QPT and SUSY in matrix model of SU(2) gauge theory

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Work done with:

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CORFU 2024

- Gague field (glue): M_{ia} (where $i = 1, 2, 3$ and $a = 1, 2, 3$).
- Chromoelectric Field: $E_{ia} = -i\frac{\partial}{\partial M_{ia}}$ $[M_{ia}, E_{jb}] = i\delta_{ij}\delta_{ab}$.
- Chromomagnetic field: $B_{ia} = \frac{1}{2}\epsilon_{ijk}F_{jk} = -M_{ia} - g\epsilon_{ijk}\epsilon_{abc}M_{jb}M_{kc}$
- Glue Hamiltonian:

$$\begin{aligned} H_{YM} &= \frac{1}{2}(E_{ia}E_{ia} + B_{ia}B_{ia}) = \text{ 9d-Harmonic osc. } + \text{cubic} + \text{quartic} \\ &= \frac{1}{2}E_{ia}^2 + \frac{1}{2}M_{ia}^2 - g\epsilon_{abc}\epsilon_{ijk}M_{ia}M_{jb}M_{kc} + \frac{g^2}{2}\epsilon_{abc}\epsilon_{ade}M_{ib}M_{jc}M_{id}M_{je} \end{aligned}$$

- Fermions in matrix model: ψ time-dependent Grassmann-valued matrices
- A left-weyl fermion transforming in adjoint rep of $SU(2)$:

$$\text{Gluino:} \quad \psi = \begin{pmatrix} b_{\alpha a} \\ 0 \end{pmatrix}, \qquad \text{spin-index: } \alpha = 1, 2,$$

$$\{b_{\alpha a}, b^{\dagger}_{\alpha' a'}\} = \delta_{\alpha\alpha'}\delta_{aa'} \qquad \text{color-index: } a = 1, 2, 3$$

- Total Hamiltonian Diez-Pandey-Vaidya (2020):

$$H = H_{YM} + H_f = H_{YM} + \underbrace{g\epsilon_{abc}b^{\dagger}_{\alpha a}\sigma^{i}_{\alpha\beta}b_{\beta b}M_{ic}}_{\text{fermion-glue interaction}} + b^{\dagger}_{\alpha a}b_{\alpha a}$$

Rotational and Gauge symmetries

	Under spatial rotations	Under color $SU(2)$
M_{ia}	spin-1 rep Generated by: $L_i \equiv -\epsilon_{ijk} E_{ja} M_{ka}$ $[L_i, L_j] = i\epsilon_{ijk} L_k$	adjoint rep Generated by: $G^a_g \equiv -\epsilon_{abc} \Pi_{ib} M_{ic}$ $[G^a_g, G^b_g] = i\epsilon_{abc} G^c_g$
ψ	spin-$1/2$ rep Generated by: $S_i \equiv \frac{1}{2} b^\dagger_{\alpha a} \sigma^i_{\alpha\beta} b_{\beta a}$ $[S_i, S_j] = i\epsilon_{ijk} S_k$	adjoint rep Generated by: $G^a_f \equiv -i\epsilon_{abc} b^\dagger_{\alpha b} b_{\alpha c}$ $[G^a_f, G^b_f] = i\epsilon_{abc} G^c_f$

$SO(3)_{rot}$ & color $SU(2)$ symmetry

- $J_i = L_i + S_i, \quad G^a = G^a_g + G^a_f$
- $[J_i, J_j] = i\epsilon_{ijk} J_k,$
- $[G^a, G^b] = i\epsilon_{abc} G^c$
- H commutes with J_i and G^a:

$$[H, J_i] = 0, \quad [H, G^a] = 0$$

- H also formally commutes with $N_f = b^\dagger_{\alpha a} b_{\alpha a}$:

$$[H, N_f] = 0$$

$U(1)_R$ NOT a symmetry: Anomaly

N_f is observable: we can find $\langle N_f \rangle$

Acharyya-Pandey-Vaidya (2021)

Physical Hilbert Space

Fermionic Hilbert space $\mathcal{H}_f$:

- Max. no of fermion number in a state=6
- No. of states=$2^6 = 64$.
- $\mathcal{H}_f$ is finite-dim: states can be arranged in reps of $\{S_i\}$ and $\{G_f^a\}$.

