Counting states and negative specific heat capacities in matrix and tensor quantum mechanics

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Based on : Denjoe O' Connor and Sanjaye Ramgoolam, "Gauged permutation invariant matrix quantum mechanics :path integrals" e-Print: 2312.12397 [hep-th], JHEP 04(2024)
Denjoe O' Connor and Sanjaye Ramgoolam, "Gauged permutation invariant matrix quantum mechanics :partition functions. " arXiv:2312.12398 [hep-th] ;
Denjoe O' Connor and Sanjaye Ramgoolam, "Gauged permutation invariant matrix quantum thermodynamics and negative specific heat capacities in large N systems" arXiv:2405.13150v1 [hep-th]

Introduction

- Matrix quantum mechanics models have interesting thermodynamics which have been interpreted in the context of gauge-string duality, e.g. the AdS/CFT correspondence.
- We find that the counting of polynomial functions of matrices or tensors invariant under appropriate symmetry groups for appropriate choices of matrix/tensor content and invariance group lead to negative specific heat capacities. Some of these examples are of interest in the context of small black holes in AdS/CFT.

AdS/CFT conjecture and Matrix combinatorics

$\mathcal{N} = 4$ SYM in 4D with $U(N)$ gauge group.

String theory in 10 dimensions on $AdS_5 \times S^5$ background.

The SYM theory contains six hermitian matrices $X_1, X_2, \cdots, X_6$. Transform as a vector of $so(6) \subset psu(2,2|4)$ super-algebra.

The $so(6)$ is the isometry group of the S^5 in the space-time.

The counting and correlators of gauge-invariant functions of multi-matrices is important in the study of the emergence of ten dimensions.

Half-BPS states - Distinguished by supersymmetry

Maximally supersymmetric (half-BPS) operators are holomorphic gauge invariant functions of a complex matrix $Z = (X + iY)$.

Half-BPS representations contain single graviton and multi-graviton states.

CFT gives an inner product on the space of states. The free field inner product is exact because of the BPS condition. As a result this simple inner product contains a lot of physical information about the duality.

BPS states $\rightarrow$ Complex matrix quantum mechanics

The inner product is the same as in the quantum mechanics of a complex matrix harmonic oscillator.

Very simple action

$$S = \int dt \sum_{i,j} (\partial_t Z^i_j)(\partial_t (Z^i_j)^*) + Z^i_j (Z^i_j)^*$$

Two sets of N^2 matrix creation operators $(A^\dagger)^i_j, (B^\dagger)^i_j$

BPS states = holomorphic = traces of one matrix oscillator

The $U(N)$ invariant states are traces, e.g. at Energy $k = 3$, assuming N large :

$$\begin{aligned} \mathrm{tr}(A^\dagger)^3|0\rangle \\ \mathrm{tr}(A^\dagger)^2\mathrm{tr}(A^\dagger)|0\rangle \\ \mathrm{tr}A^\dagger)^3|0\rangle \end{aligned}$$

are the linearly independent states.

Number of invariants states at $k = 3$ is the same as the partitions of 3.

$$\begin{aligned} 3 = 3 \\ 3 = 2 + 1 \\ 3 = 1 + 1 + 1 \end{aligned}$$

If $N = 1$ all the states are the same : $(A^\dagger)^3|0\rangle$

Counting, stable region, finite N exclusions

$Z(N,k) =$ number of young diagrams with k boxes and no more than N rows

$$Z(N,k) = p(k) \quad \text{for} \quad k \leq N$$
$$Z(N,k) < p(k) \quad \text{for} \quad k > N$$

Stable range - counting independent of N :

$$Z_{\rm stab}(k) = p(k)$$

$$Z_{\rm stab}(k) = p(k) \sim e^{\sqrt{k}}$$

Asymptotics of partition numbers – Hardy, Ramanujan ...
Sub-exponential ...

Stable limits generic

This is a very general fact about counting the dimensions of state spaces of one-matrix, multi-matrix, tensor $U(N)$ invariants.

There is a stable counting for $k < N$ given in terms of Young diagram data ; And finite N effects come from simple cut-offs on number of rows of Young diagrams (more on the multi-matrix and tensor cases later ...)

The different kinds of large k asymptotics in the stable range, arising from choices of matrix/tensor content and choice of symmetry, lead to different phase structures for the corresponding quantum thermodynamics.

