Differential geometrical methods for locally compact quantum groups

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Formal Drinfel'd twists

Let $\mathcal{G}$ be a real Lie group. Let $\mathfrak{G}$ be its Lie algebra and $\mathcal{U}(\mathfrak{G})$ the universal enveloping Hopf algebra.

Definition. A formal Drinfel'd twist based on $\mathcal{U}(\mathfrak{G})$ is an element $F \in \mathcal{U}(\mathfrak{G}) \otimes \mathcal{U}(\mathfrak{G})[[\hbar]]$:

$$F \;=:\; 1 \otimes 1 \;+\; \sum_{k=1}^{\infty} \hbar^k \, F_k$$

such that

$$(I \otimes \Delta)(F).(1 \otimes F) \;=\; (\Delta \otimes I)(F).(F \otimes 1)$$

Star products

Proposition [Drinfel'd]. A Drinfel'd twist corresponds to a left-invariant formal star-product $\star^F$ on $C^\infty(\mathcal{G})[[\hbar]]$:

$$f \star^F g \;:=\; m_0\left(\widetilde{F}(f \otimes g)\right)$$

Corollary. The first order term defines a left-invariant Poisson structure on $\mathcal{G}$:

$$\tilde{P}^F(f, g) \;:=\; \frac{1}{2}\, m_0\left(\widetilde{F}_1(f \otimes g) \;-\; \widetilde{F}_1(g \otimes f)\right)$$

Proposition. The symplectic leaf of $\tilde{P}^F$ through the unit e of $\mathcal{G}$ is an immersed Lie subgroup G of $\mathcal{G}$.

Symplectic Lie groups

Definition [Lichnérowicz]. A symplectic Lie group is a pair $(G, \tilde{\omega})$ where G is a real Lie group and where $\tilde{\omega}$ is a left-invariant symplectic structure on G.

Example. $\mathbb{R}^{2n} = T^\star(\mathbb{R}^n)$ is a symplectic Lie group ($\tilde{\omega} = \mathrm{d}p \wedge \mathrm{d}q$).

Fröbenius Lie groups and Lie algebras

Proposition. The left action of G on itself is Hamiltonian w.r.t. $\tilde{\omega}$ iff G admits an open coadjoint orbit.

Definition. A Fröbenius Lie group is a Lie group which admits an open coadjoint orbit.

Example. The affine group

$$G \ := \ GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$$

is a Fröbenius Lie group.

A bit of structure theory

Proposition.

1. A Fröbenius Lie group G contains a non-discrete Abelian normal subgroup J.
2. The left-invariant distribution symplectic-orthogonal to $T(G/J)$ is integrable.
3. Let $J^{\perp}$ the leaf through unit $e \in G$. It is an immersed Lie subgroup of G.

Co-splitings

Definition. The Lie group J is cosplit in G if the exact sequence

$$\{e\} \longrightarrow J \longrightarrow J^{\perp} \longrightarrow J^{\perp}/J \longrightarrow \{e\}$$

splits through a matched pair summand, H, of G.

Corollary. Under cosplit condition,

1. The natural action of G on $T^{\star}(J)$ induces a (measurable) isomorphism of G-spaces:

$$G/H \longrightarrow T^{\star}(J)$$

2. Considering a matched pair (H, L),

$$L \simeq G/H \longrightarrow T^{\star}(J)$$

is a G-equivariant measurable isomorphism (H acts by dressing).

The dual orbit condition

Let Q a locally compact group acting on a locally compact Abelian group V: $Q \times V \to V$, and set

$$G := Q \ltimes V$$

Definition. (Q, V) satisfy the dual orbit condition (DOC) if there exists an element $\eta \in \widehat{V}$ such that

$$Q \to \widehat{V} : q \mapsto q.\eta$$

is a measurable equivalence.

Proposition. When DOC, the map

$$G \to V \times \widehat{V} \; : \; g \mapsto g.(0, \eta)$$

is a measurable G-equivalence.

