: UV & IR cutoffs or dim reg + renormalization

Now there is a choice to be made! Comoving vs physical IR cutoff.

Is t_{i} part of the theory?

Let us choose a physical cutoff. Then

$$\sigma_0^2 = \left(\frac{H}{2\pi}\right)^2 \int_{\Lambda_{\rm IR}}^{\Lambda_{\rm UV}} \frac{\mathrm{d}p}{p} = \left(\frac{H}{2\pi}\right)^2 \ln \frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}$$

SK, *in-in*, *WFU*: n-pt functions loop-resummed to:

$$\left\langle \varphi^n(\tau) \right\rangle_c = -\frac{8\pi}{H^4} \,\mathrm{Im}\left\{ \int_0^\infty \mathrm{d}x \int_{\tau_{\rm i}}^{\tau} \mathrm{d}\bar{\tau}\, \frac{x^2}{\bar{\tau}^4} \lambda_n^{\rm obs} \left[g(x, \bar{\tau}, \tau) \right]^n \right\}$$

with g the propagator and

$$\lambda_n^{\rm obs} = \sum_{L=0} \frac{\lambda_{n+2L}^{\rm bare}}{L!} \left(\frac{\sigma_0^2}{2}\right)^L$$

(This is nothing but the coefficients of a Weierstrass transform)

The PDF of long modes

Edgeworth expansion of the PDF truncated to linear order in the interactions (single-vertex diagrams):

$$\rho(\varphi, t) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}}\left[1 + \Delta(\varphi, t)\right]$$

with

$$\Delta(\varphi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left\langle \varphi^n(t) \right\rangle_c}{\sigma^n} \mathrm{He}_n(\varphi/\sigma)$$

Perturbative QFT = Gaussian variables =

Feynman rules = Wick's theorem

Upon using

$$\mathcal{O}_\varphi \mathrm{He}_n(\varphi/\sigma) = -n\mathrm{He}_n(\varphi/\sigma)$$

the PDF can be resummed:

$$\Delta(\varphi,\tau) = \frac{8\pi}{H^4}\mathrm{Im}\int_0^\infty \mathrm{d}x \int_{\tau_\mathrm{i}}^{\tau} \mathrm{d}\bar{\tau}\, \frac{x^2}{\bar{\tau}^4}\left(\frac{g(x,\bar{\tau},\tau)}{\sigma^2}\right)^{-\mathcal{O}_\varphi} \underbrace{e^{-\frac{\sigma_0^2}{2}\frac{\partial^2}{\partial\varphi^2}}\mathcal{V}(\varphi)}_{\mathcal{V}_\mathrm{ren}(\varphi)}$$

such that $\rho = \rho_0\left[1+\Delta\right]$

This is the PDF of long modes

Emergence of stochastic dynamics

$k\tau \to 0$ limit:

$$\Delta(\varphi, t) = \frac{H t}{3H^2}\left(\mathcal{V}''_{\rm ren}(\varphi) - \frac{\varphi}{\sigma^2}\mathcal{V}'_{\rm ren}(\varphi)\right)$$

and we then take a time derivative of the PDF:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\left(1 - \frac{2Ht}{3H^2}\mathcal{V}''_{\rm ren}\right)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)$$

Fokker-Planck equation

Performing the same computation with a comoving IR cutoff:

$$\left\langle \varphi^n(t) \right\rangle_c = -\frac{4\pi^2 n}{3H^4}\sigma^{2n}(t)\sum_L^{\infty} \frac{\lambda_{n+2L}}{(n+L)L!}\left(\frac{\sigma^2(t)}{2}\right)^L$$

leads to a PDF that satisfies the autonomous Fokker-Planck eq:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2 \rho}{\partial \varphi^2} + \frac{1}{3H}\frac{\partial}{\partial \varphi}\left(\rho \mathcal{V}'_{\rm ren}\right) \quad || \quad \left[\rho_\infty \propto e^{-\mathcal{V}/H^4}\right]$$

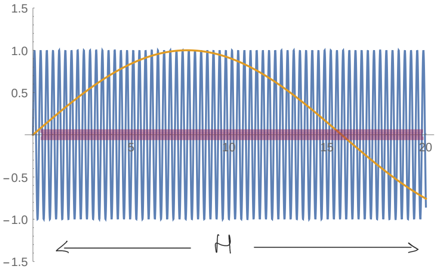
Starobinsky '86; Tsamis & Woodard '05; +++

Physical cutoff: loops renormalise the vertex

Comoving cutoff: Loops enhance secular growth

The stochastic approach: Starobinski-Yokoyama 90's

Long modes: nonlin but classical
Short modes: lin but QM



How to model the effect of short modes at large scales?

$$\hat{\varphi}_k = \varphi_{\rm lin} \hat{a}_k + \varphi^*_{\rm lin} \hat{a}^\dagger_k \rightarrow \hat{\xi}_k = \varphi_{\rm lin}(\hat{a}_k - \hat{a}^\dagger_k)$$

The EOM for long modes now becomes a stochastic Langevin equation: for $\Delta t \gg 1/H$,

$$\dot{\varphi}_L = H\hat{\xi}(t) - \frac{1}{3H}\mathcal{V}'_{\rm ren}(\varphi_L)$$

where $\hat{\xi}(t)$ is a Gaussian stochastic noise representing the short-mode bath:

$$\hat{\xi} \equiv H^{-1} \int_k \dot{W}(k)\, \tilde{\varphi}_k \,, \quad \left\langle \hat{\xi}(t_1)\hat{\xi}(t_2) \right\rangle \propto \delta(t_1 - t_2)$$