Free 2-matrix models

Invariants of two-matrix quantum mechanics - include $A^\dagger$ and $B^\dagger$ – Now we have different orderings of the two matrices within a trace

$$\mathrm{tr}(A^\dagger)^2(B^\dagger)^2$$
$$\mathrm{tr}(A^\dagger B^\dagger A^\dagger B^\dagger)$$

The stable limit has a generating function, giving the number of operators with k_1 copies of $A^\dagger$ and k_2 copies of $B^\dagger$

$$\prod_{i=1}^{\infty} \frac{1}{(1 - x_1^i - x_2^i)} = \sum_{k_1,k_2} Z(k_1,k_2) x_1^{k_1} x_2^{k_2}$$

$$Z(k) = \sum_{k_1=0}^{k} Z(k_1, k-k_1) \sim 2^k \quad \text{for} \quad k \to \infty$$

This means that the canonical partition function, in the $N = \infty$ limit

$$\sum_{k=0} Z(k) e^{-\beta k} = \sum_{k=0}^{\infty} e^{k \log 2 - \beta k}$$

diverges at $\beta < \log 2$, i.e. $T > \frac{1}{\log 2}$

O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, M. Van Raamsdonk, “The Hagedorn/Deconfinement Phase Transition in Weakly Coupled Large N Gauge Theories” 2003

One matrix model - Wigner + subsequent :
nuclei, condensed matter physics, finance

Beyond AdS/CFT and gauge-string duality, matrix models have a life in the statistics of real world matrix data.

$$\int dM e^{-trM^2}$$

Probability distribution over matrices,e.g. above, which can be compared with real world large matrices

Energy levels of nuclei in original works of Dyson, Wigner

Over the years, a very wide range of applications : condensed matter physics, statistical finance etc.

Matrix data applications motivate : $U(N) \to S_N$

Instead of Gaussian models with simple measures $e^{\mathrm{tr} M^2}$ which are invariant under $U(N)$ or $O(N)$ for real matrices, we can consider more general actions invariant under a reduced symmetry of S_N.

2 invariants at linear order :

$$\sum_{i=1}^{N} M_{ii}, \sum_{i,j=1}^{N} M_{ij}$$

At any degree k, there is a graph-theory interpretation - directed graphs with k edges and any number of nodes. Quadratic order : 11 invariants.

Quadratic invariants

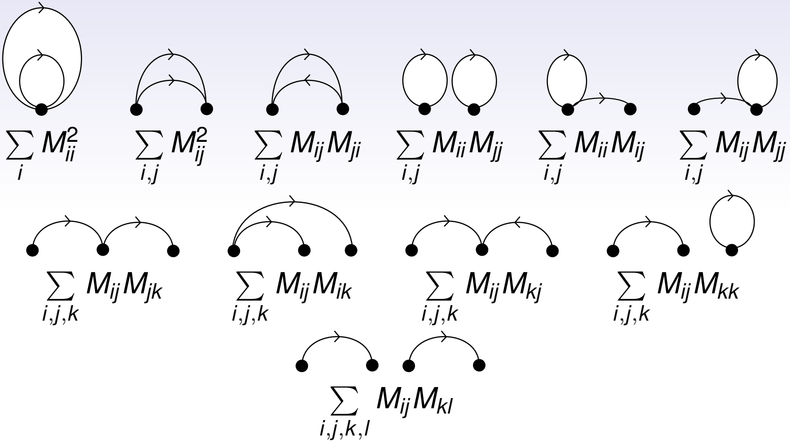


Figure: S_D invariant functions and corresponding graphs illustrated for the 11 quadratic invariants

S_N invariant matrix models : useful tool for matrix data analysis

The general Gaussian models have been developed and used to characterise the Gaussianity and small departures from Gaussianity in matrix data in computational linguistics and finance ...

Kartsaklis, Ramgoolam, Sadrzadeh, “Linguistic matrix theory,” 2017 ;
Ramgoolam “Permutation invariant Gaussian matrix models,” 2018
Ramgoolam, Sadrzadeh, Sword, “Gaussianity and typicality in matrix distributional semantics,” 2019

Barnes, Ramgoolam, Stephanou, “Permutation invariant Gaussian matrix models for financial correlation

matrices,”Physica A, 2024

Rapid growth with degree k of S_N invariant matrix polynomials

Useful point we will use :

$$1, 2, 11, 52, 296, ...$$

grows a lot faster than

$$1, 2, 3, 5, 7, ...$$

Number of directed graphs (S_N invariant polynomial functions of a matrix) grow a lot faster than partitions ($U(N)$ invariant poly fns of a matrix) ....