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Glue Hilbert space $\mathcal{H}_G$:

- $\mathcal{H}_G$ is infinite dimensional.
- $\mathcal{H}_G = \{\text{square-integrable } f(M_{ia})\}$
- inner-prod measure= $dM_{11}..dM_{33}$

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Physical Hilbert space $\mathcal{H}_{phys}$:

- $\mathcal{H}_{phys} \subset \mathcal{H}_f \otimes \mathcal{H}_G$
- $|\Psi\rangle \in \mathcal{H}_{phys}$ satisfies: $G^a|\Psi\rangle = 0$
- All states in $\mathcal{H}_{phys}$:
  - are color-singlets
  - can have spin $J = 0, \frac{1}{2}, 1 \ldots$
- spanned by color-singlet eigenstates of H

The colorless eigenstates of H can be labelled its spin J: $H|\Psi_n^J\rangle = E_n^J|\Psi_n^J\rangle$

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A variational ansatz to find E_n^J and $|\Psi_n^J\rangle$:

$$|\Psi_n^J\rangle = \sum_{k=0}^{\infty} \sum_{l} c_n^{l,k} |F_{\ldots}^l\rangle \otimes |\phi_{\ldots}^k\rangle$$

$$\{|F_{\ldots}^l\rangle\} = \text{basis of } \mathcal{H}_f$$

$$\{|\phi_{\ldots}^k\rangle\} = \text{basis of } \mathcal{H}_G = \text{eigenfunctions of 9d } H_{osc}$$

Truncated: $|\Psi_n^J\rangle = \sum_{k=0}^{N_{max}} \sum_{l} c_n^{l,k} |F_{\ldots}^l\rangle \otimes |\phi_{\ldots}^k\rangle$

Obtain $c_n^{l,k}$ by minimizing E_n^J
Increase N_{max} till E_n^J converged.

$\mathcal{N} = 1$ Supersymmetry

- Super charges: $\mathcal{Q}_\alpha = b^\dagger_{\beta a}\sigma^i_{\beta\alpha}(E_{ia} + iB_{ia}), \qquad \alpha = 1, 2$
- $[H, \mathcal{Q}_\alpha] = -igb^\dagger_{\alpha a}G_a$. $\Longleftarrow$ Commutes in $\mathcal{H}_{phys}$
- $\{\mathcal{Q}_\alpha, \mathcal{Q}^\dagger_\beta\} = \delta_{\alpha\beta}(2H + N_f - 3) - 2\sigma^i_{\beta\alpha}(J_i + M_{ia}G_a)$,
- For any energy eigenstate $|\Psi^J_n\rangle$ satisfying $H|\Psi^J_n\rangle = E^J_n|\Psi^J_n\rangle$:

$$\langle\Psi^J_n|\{\mathcal{Q}_\alpha, \mathcal{Q}^\dagger_\alpha\}|\Psi^J_n\rangle \geq 0 \implies \underbrace{E^J_n \geq -\frac{1}{2}(\langle N_f\rangle - 3) + J_3}_{\text{Bound saturated for any SUSY-Singlet}} \Longleftarrow$$

