QCD in higher orders

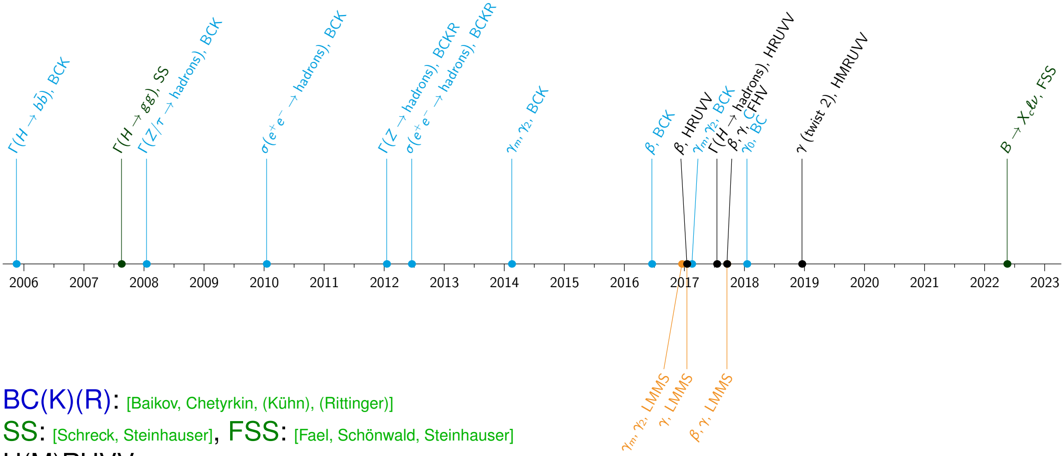
Peter Marquard

in collaboration with

A. Maier, Y. Schröder

Corfu, August 2024

Towards QCD at five loops



BC(K)(R): [Baikov, Chetyrkin, (Kühn), (Rittinger)]

SS: [Schreck, Steinhauser], FSS: [Fael, Schönwald, Steinhauser]

H(M)RUVV: [Herzog, (Moch), Ruijl, Ueda, Vermaseren, Vogt]

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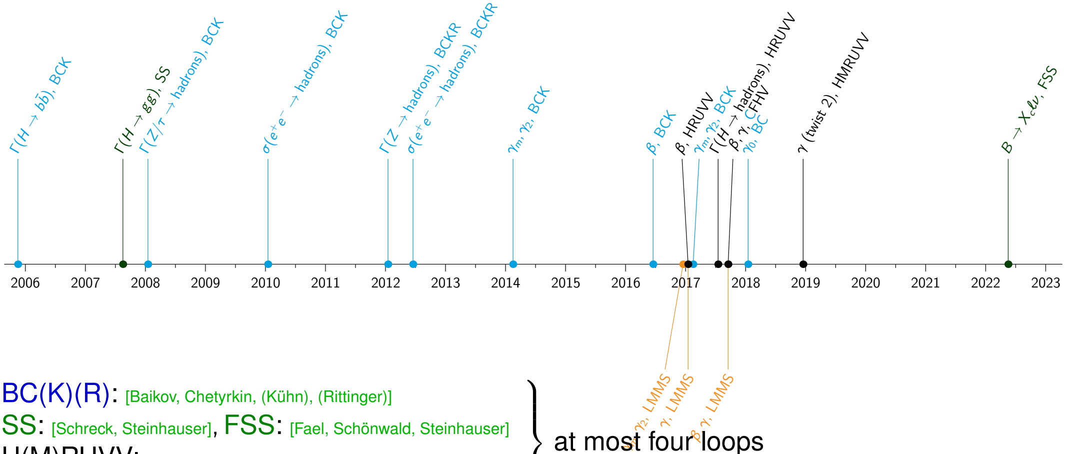

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} at most four loops

not quite QCD: massive gluons

What can we learn at five loops?