Perm Invt matrix quantum mechanics model - Path Integral

$$S[X] = \int_0^\beta d\tau \, \mathbf{tr} \left(\frac{1}{2} (\mathcal{D}_\tau X)^2 + \frac{1}{2} m^2 X^2 \right)$$

$$\mathcal{D}_\tau X \to \frac{g_{n,n+1} X_{n+1} g_{n+1,n} - X_n}{a} \quad = \frac{\mathrm{e}^{a\mathcal{D}_\tau} - 1}{a^2} X_n$$

Discretize the Euclidean time direction. Group-valued parallel transport operators.

Λ lattice sites. In the exponent, variables X_n and a quadratic form described by a matrix of size $\Lambda \times \Lambda$.

D. O' Connor, S. Ramgoolam "Gauged Permutation invariant matrix quantum mechanics : path integrals, e-Print: 2312.12397 [hep-th], JHEP 04(2024)

Continuum result : Molien-Weyl determinant formula for counting invariants :

$$Z_{N,\infty} = \int \mu(g) \frac{\mathrm{e}^{-\frac{N^2 \beta m}{2}}}{\mathbf{det}[\mathbf{1} - \mathrm{e}^{-\beta m} g \otimes g^{-1}]} \, .$$

Outline

Thermodynamic partition functions for S_N gaussian matrix quantum mechanics.

Thermodynamics in the canonical ensemble.

Negative specific heat capacity in the micro-canonical ensemble and Ensemble in-equivalence.

Asymptotic forms of degeneracies in stable range and negative SHC : 2-matrix models and tensor-models.

Gravitational systems, small black holes .. AdS/CFT.

Denjoe O'Connor, Sanjaye Ramgoolam, Gauged permutation invariant matrix quantum mechanics: Partition functions, arXiv:2312.12398 [hep-th]
Denjoe O'Connor, Sanjaye Ramgoolam, Permutation invariant matrix quantum thermodynamics and negative specific heat capacities in large N systems, arXiv:2405.13150v1 [hep-th]

Part 1 : PIMQM - Degeneracies and Partition functions

System of N^2 matrix harmonic oscillators

$$A^\dagger_{i_1 j_1} A^\dagger_{i_2 j_2} \cdots A^\dagger_{i_k j_k} |0\rangle$$

Commutation relations derived from Lagrangian

$$[A_{ij}, A^\dagger_{kl}] = \delta_{ik}\delta_{jl}$$

Hamiltonian is the number operator

$$H = \sum_{i,j} A^\dagger_{ij} A_{ij}$$

The $\beta = \frac{1}{T}$ is the inverse temperature.

$$\mathcal{Z}^{\text{ungauged}}(N, T) = \sum_{k=0}^{\infty} \mathcal{Z}^{\text{ungauged}}(N, k) e^{-\beta k}$$

The $\mathcal{Z}^{\text{ungauged}}(N, k)$ are the degeneracies. Starting from $k = 0$ they are :

$$1, N^2, \frac{N^2(N^2+1)}{2}, \cdots$$

The sum can be done. $x = e^{-\beta}$

$$\mathcal{Z}^{\text{ungauged}}(N, x) = \frac{1}{(1-x)^{N^2}}$$

This can be written as trace in the Hilbert space

$$\mathcal{Z}^{\text{ungauged}}(N, x) = \text{Tr}_{\mathcal{H}} e^{-\beta H}$$

The S_N gauged theory partition function :

$$\mathcal{Z}^{\text{gauged;S}_\text{N}}(N, x) = \text{Tr}_{\mathcal{H}} e^{-\beta H} P_0^{(S_N)}$$

$P_0^{S_N} = \frac{1}{N!} \sum_{\sigma \in S_N} \sigma$

Expanding in powers of $x = e^{-\beta}$

$$\mathcal{Z}^{\mathrm{gauged;S_N}}(N, x) \qquad = \mathrm{Tr}_{\mathcal{H}} e^{-\beta H} P_0^{(S_N)}$$

$$\mathcal{Z}^{\mathrm{gauged;S_N}}(N, x) \qquad = \sum_{k=0}^{\infty} \mathrm{Tr}_{\mathcal{H}^{(k)}} P_0^{(S_N)} e^{-\beta k}$$

At fixed oscillator number, i.e. fixed energy, the degeneracy is the micro-canonical partition function as a trace :

$$\mathrm{Tr}_{\mathcal{H}^{(k)}} P_0^{(S_N)} = \mathcal{Z}(N, k)$$

The k-oscillator space is

$$P_0^{(S_k)}(V_N \otimes V_N)^{\otimes k}$$

$\mathcal{Z}(N,k)$ can then be computed as

$$\mathcal{Z}(N,k) = \mathrm{Tr}_{(V_N \otimes V_N)^{\otimes k}} P_0^{(S_N)} P_0^{(S_k)}$$