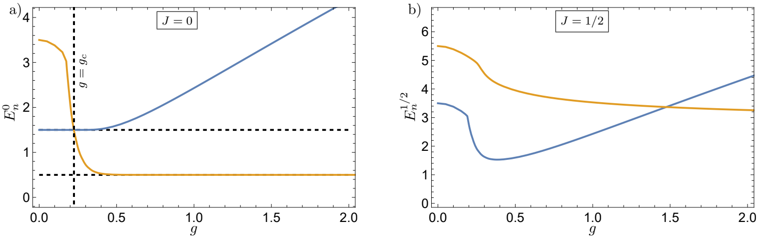
spectrum of H bounded from below

- Strong coupling limit: well studied both theoretically and numerically
- Numerical estimates of the spectrum of H at $g \to \infty$: flat directions and continuous spectra

  Campostrini-Wosiek (2004), Anous-Cogburn (2019), Han-Hartnoll (2020)...
- Theoretical computation of the Witten index at $g \to \infty$

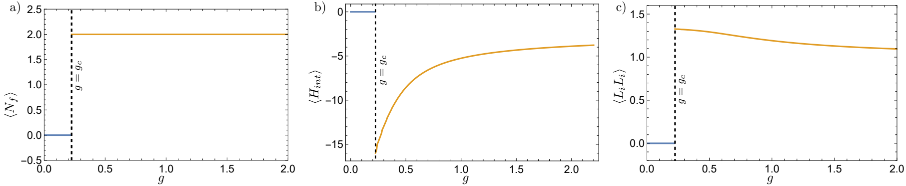
  Yi (1997), Sethi-Stern (1998) ...

- We construct the energy eigenstates for both weak and strong coupling regime
- Provide numerical evidence for:
  - QPT at weak coupling
  - a crossover to a non-supersymmetric phase in strong coupling



- We obtained the low-lying energy eigenstates for $J = 0$ and $J = 1/2$.
- The ground state:
  - has spin-0 and is unique
  - undergoes level crossing at $g = g_c \Longleftarrow$ Quantum Phase Transition
- Numerical estimate of $g_c \approx 0.225$.
- There is no other level crossing in the ground state even in the strong coupling regime

- The properties of the phases and QPT captured by ground state expectation of observables:

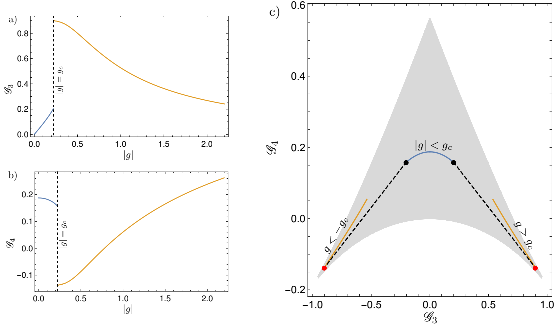


$g < g_c$	$g > g_c$
• $\langle N_f \rangle = 0$	• $\langle N_f \rangle = 2$
• "Non-interacting": $\langle H_{int} \rangle = 0$	• Interacting: $\langle H_{int} \rangle \neq 0$
• Only spin-0 glue: $\langle L_i L_i \rangle = 0$	• Glue with non-zero spin: $\langle L_i L_i \rangle \neq 0$

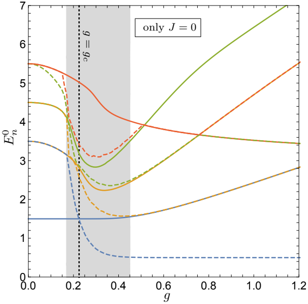
- The QPT is also captured in the third and fourth order Binder cumulants

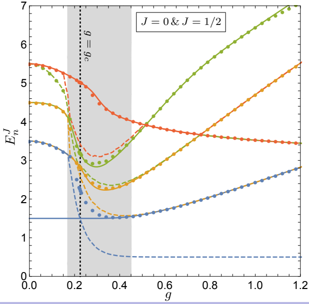
$$\mathscr{G}_3 \equiv \frac{\sqrt{3}}{2} \frac{\epsilon_{ijk}\epsilon_{abc}\langle M_{ia}M_{jb}M_{kc}\rangle}{\langle M_{ia}M_{ia}\rangle^{\frac{3}{2}}}, \qquad \mathscr{G}_4 \equiv \frac{9}{8}\left[\frac{\langle M_{ia}M_{ib}M_{ja}M_{jb}\rangle}{\langle M_{ia}M_{ia}\rangle^2} - \frac{1}{2}\right],$$

