# $\mathcal{N} = 4$ Supergravity with Local Scaling Symmetry in Four Dimensions

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# Outline

# Introduction

The first instances of four-dimensional pure $\mathcal{N} = 4$ supergravities were constructed more than 40 years ago by [Das (1977), Cremmer and Scherk (1977), Cremmer, Scherk and Ferrara (1978), Freedman and Schwarz (1978)].

The coupling of $\mathcal{N} = 4$ supergravity to vector multiplets, as well as some of its gaugings, were analyzed a few years later, by [de Roo (1985), Bergshoeff, Koh and Sezgin (1985), de Roo and Wagemans (1985), Perret (1988)].

More recently, various gauged $\mathcal{N} = 4$ supergravity models originating from orientifold compactifications of type IIA or IIB supergravity were studied [D'Auria, Ferrara and Vaula (2002), D'Auria, Ferrara, Gargiulo, Trigiante and Vaula (2003), Angelantonj, Ferrara and Trigiante (2003,2004), Dall'Agata, Villadoro and Zwirner (2009)].

A systematic parametrization of all the consistent gaugings of four-dimensional $\mathcal{N} = 4$ matter-coupled supergravity is provided by [Schön and Weidner (2006)] by means of an appropriately constrained embedding tensor.

The full Lagrangian for the most general gauged $D = 4$, $\mathcal{N} = 4$ matter-coupled supergravity in an arbitrary symplectic frame is given by [Dall'Agata, Liatsos, Noris and Trigiante (2023)].

**Objective**: construction of all possible gaugings of $D = 4$, $\mathcal{N} = 4$ supergravity coupled to an arbitrary number $n$ of vector multiplets that involve the global scaling symmetry $\mathbb{R}^+$ of the equations of motion of the ungauged theory, in addition to a subgroup of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, n)$.

**Earliest instance of a supergravity theory with local scaling symmetry:** massive $10D$ IIA theory constructed by [Howe, Lambert and West (1998), Lavrinenko, Lu and Pope (1998)] by a generalized dimensional reduction [Scherk and Schwarz (1979)] of $11D$ supergravity, **different** from Romans' massive IIA supergravity [Romans (1986)].

Later, $9D$ and $6D$ supergravity theories with local scaling symmetry were constructed by [Bergshoeff, de Wit, Gran, Linares and Roest (2002)] and [Kerimo and Lu (2003), Kerimo, Liu, Lu and Pope (2004)] respectively.

A general framework for the construction of supergravity theories with local scaling symmetry that makes use of the embedding tensor formalism was established by [Le Diffon and Samtleben (2009)]. Such theories **do not** posses an action.

We use this formalism to construct the most general $D = 4\,, \mathcal{N} = 4$ supergravity theory coupled to $n$ vector multiplets with a gauge symmetry that is the direct product of a subgroup of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, n)$ and the on-shell scaling symmetry of the corresponding ungauged theory.

# The Ingredients of $D=4\,,\mathcal{N}=4$ Supergravity

$\mathcal{N}=4$ supergravity multiplet:

- graviton $g_{\mu\nu}$
- 4 gravitini $\psi^i_\mu$, $i=1,\ldots,4$
- 6 vector fields $A^{ij}_\mu = -A^{ji}_\mu$
- 4 spin-1/2 fermions $\chi^i$ (dilatini)
- 1 complex scalar $\tau$

$n$ vector multiplets:

- $n$ vector fields $A^{\underline{a}}_\mu$, $\underline{a}=1,\ldots,n$
- $4n$ gaugini $\lambda^{\underline{a}i}$
- $6n$ real scalar fields $\phi^{\underline{a}\underline{m}}$, $\underline{m}=1,\ldots,6$

# The scalar sector of the supergravity multiplet

The complex scalar of the $\mathcal{N} = 4$ supergravity multiplet parametrizes the coset space SL(2,$\mathbb{R}$)/SO(2).