Micro-canonical partition function:

$$\mathcal{Z}(N,k) = \sum_{p \vdash N} \sum_{q \vdash k} \frac{1}{\mathrm{Sym}\ p} \frac{1}{\mathrm{Sym}\ q} \prod_{i=1}^{k} (\sum_{l|i} l p_l)^{q_i}$$

$$p = [\{[l^{p_l}]\} \qquad l \in \{1, \cdots, N\}$$
$$q = [\{i^{p_i}\} \qquad i \in \{1, \cdots, k\}$$

- Sum over conjugacy classes of S_N, S_k, labelled by cycle structures p, q.
- l is a cycle length for p. p_l is a multiplicity of cycle length l,
- i is a cycle length for q.
- Number-theoretic characteristics : l is a divisor of i.

"Linguistic Matrix Theory" 2017, Kartsaklis, Ramgoolam, Sadrzadedh

Stability property :

$$\mathcal{Z}(2k, k) = \mathcal{Z}(N, k) \text{ for } N \geq 2k$$

Canonical partition function:

$$\mathcal{Z}(N,x) = \sum_{p \vdash N} \frac{1}{\mathrm{Sym}\, p} \prod_{i=1}^{N} \frac{1}{(1-x^i)^{i p_i^2}} \prod_{i<j} \frac{1}{(1-x^{LCM(i,j)})^{2GCD(i,j) p_i p_j}}$$

"Gauged permutation invariant matrix quantum mechanics : Partition functions " 2023, D. O' Connor, S. Ramgoolam

Extracting the coefficients also gives an efficient way to get $\mathcal{Z}(N,k)$.

Part 2 : PIMQM - Thermodynamics in the canonical ensemble.

Energy per particle :

$$U = -\frac{\frac{\partial}{\partial \beta} Tr(e^{-\beta H})}{N^2 \, Tre^{-\beta H}}$$

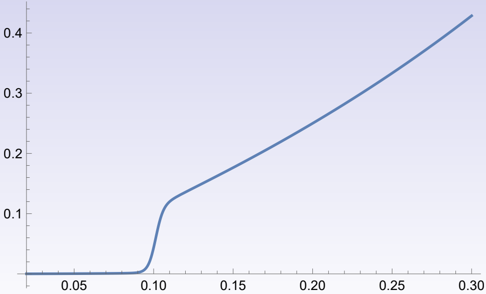
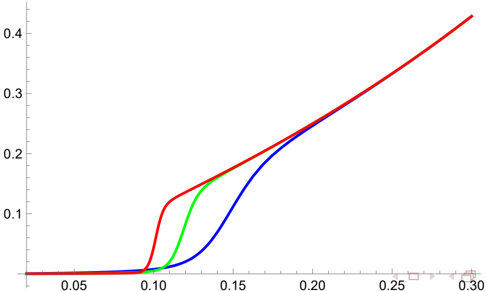


Figure: Energy versus temperature, parameterised by $x = e^{-\beta} = e^{\frac{-1}{T}}$ for $N = 20$: showing a cross-over



$$C_{\mathrm{shc}} = \frac{\partial U}{\partial T}$$

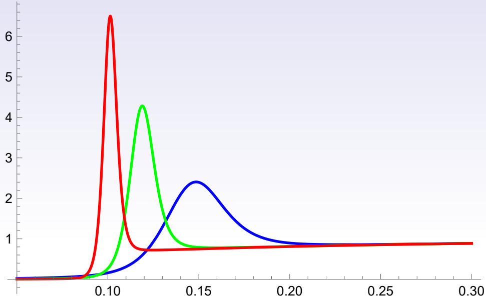


Figure: Specific heat capacity versus temperature : Sharp peak approaches zero temperature as N increases. Blue, Green and Red curves are for $N = 10, 15, 20$

As $N \to \infty$, the critical temperature - location of maximum in the SHC - goes to zero, while the height of the peak goes to infinity : $x_c \sim \frac{\log N}{N}$.

The initial low value of U, followed by the start of the steep rise is captured by the **stable part.**

$$\mathcal{Z}(2k, k) \sim k!$$

In the $N = \infty$ limit

$$\sum_{k=0}^{\infty} \mathcal{Z}(2k, k) e^{-\beta k} \sim \sum_{k=0}^{\infty} k! e^{-\beta k}$$

diverges for any finite β, hence any finite x.

