Reduction of Quantum Principal Bundles

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Plan of the Talk

1. Motivation
2. Principal bundles
3. Quantum Principal bundles
4. Reduction of principal bundles
5. Reduction of quantum principal bundles
6. Examples

Joint work with Latini and Pagani:

Reduction of Quantum Principal Bundles over non affine bases,
https://arxiv.org/abs/2403.06830

0. Motivation

Motivation

- **Quantum Groups**: are born to encode quantum symmetries. We treat physical geometric objects as *homogeneous spaces*.
- **Quantum Cartan Geometry**. We go towards a quantum theory of Cartan connections and (bi)covariant objects (e.g. covariant hamiltonian).
- **Non commutative (super)gravity**. The language of (quantum) differential geometry is natural for any geometric theory like (super)gravity.

Quantum space and Quantum Groups

- **Classical space:** $x^\mu x^\nu = x^\nu x^\mu$
- **Action of a classical group:** $x^\mu \mapsto a^\mu_\nu x^\nu$
- **Quantum space:** $x^\mu x^\nu = q x^\nu x^\mu$, $\mu > \nu$, $q = e^h \in \mathbb{C}$
- **Coaction of a quantum group:** $x^\mu \mapsto a^\mu_\nu \otimes x^\nu$

The quantum deformation of the space imposes a quantum deformation of the group coacting on the space (Manin).

Manin commutation relations and quantum $\mathrm{SL}_2(\mathbb{C})$. Assume:

$$yx = qxy$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{array}{c} ab = q^{-1} ba,\ ac = q^{-1} ca,\ bd = q^{-1} db, \\ \\ cd = q^{-1} dc\ bc = cb\ ad - da = (q^{-1} - q) bc \end{array}$$

1. Principal Bundles

Principal bundles: Classical definition

Definition

$(E, M, \wp, P)$ is a P-principal bundle if

1. *$\wp : E \longrightarrow M$ is surjective.*
2. *P acts freely from the right on E.*
3. *P acts transitively on the fiber $\wp^{-1}(m)$, $m \in M$.*
4. *(E is locally trivial over M).*

Example

$$E = \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathbb{P}^1 \cong \mathrm{SL}_2(\mathbb{C})/B, \quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

This is a principal bundle with fiber B.

Principal Bundles: Sheaf theoretic definition

Definition ((Pflaum))

$p : E \longrightarrow M$ is a P-principal bundle if and only if

- *$\mathcal{F} = C^\infty_E$ is a sheaf of $H = \mathcal{O}(P)$ comodule algebras;*
- *There exists an open covering $\{\overline{U}_i\}$ of M such that:*
  1. $\mathcal{F}(U_i)^{\mathrm{coinv}H} \simeq \mathcal{O}_M(\overline{U}_i)$
  2. $\mathcal{F}(U_i) \simeq \mathcal{F}(U_i)^{\mathrm{coinv}H} \otimes H$*, as left* $\mathcal{F}(U_i)^{\mathrm{coinv}H}$*-modules and right* H*-comodules for all* i*,*

  $\mathcal{F}(U_i)^{\mathrm{coinv}H} := \{f \in \mathcal{F}(U_i) \,|\, \delta_H(f) = f \otimes 1\}$
  $\delta_H : \mathcal{F}(U_i) \to \mathcal{F}(U_i) \otimes H$ *the* H*-coaction.*

2. Quantum Principal Bundles

Quantum Principal bundles

Definition

$(M, \mathcal{O}_M)$ is a quantum ringed space if

- *M: classical topological space*
- *$\mathcal{O}_M$: sheaf over M of non commutative algebras.*

Definition

The sheaf $\mathcal{F}$ on M is a H-quantum principal bundle over the quantum ringed space $(M, \mathcal{O}_M)$ if:

- *$\mathcal{F}$ is a sheaf of H comodule algebras;*
- *There exists an open covering $\{U_i\}$ of M such that:*
  1. $\mathcal{F}(U_i)^{\mathrm{coinv}H} = \mathcal{O}_M(U_i)$,
  2. *$\mathcal{F}$ is locally cleft, i.e. $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{coinv}H} \otimes H$.*

Example of a classical principal bundle

$$\wp : E = \mathrm{SL}_2(\mathbb{C}) \longrightarrow M = \mathrm{SL}_2(\mathbb{C})/P \simeq \mathbb{P}^1(\mathbb{C})$$

