Carroll gravity from the conformal approach

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Introduction

- Carroll field theory = ultralocal ($c \to 0$) limit of relativistic field theory. Invariant under Carroll algebra = $c \to 0$ contraction of Poincaré algebra. (Lévy-Leblond, Gupta)
- Relevant for e.g.:
  - Flat space/celestial holography (Duval, Gibbons, Horvathy; Bagchi, Grumiller et al.; Ciambelli, Marteau, Petkou, Petropoulos, Siampos; Donnay, Fiorucci, Herfray, Ruzziconi;...)
  - Physics on null hypersurfaces, black hole horizon (Donnay, Marteau; Bagchi, Banerjee, Hartong, Have, Kolekar, Mandlik)
  - Tensionless limit of strings (Bagchi et al.), new decoupling limits of string theory (Bidussi, Harmark, Hartong, Obers, Oling; Blair, Lahnsteiner, Obers, Yan)
  - Dark energy and inflation (de Boer, Hartong, Obers, Sybesma, Vandoren), fluid dynamics (de Boer, Hartong, Obers, Sybesma, Vandoren), ...
- This talk: Carroll gravity. At two-∂ level, usually studied by carefully defining $c \to 0$ limit of General Relativity. (Henneaux; Dautcourt; Bergshoeff, Gomis, Rollier, JR, ter Veldhuis; Campoleoni, Henneaux, Pekar, Perez, Salgado-Rebolledo)
- Can be subtle. More complete understanding and generalizations require intrinsic constructions.
- Here: conformal approach.

Outline

- Einstein gravity from the conformal approach
- Carroll gravity from the $c \to 0$ limit of General Relativity
- Carroll gravity from the conformal approach
- Conclusions

Einstein gravity from the conformal approach

- Einstein gravity is not invariant under local dilatations. Can be reintroduced using a compensating scalar ϕ:

$$E_\mu{}^{\hat{A}} \to \phi^{1/w} E_\mu{}^{\hat{A}} \,.$$

Invariant under

$$\delta\phi = w\Lambda_D(x)\phi \,, \qquad \delta E_\mu{}^{\hat{A}} = -\Lambda_D(x) E_\mu{}^{\hat{A}} \,.$$

- This replacement gives the Lagrangian of a conformally coupled scalar:

$$ER \quad \to \quad E\left(\frac{(D-1)(D-2)}{w^2}\partial_{\hat{A}}\phi\partial^{\hat{A}}\phi + R\phi^2\right) .$$

$\phi = 1$ dilatation gauge fixing gives Einstein-Hilbert.

- Strategy: use this to construct gravity theories by reversing the logic.
  1. Construct action of conformally coupled scalar intrinsically, by coupling ϕ to appropriate gauge fields of the conformal algebra.
  2. Retrieve gravity by gauge-fixing superfluous conformal symmetries.

Einstein gravity from the conformal approach

- Step 1: obtain "minimal" representation of gauge fields of relativistic conformal algebra.
- Introduce gauge fields of relativistic conformal algebra

$$P_{\hat{A}} \to E_\mu{}^{\hat{A}}\,, \qquad M_{\hat{A}\hat{B}} \to \Omega_\mu{}^{\hat{A}\hat{B}}\,, \qquad K_{\hat{A}} \to F_\mu{}^{\hat{A}}\,, \qquad D \to B_\mu\,,$$

with corresponding field strengths $R_{\mu\nu}(P^{\hat{A}})$, $R_{\mu\nu}(M^{\hat{A}\hat{B}})$, $R_{\mu\nu}(K^{\hat{A}})$, $R_{\mu\nu}(D)$.

- "Reducible" representation, realized by too many *independent* gauge fields. To get a minimal representation, impose so-called conventional constraints:

$$R_{\mu\nu}(P^{\hat{A}}) \equiv 2\,\partial_{[\mu}E_{\nu]}{}^{\hat{A}} + 2\,\Omega_{[\mu}{}^{\hat{A}\hat{B}}\,E_{\nu]\hat{B}} + 2\,B_{[\mu}\,E_{\nu]}{}^{\hat{A}} = 0\,,$$
$$E_{\hat{B}}{}^{\nu}R_{\mu\nu}(M^{\hat{A}\hat{B}}) \equiv E_{\hat{B}}{}^{\nu}\left(2\,\partial_{[\mu}\Omega_{\nu]}{}^{\hat{A}\hat{B}} + 2\,\Omega_{[\mu}{}^{[\hat{A}}{}_{|\hat{C}|}\,\Omega_{\nu]}{}^{|\hat{C}|\hat{B}]} + 8\,F_{[\mu}{}^{[\hat{A}}\,E_{\nu]}{}^{\hat{B}]}\right) = 0\,.$$