Thus we expect $x \to 0$ singularity arising from factorial growth of degeneracies. This type of zero temperature Hagedorn transition in the large N limit has been recognised as a feature of some $U(N)$ and $O(N)$ tensor quantum mechanics models at $N = \infty$. With S_N we can very explicitly study the approach to $x = 0$ from finite N in one-matrix model.

M. Beccaria and A. A. Tseytlin, 2017 ;
K. Bulycheva, I. R. Klebanov, A. Milekhin and G. Tarnopolsky, 2018

Part 3 : PIMQM - Thermodynamics in the micro-canonical ensemble.

Degeneracies $\Omega(N,k) = \mathcal{Z}(N,k)$.

The micro-canonical entropy

$$S(k) = \log(\Omega(N,k))$$

$$TdS = dU$$
$$\frac{dS}{dU} = T^{-1}$$

$$T_{\text{micro}}^{-1}(k) = D_k S(k)$$

For $\Omega(k) \sim k!$, $S(k) \sim k \log k$,

$$T_{\text{micro}}^{-1}(k) = D_k(k \log k) = 1 + \log k$$

$\Longrightarrow$ as $k \uparrow$ $\quad T_{\text{micro}}^{-1} \uparrow$ $\quad$ and $T_{\text{micro}} \downarrow$

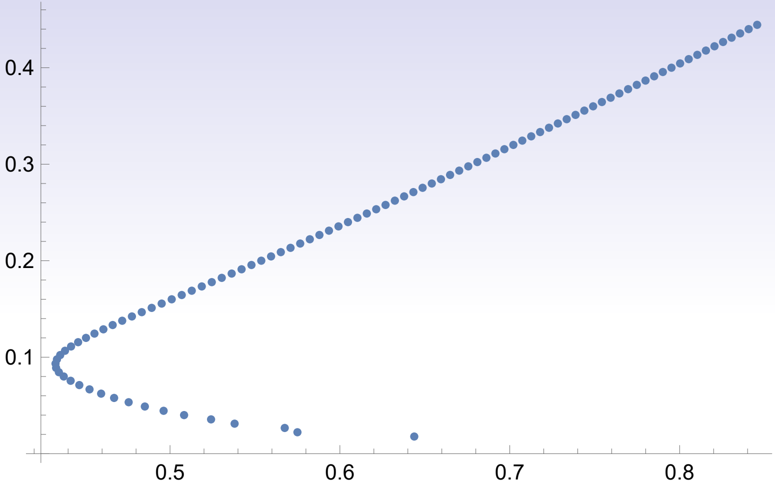


Figure: Plot of micro-canonical energy $E = \frac{k}{N^2}$ versus micro-canonical temperature at $N = 15$ for $k_{min} = 4, k_{max} = 100$ - using descending derivative – produces consistent negative SHC trend below critical E

Negative specific heat capacity on the lower branch.

Minimum temperature occurs at $k \sim \frac{N \log N}{2}$.

Stable region large k behaviour of $k!$ determines the lower branch. $k \sim N/2$: first signals of finite N taming of the large growth. But at higher N, the taming becomes thermodynamically significant.

This kind of small finite N corrections to stable behaviour at $k \sim N$ then thermodynamically significant effect at much higher N, has been seen in work on $U(N)$ invt multi-matrix models>

Berenstein, 2018
Denjoe O' Connor, 2022

The SHC cannot be negative in the canonical ensemble.

$$C_{\mathrm{shc}} = \langle H^2 \rangle - \langle H \rangle^2$$

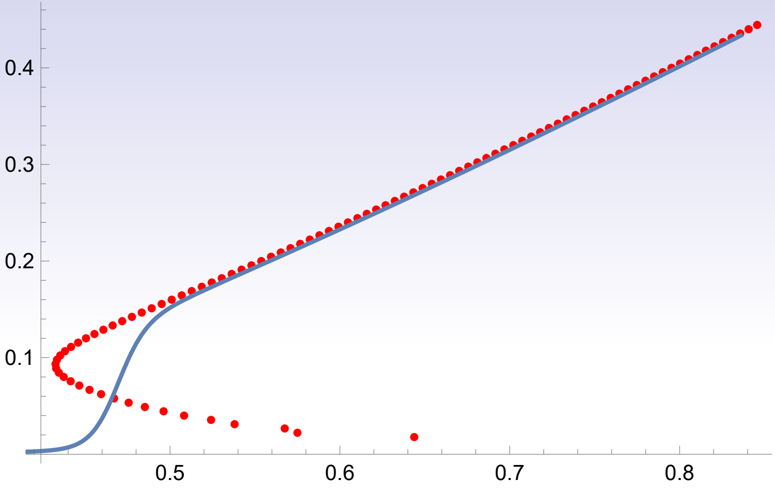


Figure: Plot of the expectation value of the energy $U = \mathcal{U}/N^2$ in the canonical ensemble versus canonical temperature T, superposed upon $E = \frac{k}{N^2}$ versus identification of $T_{\rm micro}$ in micro-canonical ensembles : Equivalence of ensemble above the transition region. This plot is for $N = 15$. The micro-canonical data starts at $k = 4$ and ends at $k = 100$.