On the coordinate algebras:

$$\pi : \mathbb{C}[\mathrm{SL}_2] = \mathbb{C}[a,b,c,d]/(ad-bc-1) \quad\longrightarrow\quad \mathbb{C}[\mathrm{SL}_2]/(c) = \mathbb{C}[t,p,t^{-1}]$$

$V_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; a \neq 0 \right\}$, $V_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; c \neq 0 \right\}$ open cover of $\mathrm{SL}_2(\mathbb{C})$.
Let $U_i = \wp(V_i)$. Define the sheaf $\mathcal{F}$ of $\mathcal{O}(P)$-comodule algebras

$$\mathcal{F}(U_1) := \mathbb{C}[\mathrm{SL}_2][a^{-1}], \qquad \mathcal{F}(U_2) := \mathbb{C}[\mathrm{SL}_2][c^{-1}]$$

$$\mathcal{F}(U_{12}) := \mathbb{C}[\mathrm{SL}_2][[a^{-1}, c^{-1}] \qquad \mathcal{F}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}.$$

$\mathcal{F}$ is a (quantum) principal bundle on $\mathbb{P}^1(\mathbb{C})$.

The Manin bialgebra

Define the *quantum special linear group*:

$$\mathbb{C}_q[\mathrm{SL}_2] = \mathbb{C}_q\langle a, b, c, d\rangle / I_M + (ad - q^{-1}bc - 1) .$$

I_M is the ideal of the *Manin relations*

$$ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad bd = q^{-1}db, \quad cd = q^{-1}dc,$$

$$bc = cb \qquad ad - da = (q^{-1} - q)bc$$

Examples of a Quantum Ringed space and a Quantum Principal Bundle

U_i cover of $X = \mathrm{SL}_2(\mathbb{C})/P$ as above.

Define the quantum ringed space:

$$\mathcal{O}_{q,\,\mathbb{P}^1(\mathbb{C})}(U_1) = \mathbb{C}_q[a^{-1}c] \simeq \mathbb{C}_q[u], \quad \mathcal{O}_{q,\,\mathbb{P}^1(\mathbb{C})}(U_2) = \mathbb{C}_q[c^{-1}a] \simeq \mathbb{C}_q[v]$$

On the quantum ringed space $(\mathbb{P}^1(\mathbb{C}), \mathcal{O}_{q\,,\mathbb{P}^1(\mathbb{C})})$ *define the quantum principal bundle* $\mathcal{F}$*, with respect to the covering* U_1*,* U_2*:*

$$\mathcal{F}(U_1) := \mathbb{C}_q[\mathrm{SL}_2][a^{-1}] \qquad \mathcal{F}(U_1) := \mathbb{C}_q[\mathrm{SL}_2][a^{-1}]$$

$$\mathcal{F}(U_{12}) := \mathbb{C}_q[\mathrm{SL}_2][a^{-1}, c^{-1}]$$

This is a sheaf of $\mathcal{O}_q(P)$*-comodule algebras on* $\mathbb{P}^1(\mathbb{C})$*.*

3. Reduction of Principal Bundles

Reduction of Principal Bundles

Definition

Let $\xi = (E, \pi, M)$ *be a principal* P*-bundle* $K \subset P$ *a subgroup.*
Let $\xi_0 = (E_0, \pi_0, M)$ *be a principal* K*-bundle.*
ξ_0 *is a reduction of* ξ *if there is a* K*-equivariant homeomorphism*

$$\varphi : E_0 \to \phi(E_0) \subset E, \qquad \varphi(xk) = \varphi(x)k, \quad x \in E_0,\ k \in K,$$

Proposition

A principal P*–bundle* $\xi = (E, \pi, M)$ *is reducible to a principal* K*–bundle* $\xi_0 = (E_0, \pi_0, M)$ *if and only if the bundle* $\xi_K := (E \setminus K, \pi_K, M)$*, with* π_K *being the projection induced by* π *on* $E \setminus K$*, admits a global section.*

Example of a reduction

Consider:
$\pi : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/B$
B is the subgroup of upper triangular matrices.
$\pi_0 : \mathrm{SL}_2(\mathbb{C})/N \longrightarrow \mathbb{P}^1(\mathbb{C})$
$N \subset B$ is the unipotent subgroup.

$$E = \mathrm{SL}_2(\mathbb{C}), \quad P = B, \quad K = T \subset B, \quad E_0 = \mathrm{SL}_2(\mathbb{C})/N$$

where $T = B/N$ is the torus of diagonal matrices.