Allows to solve $\Omega_\mu{}^{\hat{A}\hat{B}}$ and $F_\mu{}^{\hat{A}}$ in terms of $E_\mu{}^{\hat{A}}$ and B_μ. E.g.:

$$\Omega_\mu{}^{\hat{A}\hat{B}}(E,B) = \Omega_\mu{}^{\hat{A}\hat{B}}(E) + 2\,E_\mu{}^{[\hat{A}}E^{\hat{B}]\nu}B_\nu\,, \qquad F_{\hat{A}}{}^{\hat{A}} = -\frac{1}{4(D-1)}R'_{\hat{A}\hat{B}}(M^{\hat{A}\hat{B}})\,.$$

where $\Omega_\mu{}^{\hat{A}\hat{B}}(E)$ is the usual Levi-Civita spin-connection and

$$R'_{\mu\nu}(M^{\hat{A}\hat{B}}) = 2\,\partial_{[\mu}\Omega_{\nu]}{}^{\hat{A}\hat{B}}(E,B) + 2\,\Omega_{[\mu}{}^{[\hat{A}}{}_{|\hat{C}|}(E,B)\,\Omega_{\nu]}{}^{|\hat{C}|\hat{B}]}(E,B)\,.$$

Einstein gravity from the conformal approach

- Step 2: Construct action of a conformally coupled compensating scalar field ϕ with dilatation weight w

$$\delta\phi = w\,\Lambda_D\,\phi \qquad \Rightarrow \qquad D_{\hat{A}}\phi \equiv E_{\hat{A}}{}^{\mu}\left(\partial_\mu - w\,B_\mu\right)\phi\,.$$

- Under $K_{\hat{A}}$, $B_{\hat{A}}$ transforms with a shift:

$$\delta B_{\hat{A}} = 2\,\Lambda_{K\hat{A}} \qquad \Rightarrow \qquad \delta D_{\hat{A}}\phi = -2w\,\Lambda_{K\hat{A}}\phi\,.$$

The proper definition of a covariant conformal box operator on ϕ is then

$$\Box^C\phi \equiv E^{\hat{A}\mu}\left[\partial_\mu\left(D_{\hat{A}}\phi\right) + \Omega_{\mu\hat{A}}{}^{\hat{B}}(E,B)\,D_{\hat{B}}\phi - (w+1)\,B_\mu\,D_{\hat{A}}\phi + 2\,w\,F_\mu{}^{\hat{A}}(E,B)\,\phi\right]$$

- For $w = (D-2)/2$, the following action is conformally invariant:

$$\mathcal{L}_{\rm conf} = -\frac{1}{2}\,E\,\phi\,\Box^C\phi\,,$$

B_μ does not appear!
$\Rightarrow$ $\mathcal{L}_{\rm conf}$ describes the conformally coupled scalar and is thus gauge-equivalent to Einstein-Hilbert.

Carroll gravity

- Carroll gravity = $c \to 0$ limit of General Relativity. Taken by reintroducing c as

$$\hat{A} \quad \to \quad \{0, A = 1, \cdots, D-1\}$$

$$E_\mu{}^0 = c\,\tau_\mu\,, \qquad E_\mu{}^A = e_\mu{}^A\,, \qquad E_0{}^\mu = c^{-1}\tau^\mu\,, \qquad E_A{}^\mu = e_A{}^\mu\,.$$

Redefining also the local Lorentz transformation parameters $\Lambda^{\hat{A}\hat{B}}$ as

$$\Lambda^{0A} = c\,\lambda^{0A}\,, \qquad \Lambda^{AB} = \lambda^{AB}\,,$$

one finds

$$\delta E_\mu{}^{\hat{A}} = -\Lambda^{\hat{A}}{}_{\hat{B}} E_\mu{}^{\hat{B}} \quad \xRightarrow{c\to 0} \quad \begin{cases} \delta\tau_\mu = -\lambda^{0A} e_{\mu A}\,, \\ \delta e_\mu{}^A = -\lambda^A{}_B e_\mu{}^B \end{cases} .$$

- τ_μ, $e_\mu{}^A$ are Vielbeine for a Carrollian geometry. Give along with their dual “inverse” Vielbeine τ^μ, $e_A{}^\mu$ a degenerate Carroll metric structure

$$\tau^{\mu\nu} = \tau^\mu \tau^\nu\,, \qquad h_{\mu\nu} \equiv e_\mu{}^A \delta_{AB} e_\nu{}^B\,.$$

τ^μ, $e_A{}^\mu$ used to turn curved into flat indices:

$$X_0 \equiv \tau^\mu X_\mu\,, \qquad X_A \equiv e_A{}^\mu X_\mu\,, \qquad X_{0A} \equiv \tau^\mu e_A{}^\nu X_{\mu\nu}\,, \qquad X_{AB} \equiv e_A{}^\mu e_B{}^\nu X_{\mu\nu}\,.$$

Carroll gravity

- Einstein-Hilbert action can be expanded as ($e = \det(\tau_\mu, e_\mu{}^a)$)

$$\mathcal{L}_{\rm EH} = c^{-2}\,\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + c^2\mathcal{L}^{(2)} + \mathcal{O}(c^4)\,.$$

  $c \to 0$ limit can then be taken in two ways.