- Total decay widths, $\sigma(e^+e^- \to \text{hadrons})$ at order α_s^5
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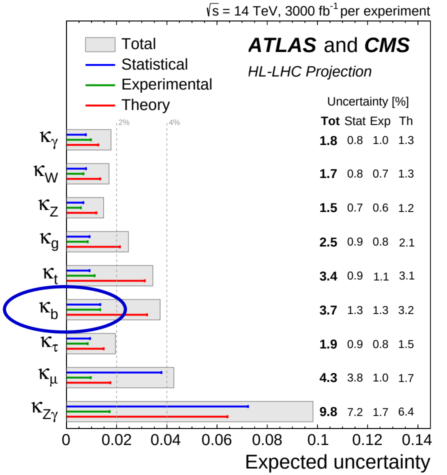
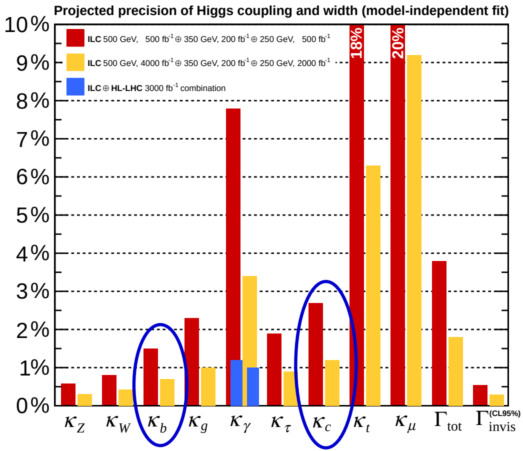
- Renormalisation group evolution at six loops, including five-loop decoupling
- Moments of $\sigma(e^+e^- \to \text{hadrons})$ at order α_s^5
  $\hookrightarrow$ heavy quark mass determinations
- ...

Why precise quark masses?

Higgs coupling measurements



[arXiv:1902.00134]



[arXiv:1506.05992]

Heavy quark masses at four loops

[Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009 + 2017]

$$m_b(10\,\text{GeV}) = \big(3610 \pm 10(\text{exp}) \pm 12(\alpha_s) \pm 3(\mu)\big)\,\text{MeV}$$

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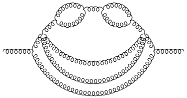
Need masses at five loops

Where to start?

1. Generate diagrams (`qgraf`) [Nogueira 1991]
2. Identify integral families (`nauty` and `Traces`) [McKay, Piperno 2014]
3. Insert Feynman rules (`FORM`) [Vermaseren et al.]
4. Reduce to master integrals
5. Calculate & insert master integrals

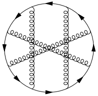
Where to start?

One momentum or one mass?



- Dedicated methods
  - Triangle / diamond reduction rules

    [Chetyrkin, Tkachov 1981; Ruijl, Ueda, Vermaseren 2015]
  - Graphical functions
  - ...
- 64 integral families
- 15 propagators
- 20 possible scalar products



- Standard reduction methods: Integration-by-parts reduction over finite fields

  [Chetyrkin, Tkachov 1981; Laporta 2000; von Manteuffel, Schabinger 2014; Peraro 2016; ...]
- 34 integral families
- 12 propagators
- 15 possible scalar products

Where to start?

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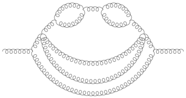

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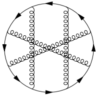

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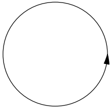
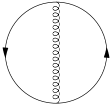
Heavy-quark condensate

$$\langle \bar{\psi}\psi \rangle =$$  $$+$$  $$+ \ \ldots$$

- Leading non-analytic contribution in
  - Operator Product Expansion ($m \sim \Lambda_{\rm QCD}$)
  - Asymptotic small-mass expansion ($m \gg \Lambda_{\rm QCD}$)

  Extrapolation from heavy to light quarks via
  Renormalisation Group Optimised Perturbation Theory [Kneur, Neveu 2010-2020]