$(\mathrm{SL}_2(\mathbb{C})/N, \pi_0, \mathbb{P}^1(\mathbb{C}))$ *is a reduction of* $(\mathrm{SL}_2(\mathbb{C})/B, \pi, \mathbb{P}^1(\mathbb{C}))$

Key point: $\mathrm{SL}_2(\mathbb{C})/N \cong \mathbb{C}^2 \setminus \{(0,0)\}$ *is* **not** *an affine/projective algebraic variety, there is no algebra associated with it.*
We need a sheaf theoretic approach to quantize it!

4. Reduction of Quantum Principal Bundles

Affine Reductions of Quantum Principal Bundles

Definition

Let H be a Hopf algebra. An H-comodule algebra A is *principal* if:

1. $A^{\mathrm{co}H} \subset A$ is H-Galois,
2. A is a faithfully flat $A^{\mathrm{co}H}$-module.

Definition

Let A be a principal H-comodule algebra with $B := A^{\mathrm{co}H}$ and J be a Hopf ideal of H such that H is a principal left H_0-comodule algebra for $H_0 := H/J$.
Let A_0 be a principal H_0-comodule algebra with $B_0 := A_0^{\mathrm{co}H_0}$.
We say that A_0 is a *reduction* of A if

1. $B \cong B_0$ as algebras;
2. there exists a surjective H_0-comodule morphism, $\phi : A \longrightarrow A_0$, $\phi(B) = B_0$, where A carries the induced H_0-coaction.

Sheaf Reduction of Quantum Principal Bundles

Definition

Let $\mathcal{F}$ and $\mathcal{F}_0$ be quantum principal bundles over the quantum ringed space $(M, \mathcal{O}_M)$ for Hopf algebras H and $H_0 = H/J$, respectively. We say that $\mathcal{F}_0$ is a reduction (resp. algebraic reduction) of $\mathcal{F}$ if there exists an open covering $\{U_i\}$ of M such that:

1. *$\mathcal{F}$ and $\mathcal{F}_0$ are quantum principal bundles with respect to such cover,*
2. *there exists an H_0-comodule (resp. H_0-comodule algebra) morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{F}_0$ such that $\varphi(\mathcal{F}(U_i)^{\mathrm{co}H}) = \mathcal{F}(U_i)^{\mathrm{co}H_0}$ for the induced coaction of $H_0 = H/J$ on the $\mathcal{F}(U_i)$.*

4. Example of Sheaf Reduction of Quantum Principal Bundles

The Takhtajan-Sudbery algebra $\widetilde{\mathcal{O}}_q(\mathrm{GL}(n))$

It is generated by elements a_{ij}, $i, j = 1, \dots, n$, with commutation relations

$$\begin{gathered} a_{ik} a_{il} = q^{-1} \, a_{il} a_{ik} \, ; \quad a_{ik} a_{jk} = q \, a_{jk} a_{ik} \\ a_{il} a_{jk} = q^2 \, a_{jk} a_{il} \, ; \quad a_{ik} a_{jl} = a_{jl} a_{ik} \, , \; i < j, k < l \end{gathered} \tag{1}$$

with D^{-1}, the inverse of the quantum determinant D,

$$a_{ik} D = q^{2(k-i)} D a_{ik} \; , \quad a_{ik} D^{-1} = q^{-2(k-i)} D^{-1} a_{ik} \; . \tag{2}$$

$\widetilde{\mathcal{O}}_q(\mathrm{GL}(2))$ it has generators a, b, c, d and D^{-1}:

$$ab = q^{-1} \, ba \, , ac = q \, ca \, , cd = q^{-1} \, dc \, , bd = q \, db \, , bc = q^2 \, cb \, , ad = da \tag{3}$$

together with

$$aD^{\pm 1} = D^{\pm 1} a \, , bD^{\pm 1} = q^{\pm 2} D^{\pm 1} b \, , cD^{\pm 1} = q^{\mp 2} D^{\pm 1} c \, , dD^{\pm 1} = D^{\pm 1} d$$

for $D = ad - q^{-1} bc$ the quantum determinant.