- First way: retain leading order term $\mathcal{L}^{(0)}$ = "electric Carroll gravity":

$$\mathcal{L}_{\rm el} = \mathcal{L}^{(0)} \propto e\left(T'_{0(A,B)}T_0'^{(A,B)} - T'_{0A}{}^{A}T'_{0B}{}^{B}\right)\,,$$

  with "intrinsic torsion" $T'_{0(A,B)} \equiv 2\tau^\mu e_{(A}{}^{\nu}\partial_{[\mu}e_{\nu]B)}$.

- Second way: first rewrite the expansion of $\mathcal{L}_{\rm EH}$ in the classically equivalent form

$$\mathcal{L}_{\rm EH} \sim \mathcal{L}^{(0)} + e\left(\lambda^{AB}T'_{0(AB)} - \lambda T'_{0A}{}^{A}\right) + c^2\left(\mathcal{L}^{(2)} - \frac{1}{4}e\lambda^{AB}\lambda_{AB} + \frac{1}{4}e\lambda^2\right) + \mathcal{O}(c^4)\,.$$

  Retaining the leading order term gives "magnetic Carroll gravity":

$$\mathcal{L}_{\rm magn} \propto \mathcal{L}^{(0)} + e\left(\lambda^{AB}T'_{0(AB)} - \lambda T'_{0A}{}^{A}\right)\,.$$

  λ^{AB}, λ are Lagrange multipliers for geometric constraints.

Carroll gravity from the conformal approach

- Can the two Carroll gravity theories be obtained via a conformal approach?
- Starting point: conformal Carroll algebra = $c \to 0$ contraction of relativistic conformal algebra ($A = 1, \cdots, D-1$)

$$\begin{aligned} H &\to \tau_\mu\,, & P_A &\to e_\mu{}^A\,, & J_{AB} &\to \omega_\mu{}^{AB}\,, & J_{0A} &\to \omega_\mu{}^{0A}\,,\\ K &\to f_\mu\,, & K_A &\to g_\mu{}^A\,, & D &\to b_\mu\,, \end{aligned}$$

  Field strengths: $R_{\mu\nu}(H)$, $R_{\mu\nu}(P_A)$, $R_{\mu\nu}(J_{AB})$, $R_{\mu\nu}(J_{0A})$, $R_{\mu\nu}(K_A)$, $R_{\mu\nu}(K)$, $R_{\mu\nu}(D)$.
- Reducible representation. Impose conventional constraints:

$$R_{\mu\nu}(H) = R_{BC}(P^A) = R_{\mu B}(J^{AB}) = R_{\mu A}(J^{0A}) = 0\,.$$

- Can be used to express

$$\omega_\mu{}^{AB}\,, \qquad \omega_0{}^{0A}\,, \qquad \omega^{[A|,0|B]}\,, \qquad b_0\,, \qquad g_\mu{}^A\,, \qquad f_\mu\,,$$

  in terms of the remaining independent gauge fields τ_μ, $e_\mu{}^A$, b_A and $\omega^{(A|,0|B)}$.

Carroll gravity from the conformal approach

- There exist two kinds of free massless Carroll scalars. Start from the Lagrangian of a massless relativistic scalar in Hamiltonian form:

$$\mathcal{L} = \Pi_\Phi \partial_t \Phi - \frac{c^2}{2}\Pi_\Phi^2 - \frac{1}{2}\partial^A \Phi \partial_A \Phi \,.$$

  $c \to 0$ limit can be taken in two ways.

- First way: rescale

$$\Pi_\Phi = c^{-1}\,\pi \,, \qquad \Phi = c\,\phi \,,$$

  and then take $c \to 0$. This gives an electric Carroll scalar:

$$\mathcal{L} = \pi \partial_t \phi - \frac{1}{2}\pi^2 \qquad \Leftrightarrow \qquad \mathcal{L} = \frac{1}{2}\partial_t \phi \partial_t \phi \,.$$

- Second way: rename

$$\Pi_\Phi = \pi \,, \qquad \Phi = \phi \,,$$

  and then take $c \to 0$. This gives a magnetic Carroll scalar:

$$\mathcal{L} = \pi \partial_t \phi - \frac{1}{2}\partial^A \phi \partial_A \phi \,.$$