Definition. A locally compact quantum group (LCQG) is a quadruple $(\mathcal{M}, \Delta, \varphi_\ell, \varphi_r)$ where

1. $\mathcal{M}$ is a von Neumann algebra,
2. $\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ is a compatible co-product,
3. φ_ℓ and φ_r are weights on $\mathcal{M}^+$ ("Haar weights")

such that
for all positive linear functional ω on $\mathcal{M}^+$ and $x \in \mathcal{M}^+$, one has

$$\varphi_\ell(\omega \otimes 1(\Delta(x))) = \omega(1)\,\varphi_\ell(x) \qquad \text{(sim. for } \varphi_r)$$

Example. On G locally compact group ($\mathcal{M} = L^\infty(G)$)

$$\varphi_\ell(f) := \int_G f(g)\, \mathrm{d}g$$

Unitary 2-cocycles

Definition. Let $(\mathcal{M}, \Delta)$ be a von Neumann bi-algebra. A unitary 2-cocycle is a unitary $\hat{\Omega} \in \mathcal{M} \otimes \mathcal{M}$ such that

$$(\Delta \otimes I)(\hat{\Omega})\,(\hat{\Omega} \otimes 1) \;=\; (I \otimes \Delta)(\hat{\Omega})\,(1 \otimes \hat{\Omega})$$

Theorem [De Commer]. Let $(\mathcal{M}, \Delta, \varphi_\ell, \varphi_r)$ be a LCQG. Let $\hat{\Omega}$ be a unitary 2-cocycle on $(\mathcal{M}, \Delta)$. Set

$$\Delta_{\hat{\Omega}} \;:=\; \hat{\Omega}\,\Delta(.)\,\hat{\Omega}^*$$

Then $(\mathcal{M}, \Delta_{\hat{\Omega}})$ underlies a LCQG.

Microlocal operators

Let V be a finite dimensional real vector space. For every symbol $a \in S^m(T^\star(V))$, consider

$$\mathrm{Op}(a)\psi(q_0) \;:=\; \int_{T^\star(V)} a(q,p)\, e^{i<p,q-q_0>}\, \psi(q)\,\mathrm{d}q\,\mathrm{d}p$$

Operator symbol composition formula:

$$a \star b(x_0) \;=\; \int K(x,x',x_0)\, a(x)\, b(x')\,\mathrm{d}x\,\mathrm{d}x'$$

with

$$K(x,x',0) \;=\; e^{-i<p,q'>}\,\delta_0(q)\,\delta_0(p')$$

DOC groups [BGNT ; J. Funct. Analysis (2021)]

Theorem. Let $G = Q \ltimes V$ be a DOC group. Through the measurable equivalence $G \to V \times \widehat{V}$, the element

$$\hat{\Omega} := \int_{G\times G} K(x,x',0)\,\lambda_x \otimes \lambda_{x'}\,\mathrm{d}x\,\mathrm{d}x'$$
$$= \int_{Q\times\widehat{V}} e^{-i<p,q>}\,\lambda_{(q,0)} \otimes \lambda_{(e,p)}\,\mathrm{d}q\,\mathrm{d}p$$

is a unitary dual 2-cocycle based on $W^\star(G)$.

Formulas

In coordinates $G = \{g = (q, v)\}_{q \in Q\,,\, v \in V}$, the associated left-invariant star-product on G is

$$f_1 \star f_2(x_0) := \int_{G \times G} K(x, x', 0)\, f_1(x_0 x)\, f_2(x_0 x')\, \mathrm{d}x\, \mathrm{d}x' =$$

$$\int_{Q \times V} e^{i \langle q.\eta - \eta\,,\, v \rangle}\, f_1(q_0, q_0.v + v_0)\, f_2(q_0 q, b_0)\, \frac{\Delta_G(q, v)}{\Delta_Q(q)}\, \mathrm{d}q\, \mathrm{d}v$$

Here $G = GL_1(\mathbb{R}) \ltimes \mathbb{R} = \{(a,b)\}_{a\in\mathbb{R}^\times\,,\,b\in\mathbb{R}}$. The unitary dual 2-cocycle is

$$\hat{\Omega} \;:=\; \int_{\mathbb{R}^\times\times\mathbb{R}} e^{i\left(\frac{1-a}{a}\right)b}\,\lambda_{(1,b)}\otimes\lambda_{(a,0)}\,\frac{1}{|a|}\,\mathrm{d}a\,\mathrm{d}b$$

The Lie algebra is generated by two elements X and Y with $[X,Y]=Y$ and the formal twist associated to $\hat{\Omega}$ is

$$\hat{\Omega} \;\sim\; F \;:=\; e^{X\otimes\log(1+Y)}$$