QPBs on the projective line $\mathbb{P}^1(\mathbb{C})$ and sheaf reductions

Let U_1, U_2 be the usual cover of the complex projective line.
We define two sheaves $\mathcal{F}_0$, $\mathcal{F}_0$ over $\mathbb{P}^1(\mathbb{C})$:

$$\mathcal{F}_0(U_1) := \mathbb{C}_q[a, c, a^{-1}, D^{\pm 1}] \subset \mathcal{F}(U_1) := \widetilde{\mathcal{O}}_q(\mathrm{GL}(2))[a^{-1}]$$

$$\mathcal{F}_0(U_2) := \mathbb{C}_q[a, c, c^{-1}, D^{\pm 1}] \subset \mathcal{F}(U_2) := \widetilde{\mathcal{O}}_q(\mathrm{GL}(2))[c^{-1}]$$

$$\mathcal{F}_0(U_1 \cap U_2) := \mathbb{C}_q[a, c, a^{-1}, c^{-1}, D^{\pm 1}] \subset \mathcal{F}(U_1 \cap U_2)$$

$$\mathcal{F}(U_1 \cap U_2) := \widetilde{\mathcal{O}}_q(\mathrm{GL}(2))[a^{-1}, c^{-1}]$$

$\mathcal{F}$ is a sheaf of H-comodule algebras, $H = \widetilde{\mathcal{O}}_q(\mathrm{GL}(2))[a^{-1}]/(b)$.
$\mathcal{F}_0$ is a sheaf of H_0-comodule algebras, $H_0 = \widetilde{\mathcal{O}}_q(\mathrm{GL}(2))[a^{-1}]/(b, c)$.

Proposition

$\mathcal{F}$ and $\mathcal{F}_0$ are QPB and $\mathcal{F}_0$ is a reduction of $\mathcal{F}$.

6. Existence of Reductions

Result on the existence of Reductions

Let $\{U_i\}$ be a finite cover of a quantum space and consider the topology induced by it.

Theorem

Let $\mathcal{F}$ be a quantum H-principal bundle over the quantum ringed space $(M, \mathcal{O}_M)$ with respect to a finite open covering $\{U_i\}$. Let $H_0 = H/J$. Assume $\{f_i : {}^{\mathrm{co}H_0}H \longrightarrow Z_{\mathcal{F}(U_i)}(\mathcal{O}_M(U_i))\}$ is a family of H-module and H-comodule algebra maps such that the following diagram commutes

$$\begin{array}{ccc} {}^{\mathrm{co}H_0}H & \xrightarrow{f_i} & Z_{\mathcal{F}(U_i)}(\mathcal{O}_M(U_i)) \subseteq \mathcal{F}(U_i) \\ \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle \rho_{i,ij}} \\ Z_{\mathcal{F}(U_j)}(\mathcal{O}_M(U_j)) \subseteq \mathcal{F}(U_j) & \xrightarrow{\rho_{j,ij}} & \mathcal{F}(U_i \cap U_j) \end{array} \tag{4}$$

Then $\mathcal{F}$ admits an algebraic reduction $\mathcal{F}_0$ to H_0.

Examples: The Taktajan-Sudbery algebra

Define $A = \widetilde{\mathcal{O}}_q(\mathrm{GL}(n))$ and:

$$H = A/(a_{s1}, s \neq 1), \quad J = (a_{1s}, s = 2, \dots, n), \quad H_0 = H/J$$

H is a left H_0-comodule algebra

$$\rho : H \to H_0 \otimes H, \qquad h \mapsto \pi(h_1) \otimes h_2, \quad (\pi : H \longrightarrow H_0 = H/J)$$

Define:

$$f_\ell : {}^{\mathrm{co}H_0}H \to A_\ell := A[a_{1,\ell}^{-1}], \quad \beta_s \mapsto a_{\ell 1}^{-1} a_{\ell s}\,, \quad s = 2, \dots, n.$$

f_ℓ satisfy the hypotheses of the previous theorem, hence we have a quantum reduction corresponding to the classical reduction:

$$\mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})/N \longrightarrow \mathrm{GL}_n(\mathbb{C})/P$$

Remark: The Manin deformation does not satisfy the hypotheses of the theorem and we cannot construct a similar reduction.

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