From this one can straightforwardly get the autonomous FP equation: integrate Langevin

$$\boxed{\varphi_L(t) = \varphi_G(t) - \tfrac{1}{3H} \int \mathrm{d}t' \, \mathcal{V}'_{\rm ren} \left[\varphi_G(t')\right]}$$

compute

$$\langle \varphi_L^n(t) \rangle_c$$

plug in Edgeworth and derive:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2 \rho}{\partial \varphi^2} + \frac{1}{3H}\frac{\partial}{\partial \varphi}\left(\rho \mathcal{V}'_{\rm ren}\right)$$

Langevin from *in-in*

$\varphi(x,t) = U(t)\varphi_I(x,t)U^\dagger(t)$, where $\varphi_I(x,t)$ is the interaction-picture field and $U = \exp\left\{-i\int \mathrm{d}t' \int \mathrm{d}^3x'\, \mathcal{V}_{\rm ren}(\varphi_I)\right\}$.

To first order in the potential:

$$\varphi(t) \simeq \varphi_I(t) - \frac{1}{3H}\int \mathrm{d}t'\, \mathcal{V}'_{\rm ren}\left[\varphi_I(t')\right]$$

Now apply $\hat{\boldsymbol{L}}$:

$$\boxed{\varphi_L(t) \simeq \varphi_G(t) - \tfrac{1}{3H}\int \mathrm{d}t'\, \hat{\boldsymbol{L}}\left\{\mathcal{V}'_{\rm ren}\left[\varphi_I(t')\right]\right\}}$$

Computing

$$\langle \varphi_L^n(t)\rangle_c\,,$$

and following the same steps now leads to

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\left(1 - \frac{2Ht}{3H^2}\mathcal{V}''_{\rm ren}\right)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)$$

The choice that was made is

$$\hat{\boldsymbol{L}} \left\{ \mathcal{V}'_{\rm ren} \left[\varphi_I(t') \right] \right\} = \mathcal{V}'_{\rm ren}(\hat{\boldsymbol{L}} \varphi_I)$$

When does this hold? When we negelct short-long mode correlations. (equivalent to comoving vs physical loop cutoffs)

To sum-up

- ⋆ Statistics of long scalar modes on dS
- ⋆ resummation of Feynman diagrams = stochastic formalism
- ⋆ Important details :

  $\mathcal{V} \to \mathcal{V}_{\rm ren}$

  Physical vs comoving reg loops: cumulative diffusion

  Need for resummation in $\mathcal{V}$ to reach equilibrium

Thanks!

Edgeworth expansion

$$\rho(\varphi) = \int \mathrm{d}J\, z(J) e^{-iJ\varphi}$$

Using $z = e^{w}$, and $w(J) = -\frac{1}{2}\sigma^2 J^2 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \langle \varphi^n \rangle_c J^n$, we obtain

$$\rho(\varphi) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2}}}{\sqrt{2\pi}\sigma} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{\langle \varphi^{n_1} \rangle_c}{n_1! \sigma^{n_1}} \cdots \frac{\langle \varphi^{n_N} \rangle_c}{n_N! \sigma^{n_N}} \mathrm{He}_{n_1+\cdots+n_N}(\varphi/\sigma)$$

The renormalized potential

Separate the IR modes

$$\varphi = \varphi_S + \varphi_L + \varphi_{\rm IR}$$

and integrate them out:

$$\mathcal{V}(\bar\varphi) = \langle \Psi_{\rm IR} | \mathcal{V}(\bar\varphi + \varphi_{\rm IR}) | \Psi_{\rm IR} \rangle$$

with

$$|\Psi_{\rm IR}\rangle = \int \mathcal{D}\varphi_{\rm IR} \Psi(\varphi_{\rm IR}) |\varphi_{\rm IR}\rangle$$

with $|\varphi_{\rm IR}\rangle$ an IR field-eigenstate and $|\Psi(\varphi_{\rm IR})|^2$ the Gaussian. This leads directly to

$$\boxed{\mathcal{V}(\bar\varphi) = \mathcal{V}_{\rm ren}(\bar\varphi)}$$

On the IR divergences in de Sitter space: loops, resummation and the semi-classical wavefunction

Sebastián Céspedes (Imperial Coll., London), Anne-Christine Davis (Cambridge U., DAMTP), Dong-Gang Wang (Cambridge U., DAMTP)
Nov 29, 2023

59 pages
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Anna Negro (Leiden U.), Subodh P. Patil (Leiden U.)
Feb 15, 2024

25 pages
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Breakdown of Semiclassical Methods in de Sitter Space

C.P. Burgess (McMaster U. and Perimeter Inst. Theor. Phys.), R. Holman (Carnegie Mellon U.), L. Leblond (Perimeter Inst. Theor. Phys.), S. Shandera (Perimeter Inst. Theor. Phys.)
May, 2010

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$\lambda\phi^4$ in dS

Victor Gorbenko (Princeton, Inst. Advanced Study and Stanford U., ITP), Leonardo Senatore (Stanford U., ITP and KIPAC, Menlo Park and SLAC)
Oct 31, 2019

95 pages
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Abstract: (arXiv)
We resolve the issue of infrared divergences present in theories of light scalar fields on de Sitter space.