- For any random 3×3 Hermitian real matrix M_{ia} , the allowed values of $\mathscr{G}_3$ and $\mathscr{G}_4$ constrained to lie in shaded region Pandey-Vaidya (2016)
- The ground state expectation values is a curve in the shaded region
- Carries info about gauge configurations in the ground state.



- For $g < g_c$: gauge config localized near "center of arrow"
- For $g = g_c$: it jumps to the "corner"
- Corners correspond to special configurations: all three singular values become equal





Except near the QPT:

- Each excited state is 4-fold degenerate:
  two spin-0 states + one spin-1/2 doublet
  $\mathcal{N} = 1$ Super-multiplets
- The ground state has energy
  $E_{gs} = -\frac{1}{2}(\langle N_f \rangle - 3) \Leftarrow$ SUSY-singlet

Near the QPT:

- The levels get rearranged.
- This leads to lifting of the degeneracy of the multiplets in the neighbourhood of g_c

Away from g_c, the system is supersymmetric in both phases.

Witten index at small g:
as the spectrum is discrete and the ground state is unique bosonic state

$$W = \lim_{\beta \to \infty} (-1)^F e^{-\beta H} = 1$$

To study the strong coupling (large g) regime:

- Re-scale: $M_{ia} \to g^{-\frac{1}{3}} M_{ia}$ and $E_{ia} \to g^{\frac{1}{3}} E_{ia}$ Define: $\nu \equiv g^{-2/3}$.
- The Hamiltonian

$$
\begin{aligned}
H &\equiv \nu^{-1}\Big[\frac{1}{2}E_{ia}^2 + \frac{1}{2}\epsilon_{abc}\epsilon_{ade}M_{ib}M_{jc}M_{id}M_{je} + \epsilon_{abc}b^{\dagger}_{\alpha a}\sigma^{i}_{\alpha\beta}b_{\beta b}M_{ia} - \\
&\qquad \nu\epsilon_{ijk}\epsilon_{abc}M_{ia}M_{jb}M_{kc} + \nu b^{\dagger}_{\alpha a}b_{\alpha a} + \frac{\nu^2}{2}M_{ia}^2\Big], \\
&\equiv \nu^{-1}\widetilde{H}
\end{aligned}
$$

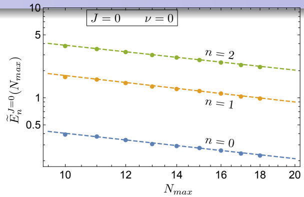
- We can now find the eigenvalues of $\widetilde{H}$
- Problem: Large g or small ν regimes are plagued with finite cut-off (finite N_{max}) error!
- Primary reason: M_{ia}^2 gets suppressed in $\widetilde{H}$.
- Most severe: At $\nu = 0$

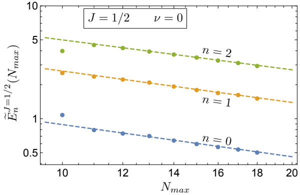
At $\nu = 0$:

- All low-lying eigenvalues of $\widetilde{H}$ has a power-law dependence on N_{max}:

$$\widetilde{E}_n \sim \frac{C_n}{(N_{max})^{\alpha}},$$

- Log-Log plots are fitted with line (dashed lines)
- Numerically obtained: $\alpha \approx 0.93$.