Part 4 : Scale $x_c \sim \frac{\log N}{N}$ from high T expansion

From the form of the $Z(N, x)$ as a sum over p, we can the degree of the pole at $x = 1$

$$\mathcal{Z}(N, p, x) = \frac{1}{(1-x)^{\mathrm{Deg}(N,p)}} \mathcal{R}(x)$$

$$\mathrm{Deg}(N, p) = \sum_i i p_i^2 + \sum_{i<j} 2G(i, j) p_i p_j$$

The most singular term : $p = [1^N]$,
Second most singular term : $p = [2, 1^{N-2}]$

Leading two terms

$$\frac{1}{N!(1-x)^{N^2}} + \frac{1}{2(N-2)!}\frac{1}{(1-x)^{(N-2)^2}(1-x^2)^{2N-2}}$$

For breakdown of the high T expansion:

$$x_{\rm bkdwn} = \frac{\log N}{2N} + \frac{1}{4N}(\log 2 - \log a) + \frac{1}{4N^2}(\log(\frac{a}{2}) - 1) - \frac{\log N}{2N^2} + \mathcal{O}(\frac{1}{N^3}) + \mathcal{O}(\frac{\log N}{N^3})$$

This identifies $x \sim \frac{\log N}{N}$ as the characteristic scale of the transition.

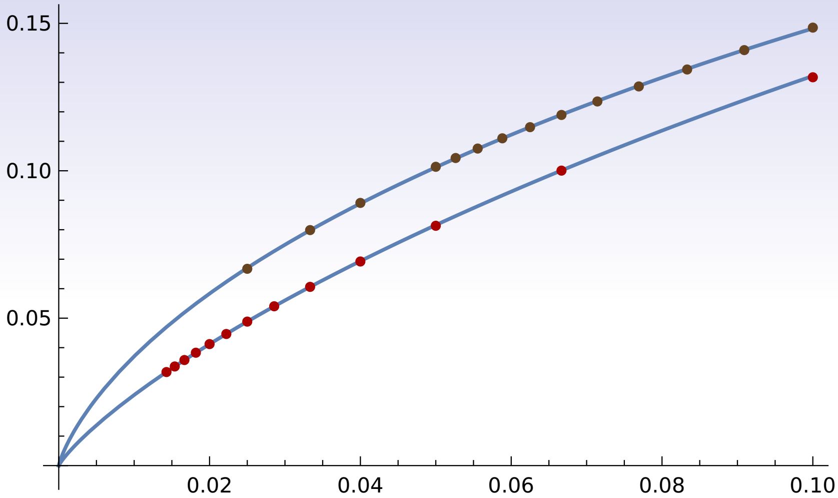


Figure: Comparison of microcanonical and canonical values for the tansition value of $x = \mathrm{e}^{-\beta}$ with $\beta = \beta_{micro}(k_{crit})$ for the microcanonical ensemble. The best fit curves, plotted against $\nu = \frac{1}{N}$ are $x_{crit}(\nu) = 0.508 \log(N)/N + 0.032/N + 0.522 \log(N)/N^2$ for the microcanonical ensemble and $x_{max}(\nu) = 1.37557 \log(N)/N - 2.86538/N + 5.1274 \log(N)/N^2$ for the canonical ensemble.

The specific heat capacity, per particle, i,e. as derived from $\frac{\log \mathcal{Z}(N,x)}{N^2}$, has a peak of $C_{\rm shc} \sim N \log N$.

N	$\frac{C_{\rm shc;max}}{N \log N}$
10	0.104449
15	0.105513
20	0.108528
25	0.108454
30	0.106822
40	0.102552

Interesting project Build a local model of the transition which explains this scaling.

To summarise.

Field	symmetry	Degen.	Thermodynamics
One matrix	$U(N)$	$\mathcal{Z}(k) \sim e^{\sqrt{k}}$	No transition.
Two matrix	$U(N)$	$\mathcal{Z}(k) \sim e^{k \log 2}$	Finite temp. Hagedorn
One matrix	S_N	$\mathcal{Z}(k) \sim e^{k \log k}$	Zero temp. Hagedorn.