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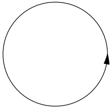
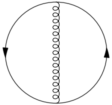
$\langle \bar{\psi}\psi \rangle =$  $+$  $+ \ \dots$

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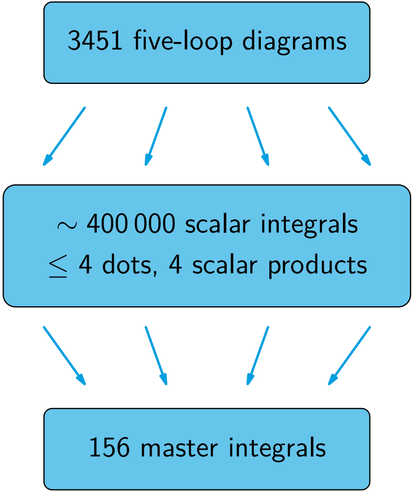
- Direct relation to vacuum anomalous dimension: [Spiridonov, Chetyrkin 1988]

$$\mu^2 \frac{d}{d\mu^2} m \langle \bar{\psi}\psi \rangle = -4 m^4 \gamma_0$$

$\hookrightarrow$ independent check of five-loop result [Baikov, Chetyrkin 2018]

Heavy-quark condensate

Calculation



Master integrals

Sector decomposition [Hepp 1966; Speer 1968; Binoth, Heinrich 2000]

Compute master integrals with `FIESTA`: [Smirnov et al. 2008-2021]

$$\langle\bar{\psi}\psi\rangle\Big|_{\left(\frac{\alpha_s}{\pi}\right)^4} = \frac{m^3}{16\pi^2}\Bigg[\frac{(-3.5\pm 3.0)\times 10^{-8}}{\epsilon^{11}} + \frac{(-2.1\pm 7.6)\times 10^{-6}}{\epsilon^{10}} + \frac{(0.2\pm 2.1)\times 10^{-4}}{\epsilon^{9}} + \frac{(-0.5\pm 7.5)\times 10^{-2}}{\epsilon^{8}} + \dots\Bigg]$$

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- $\sim$ 2 digits lost per order in ϵ $\Rightarrow$ need better than double precision
- Found no significant improvement with
  - Quasi-Monte Carlo methods [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk 2018]
  - Quasi-finite integral basis [von Manteuffel, Panzer, Schabinger 2014]
  - Above plus direct Feynman parameter integration
  - Tropical integration [Borinsky 2020]
  - `pySecDec` [Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk 2023]

Master integrals

Difference equations [Laporta 2000; Luthe, Schröder]

- Raise one denominator to symbolic power x:

$$M_i(x) = \int \frac{1}{D_1^{x} \cdots D_{15}^{\alpha_{15}}}$$

- IBP reduction $\to$ coupled difference equations in x

$$\sum_{j,k} q_{jk}(d,x) M_j(x+k) = 0$$

- Factorial series ansatz $\to$ recurrence relations for $a_{i,s}$

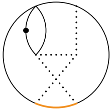
$$M_i(x) = \sum_{s=0}^{\infty} a_{i,s} \frac{\Gamma(x+1)}{\Gamma(x+s-d/2+1)}$$

- Boundary conditions from $x \to \infty$

High-precision numerical solution

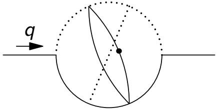
Master integrals

Direct numerical integration



$$= \int \frac{d^d q}{i\pi^{\frac{d}{2}}}$$

$$\times$$



Master integrals

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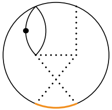
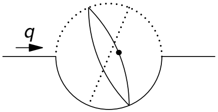


$$= -\int_0^{-\infty} \frac{dq^2}{\Gamma(2-\epsilon)} \left(-q^2\right)^{1-\epsilon}$$

$\times$



Integral is

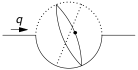
- IR finite: q only flows through massive lines
- UV finite: otherwise raise tree-level propagator to higher power

Alternative: introduce counterterms

Expansion in ϵ straightforward

Master integrals

Direct numerical integration

$P_0 =$ 

from differential equations for propagator master integrals P_i:

$$q^2 \frac{\partial}{\partial q^2} P_i = Q_{ij} P_j$$

[Kotikov 1991, Remiddi 1997,...]