Nonetheless, $\nu = 0$

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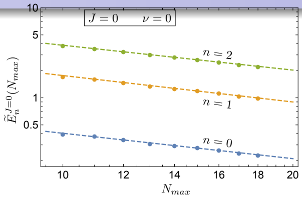
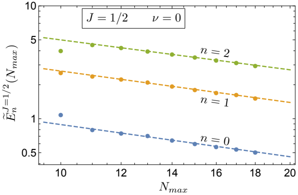
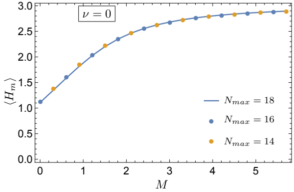




- Lightest spin-0 state 2-fold degenerate at $\nu = 0$.
- To understand the implications: add mH_m to $\widetilde{H}$ where $H_m = (\psi^T\gamma^2\gamma^0\psi + h.c)$

$$\lim_{m\to 0_+} \langle H_m \rangle \approx -1$$ Non-vanishing "Gluino condensate"



- Residual $\mathbb{Z}_2 \subset U(1)_R$ Witten (1982)

- The finite-cutoff error:

$$\Delta E_n(\nu) \equiv \widetilde{E}_n(\nu, N_{max}) - \mathcal{E}_n^J(\nu)$$

- $\Delta E_n \equiv (\nu)$ for all low-lying eigenvalues of $\widetilde{H}$ has a power-law dependence on N_{max}:

$$\Delta E_n(\nu) \sim \frac{D_n(\nu)}{(N_{max})^{\beta(\nu)}}$$

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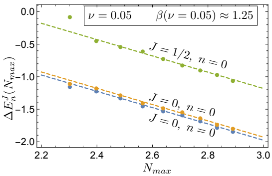

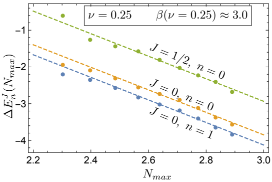

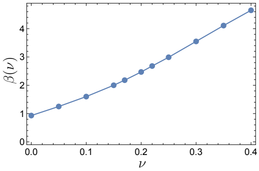

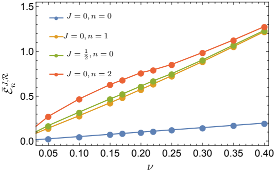

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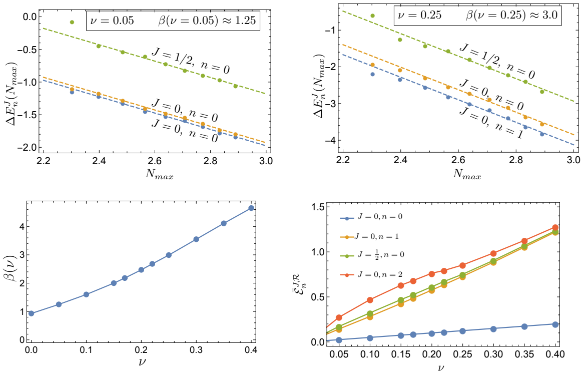

- $\mathcal{E}_n^J$ can be identified as:

$$\mathcal{E}_n^J(\nu) = \lim_{N_{max}\to\infty} \widetilde{E}_n(\nu, N_{max})$$

- The spectrum is discrete away from $\nu = 0$
- Lightest multiplet at large ν: breaking at small non-zero ν
- Cross-over to a non-supersymmetric phase
- SUSY reappears only at $\nu = 0$

- Weak and intermediate coupling:
  - QPT at $g_c \approx 0.225$
  - Observables are discontinuous at g_c
  - Away from the critical coupling, both phases are supersymmetric
  - In vicinity of g_c: SUSY breaks due rearrangement of levels
- At $\nu = 0$:
  - Power-law dependence of the energy eigenvalues
  - Non-zero Gluino Condensate
  - Continuous spectrum of H? Witten Index?
- Strong coupling regime: $\nu > 0$:
  - Spectrum is discrete
  - Lightest supermultiplet breaks: cross-over to a non-supersymmetric phase
  - Why it happens: Quantum anomalies? Smilga (1987), Casahorran-Esteve (1992) . . .

Thank You