Table: Degeneracies and negative heat capacities

$\Omega(k)$	**Parameter ranges**	**Description**
e^{ak^b}	$a > 0, b > 1$	Super-exponential
$e^{ak(\log k)^b}$	$a > 0, b > 0$	Weakly super-exponential
$e^{a \log k + k \log d}$	$d > 1, a < 0$	Sub-exponential (power-law corrected exponential)

Part 5 : Similar thermodynamics in matrix/tensor systems with $U(N)$ symmetry.

A tensor quantum mechanics model, with harmonic oscillator potential, and variables

$$\Phi_{ijk}$$

transforming in $V_N \otimes V_N \otimes V_N$ representation of $U(N)^{\times 3}$.

We gauge the $U(N)^{\times 3}$ so that the physical states are invariant.

Two tensor creation operators :

$$\Phi_{ijk} \to A^{\dagger}_{ijk}$$
$$\bar{\Phi}^{ijk} \to (B^{\dagger})^{ijk}$$

Physical states :

$$|0\rangle;$$
$$(A^{\dagger})_{ijk}(B^{\dagger})^{ijk}|0\rangle$$

Hidden S_n symmetry for degree n observables

In general

$$(A^\dagger)_{i_1 j_1 k_1} \cdots (A^\dagger)_{i_n j_n k_n} (B^\dagger)^{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}} \cdots (B^\dagger)^{i_{\sigma_1(n)} j_{\sigma_2(n)} k_{\sigma_3(n)}} |0\rangle$$

Counting :

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_L \sigma_1 \gamma_R, \gamma_L \sigma_2 \gamma_R, \gamma_L \sigma_3 \gamma_R)$$

$\sigma_i, \gamma_L, \gamma_R \in S_n$

Joseph Ben Geloun, Sanjaye Ramgoolam, “Counting Tensor Model Observables and Branched Covers of the 2-Sphere,” arXiv:1307.6490 [hep-th] Ann.Inst.H.Poincare Comb.Phys.Interact. 1 (2014) 1, 77-138

Counting

Dimension of space of invariants for general N

$$\mathcal{Z}(N,n) = \sum_{\substack{R,S,T\vdash n \\ l(R),l(S),l(T)\leq N}} (C(R,S,T))^2$$

For $n \leq N$ – stable limit – this simplifies :

$$\mathcal{Z}(n) = \sum_{p\vdash n} (\mathrm{Sym}\, p)$$

Asymptotics

$$\mathcal{Z}(n) \sim n!$$

Joseph Ben Geloun, Sanjaye Ramgoolam, "All-orders asymptotics of tensor model observables from symmetries of restricted partitions," J. Phys. A: Math. Theor. 55 435203 ; arXiv:2106.01470 [hep-th]

High temperature scaling from path integrals

$$Z_N^{(s)}(x,d) \sim \int \prod_{k=1}^{s} [dA^k] \prod_{a=1}^{d} [d\Psi^a][d\overline{\Psi}^a] \, \mathrm{e}^{-\sum_{a=1}^{d}\left(|(A^{(1)}_{i_1 i'_1} + \cdots A^{(s)}_{i_s i'_s})\Psi^a_{i'_1 \ldots i'_s}|^2 + \beta^2 m^2 |\Psi^a_{i_1,\ldots,i_s}|^2 \right)}$$

$$\begin{aligned} \Psi' &= \frac{\Psi}{\beta m} && \ldots. && dN^s \text{ fields} \\ \bar{\Psi}' &= \frac{\bar{\Psi}}{\beta m} && \ldots. && dN^s \text{ fields} \\ A' &= \beta m A && \ldots && (N^2-1)s+1 \text{ fields} \end{aligned}$$

$$Z_N^{(s)}(x;d) \sim (\beta m)^{s(N^2-1)+1-2dN^s} \sim \frac{1}{(1-x)^{2dN^s - s(N^2-1)+1}} \, .$$

Denjoe O' Connor and Sanjaye Ramgoolam, “Gauged permutation invariant matrix quantum thermodynamics and negative specific heat capacities in large N systems” arXiv:2405.13150v1 [hep-th]

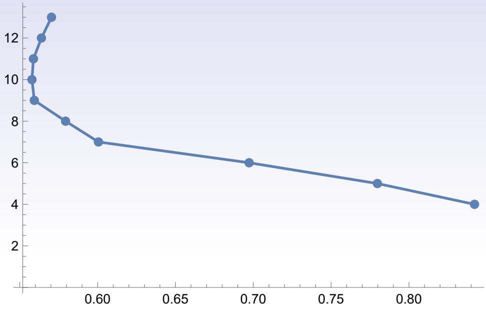


Figure: Micro-canonical energy versus temperature for 3-index tensor $N = 4$ with k equals 3 to 12 using the symmetric D_{sym} discrete derivative. Note the curve turns around, i.e. SHC become positive at higher energy.