Expansions around $q^2 \to 0$, $q^2 \to -\infty$ from power series ansätze:

$$P_i = \sum_{k=0}^{N-1} c_{ik} q^{2k} + \mathcal{O}(q^{2N}), \qquad P_i = \sum_{n=0}^{\#\text{loops}} \sum_{k=k_0}^{N-1} d_{ikn} \left(-\frac{1}{q^2}\right)^{k+n\epsilon} + \mathcal{O}\left(\frac{1}{(-q^2)^N}\right)$$

Boundary conditions $c_{i,0}$, d_{ik_0n}:
products of massive vacuum diagrams & massless propagators with ≤ 4 loops

[Schröder, Vuorinen 2005; Faisst, Maierhöfer, Sturm 2006; Baikov, Chetyrkin 2010; Lee, Smirnov, Smirnov 2011]

Master integrals

Padé approximation

Accelerate series convergence through Padé approximation $[n/m]$:

$$P_i = \sum_{k=0}^{N-1} c_{ik} q^{2k} + \mathcal{O}(q^{2N})$$

$$\Downarrow$$

$$P_i = [n/m] + \mathcal{O}(q^{2N}) \equiv \frac{a_0 + a_1 q^2 + \cdots + a_n q^{2n}}{1 + b_1 q^2 + \cdots + b_m q^{2m}} + \mathcal{O}(q^{2N})$$

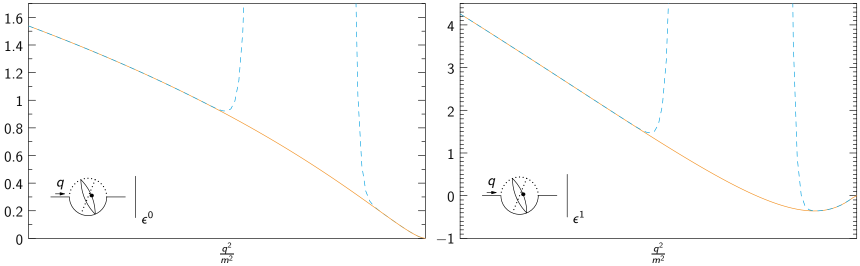
Further improvements: [Baikov, Broadhurst 1995]

- Combine low- and high-energy expansions into single Padé approximation
- Subtract high-energy logarithms $\ln^n(-q^2)$
- Conformal mapping $q^2 \to \frac{4\omega}{(1+\omega)^2} m^2$

See also [Faisst, Chetyrkin, Kühn 2004]

Master integrals

Result for $N = 20$ expansion terms



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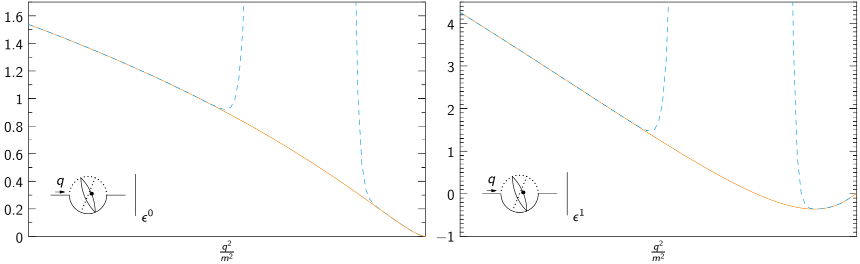

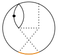 $= 5.8125309358416596949 - 31.572349480122869826\epsilon + \mathcal{O}(\epsilon^2)$

Checks:

- ✓ Change integration contour to $0 \to -i\infty \Rightarrow$ agrees within $10^{-19} + 2 \times 10^{-18}\epsilon$
- ✓ Agrees with result for $N = 25$ within uncertainty
- ✓ `FIESTA`: $5.81409 - 31.58070(87)\epsilon$
- ⚠ Compare with numerical solution of differential equations

Next Steps

Decoupling

Decoupling at five loops

Motivation

Top quarks are problematic for QCD processes with $E < m_t$:

- Diagrams with massive internal lines $\Rightarrow$ hard to calculate
- Large logarithms $\ln\left(\frac{E}{m_t}\right)$ spoil perturbative convergence

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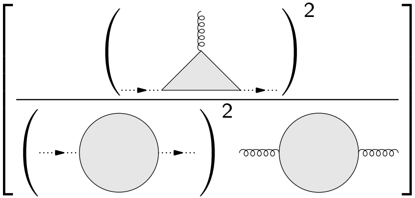
Solution: effective 5-flavour theory with coupling

$$\alpha_s^{(5)} = \alpha_s^{(6)} \times$$



$q{=}0$
$m_t{\neq}0$
$m_Q{=}0$

Relation known to four loops [Schröder, Steinhauser 2005]

Decoupling at five loops

Status

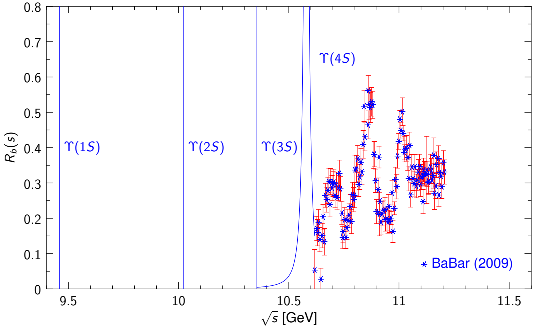
- 131 860 803 scalar integrals
- Up to 7 dots, 6 scalar products
- Reduction to master integrals with `crusher` + `tinbox` [Maier, Marquard, Seidel]

Reducing integral family 31/34

Next Steps

Quark Masses

Heavy Quark Mass Determination



Consider *moments* $\mathcal{M}_n$ of $R_Q(s) = \frac{\sigma(e^+e^- \to Q\bar{Q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$

$$\mathcal{M}_n = \int_{s_0}^{\infty} ds \, \frac{R_Q(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \underbrace{\left[\left(\frac{d}{dq^2}\right)^n \Pi_Q(q^2)\right]_{q^2=0}}_{\text{massive vacuum diagrams}}$$

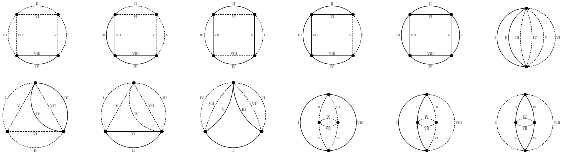
The first moment at five loops

First results

n_h closed heavy (massive) quark loops + n_l closed light (massless) quark loops

	n_l^0	n_l^1	n_l^2	n_l^3
n_h^1	✗	✗	✓	✓
n_h^2	✗	✗	✓	
n_h^3	✗	✓		
n_h^4	✓			

Master integrals:



$$\mathcal{M}_1^{5\text{ loop}} = \frac{3\pi^2}{m_Q^2}\left(\frac{\alpha_s}{\pi}\right)^4 n_h C_F \left[0.6\, T_F^3 n_l^3 + 1.2\, T_F^3 n_l^2 n_h + 0.9\, T_F^3 n_l n_h^2 + 0.2\, T_F^3 n_h^3 + (C_F - 5C_A) T_F^2 n_l^2 + \ldots\right.$$

Conclusion

Five-loop QCD work in progress

- Quark condensate
- Decoupling
- Charm & bottom mass determinations

Combination of methods for computing five-loop vacuum diagrams:

- Sector decomposition
- Recurrence equations
- Direct integration