Carroll gravity from the conformal approach

- First case: construct conformally invariant generalization of electric Carroll scalar

$$\mathcal{L} = -\frac{1}{2}\phi\partial_t\partial_t\phi\,.$$

- Take ϕ with dilatation weight w. The correct generalizations of $\partial_t\phi$ and $\partial_t\partial_t\phi$ and $\mathcal{L}$ are

$$D_0\phi \equiv \tau^\mu\left(\partial_\mu - w b_\mu\right)\phi\,, \qquad D_0 D_0\phi \equiv \tau^\mu\left[\partial_\mu\left(D_0\phi\right) - (w+1)b_\mu D_0\phi\right]\,,$$
$$\mathcal{L} = -\frac{1}{2}e\phi D_0 D_0\phi\,.$$

  Only dilatations act non-trivially. Conformally invariant for $w = (D-2)/2$.

- Gauge-fixing dilatations by setting $\phi = 1$ gives 'electric Carroll gravity':

$$\mathcal{L} = \frac{w^2}{2}e b_0^2 = \frac{w^2}{2(D-1)^2}e T'_{0A}{}^{A} T'_{0B}{}^{B} \qquad \text{with}\;\; T'_{0A}{}^{A} = 2\tau^\mu e_A{}^\nu \partial_{[\mu}e_{\nu]}{}^{A}\,.$$

- Not the most general electric Carroll gravity possible. Other possibility:

$$\mathcal{L}' \propto e T'_0{}^{\{A,B\}} T'_{0\{A,B\}} \qquad \text{with}\;\; T'_{0\{A,B\}} = 2\tau^\mu e_{\{A}{}^{\nu}\partial_{[\mu}e_{\nu]B\}}\,.$$

  Not reproduced from a dynamic scalar Lagrangian, since it transforms homogeneously under dilatations.

Carroll gravity from the conformal approach

- Second case: construct conformally invariant generalization of electric Carroll scalar

$$\mathcal{L} = \pi\partial_t\phi - \frac{1}{2}\partial^A\phi\partial_A\phi\,.$$

- First attempt to make this conformally invariant: assume ϕ has dilatation weight w and generalize $\mathcal{L}$ to

$$\begin{aligned} &\mathcal{L}_0 = \pi D_0\phi - \frac{1}{2}D^A\phi D_A\phi\,, \\ &\text{with}\quad D_0\phi = \tau^\mu\left(\partial_\mu - wb_\mu\right)\phi\,, \qquad D_A\phi = e_A{}^\mu\left(\partial_\mu - wb_\mu\right)\phi\,. \end{aligned}$$

  Invariant under dilatations and boosts for $w = (D-2)/2$ and

$$\delta\pi = \lambda^{0A}D_A\phi + \frac{1}{2}D\lambda_D\pi\,.$$

- Not invariant under K_A however:

$$\delta\mathcal{L}_0 = 2we\lambda_K{}^A\phi D_A\phi\,.$$

Carroll gravity from the conformal approach

- This can be cancelled by adding a boost invariant combination of ${g_A}^A\phi^2$ and $f_0\phi^2$:

$$\mathcal{L} = \pi D_0\phi - \frac{1}{2}D^A\phi D_A\phi + we{g_A}^A\phi^2 + wef_0\phi^2 \,.$$

Invariance under K requires assigning a K-transformation to π:

$$\delta_K\pi = (D-2)\lambda_K\phi \,.$$

- Fix dilatations, K_A and K transformations by:

$$\phi = 1\,, \qquad b_A = 0\,, \qquad \pi = 0\,,$$
$$\mathcal{L}_{\text{fixed}} = we\left({g_A}^A + f_0\right)\,.$$

- ${g_A}^A + f_0$ depends on τ_μ, ${e_\mu}^a$, but also the independent spin connection components $\omega^{(A|,0|B)}$. Can be identified with Lagrange multipliers for the constraint

$$T'_{0(A,B)} \equiv 2\tau^\mu {e_{(A}}^\nu \partial_{[\mu} e_{\nu]B)} = 0\,.$$

$\mathcal{L}_{\text{fixed}}$ then agrees with magnetic Carroll gravity Lagrangian.

Conclusions

- Carroll gravity comes in 2 guises: electric and magnetic. Both can be constructed from the conformal approach.
- Starts from a single conformal Carroll algebra, but two different "electric" and "magnetic" massless compensating scalar field theories.
- Difference with relativistic conformal approach: some invariants involving intrinsic torsion can not be reproduced by coupling to dynamic matter.
- Extra difference with relativistic conformal approach: extra independent spin connection components. Match with Lagrange multipliers in magnetic Carroll gravity.
- Outlook:
  - Including matter couplings.
  - Carroll supergravity.
  - Aristotelian gravity, coupled to fracton matter?