Another example with similar thermodynamics, but finite temperature transition as $N \to \infty$

Complex matrix model : Z^i_j - N^2 complex variables.

$$Z \to UZU^\dagger$$

$U(N)$ transformations.
$U(1)$ transformation :

$$Z \to e^{i\theta} Z$$
$$Z^\dagger \to e^{-i\theta} Z^\dagger$$

Counting of multi-traces : Total number of Z and $Z^\dagger$ in the traces equal.

Micro-canonical degeneracies in the stable high energy regime :

$$Z(n) \sim \frac{4^n}{\sqrt{n}}$$

Ramgoolam, Wilson, Zahabi, 2018

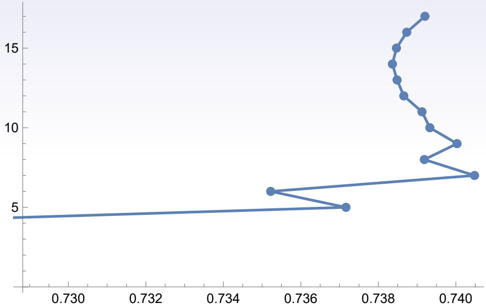


Figure: Micro-canonical energy versus temperature for zero charge complex matrix system $N = 13$ with k equals 3 to 18 using the symmetric D_{sym} discrete derivative. Note the curve has a short positive SHC branch, a negative SHC branch (expected to grow in size with N) and a positive SHC branch expected to connect to extend to the high temperature limit.

The curve comes from the group-theoretic counting formula :

$$Z(n,N) = \sum_{\substack{R,S \vdash n, T \vdash n \\ l(T) \leq N}} (LR(R,S,T))^2$$

To summarise.

Field	symmetry	Degen.	Thermodynamics
One matrix	S_N	$\mathcal{Z}(k) \sim e^{k \log k}$	Zero temp. Hagedorn. and neg SHC
Two matrix (*zerocharge*)	$U(N)$	$\mathcal{Z}(k) \sim \frac{e^{k \log 2}}{\sqrt{k}}$	Zero temp. Hagedorn. and neg SHC
Φ_{ijk}	$U(N)^{\times 3}$	$\mathcal{Z}(k) \sim e^{k \log k}$	Zero temp. Hagedorn. and neg SHC

The negative specific heat capacities here arise purely from counting in harmonic oscillator systems, without interaction.

Other recent discussions of negative SHC in matrix models seeking to understand small black holes invoked the quartic interactions away from the free limit to argue for the negative SHC. (Berenstein 2018).

Better understanding the relation between our counting results and these recent discussions is an interesting question for the future.

Some open questions :

Derive the numerical coefficients characterising the transition $x_c \sim \frac{\log N}{N}$. Locations of minimum and maximum of SHC in canonical ensemble. Values of the energy $\mathcal{U}/N^2$ at these locations. Also $k_{\rm crit}/N^2$ in the micro-ensemble.

Negative SHC are known to arise for small black holes in AdS - Hawking-Page. They deduce, from semi-classical gravity - using the mass of the black hole as function of the horizon radius, and the Hawking formula for the temperature as a function of the radius, that the black hole solution at horizon radius below a critical radius, has negative SHC.

They combine this with existence of solutions involving radiation coupled to gravity.

They build a picture of the gravitational thermodynamics in different temperature ranges.

To build a quantum mechanical model for this physics based on a $\mathcal{Z}(x, N) = Tr e^{-\beta H}$ involving negative SHC as well as the interaction of radiation, the quantum mechanics must contain some well-defined sector with negative SHC coupled with additional degrees of freedom.....

The matrix and tensor models discussed here should provide some useful ingredients ...

Negative SHCs have been discussed quite generally in gravitational thermodynamics - Thirring, Lynden-Bell - for astrophysical objects beyond black holes ...

Also in statistical physics e.g. Touchette (refs in paper) — attributed to long range forces ...

The zero-charge 2-matrix sector is a promising set-up to investigate the gravitational side of the physics further ... for some range of parameters, has an interpretation in terms of branes and anti-branes ... hence attractive forces ; has a known gravitational dual via standard AdS/CFT ....