**Coset representative:** complex SL(2,$\mathbb{R}$) vector $\mathcal{V}_\alpha$, $\alpha = +, -$, which satisfies

$$\mathcal{V}_\alpha \mathcal{V}^*_\beta - \mathcal{V}^*_\alpha \mathcal{V}_\beta = -2i\epsilon_{\alpha\beta} \, , \tag{1}$$

where $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{+-} = 1$.

$\mathcal{V}_\alpha$ carries SO(2) charge $+1$.

We also define

$$M_{\alpha\beta} = \mathrm{Re}(\mathcal{V}_\alpha \mathcal{V}^*_\beta) \, . \tag{2}$$

# The scalar sector of the vector multiplets

The $6n$ real scalars of the $n$ vector multiplets parametrize the coset space SO(6,$n$)/(SO(6)$\times$SO($n$)).

**Coset representative:** $(n+6) \times (n+6)$ matrix $L$ with entries $L_M{}^{\underline{M}} = (L_M{}^{\underline{m}}, L_M{}^{\underline{a}})$, where $M = 1, \ldots, n+6$, $\underline{m} = 1, \ldots, 6$, $\underline{a} = 1, \ldots, n$, which is an element of SO(6,$n$):

$$\eta_{MN} = \eta_{\underline{MN}} L_M{}^{\underline{M}} L_N{}^{\underline{N}} = L_M{}^{\underline{M}} L_{N\underline{M}} = L_M{}^{\underline{m}} L_{N\underline{m}} + L_M{}^{\underline{a}} L_{N\underline{a}} \,, \quad (3)$$

where $\eta_{MN} = \eta_{\underline{MN}} = \mathrm{diag}(-1,-1,-1,-1,-1,-1,1,\ldots,1)$.

We also introduce the positive definite symmetric matrix $M = LL^T$ with elements

$$M_{MN} = -L_M{}^{\underline{m}} L_{N\underline{m}} + L_M{}^{\underline{a}} L_{N\underline{a}} \,. \quad (4)$$

We can trade $L_M{}^{\underline{m}}$ for the antisymmetric SU(4) tensors $L_M{}^{ij} = -L_M{}^{ji}$, $i, j = 1, \ldots, 4$, defined by

$$L_M{}^{ij} = \Gamma_{\underline{m}}{}^{ij} L_M{}^{\underline{m}}, \tag{5}$$

where $\Gamma_{\underline{m}}{}^{ij}$ are six antisymmetric $4\times4$ matrices that realize the isomorphism between the fundamental representation of SO(6) and the twofold antisymmetric representation of SU(4).

$$\text{Pseudoreality} : L_{Mij} = (L_M{}^{ij})^* = \frac{1}{2}\epsilon_{ijkl} L_M{}^{kl} \tag{6}$$

# Fermionic fields

| Field | SO(2) charge |
|---|---|
| $\psi_\mu^i$ | $-\frac{1}{2}$ |
| $\chi^i$ | $+\frac{3}{2}$ |
| $\lambda^{ai}$ | $+\frac{1}{2}$ |

$$\gamma_5 \psi_\mu^i = \psi_\mu^i, \quad \gamma_5 \chi^i = -\chi^i, \quad \gamma_5 \lambda^{ai} = \lambda^{ai}. \tag{7}$$

$\psi_{i\mu} = (\psi_\mu^i)^c$, $\chi_i = (\chi^i)^c$ and $\lambda_i^a = (\lambda^{ai})^c$ have opposite SO(2) charges and chiralities.

# Symplectic frames

The Lagrangian describing the ungauged four-dimensional $\mathcal{N}=4$ Poincaré supergravity coupled to $n$ vector multiplets contains $n+6$ abelian vector fields $A_\mu^\Lambda$, $\Lambda = 1, \ldots, n+6$, referred to as **electric vectors**.

These fields combine with their **magnetic duals**, $A_{\Lambda\mu}$, into an SL(2,$\mathbb{R}$) $\times$ SO(6,$n$) vector $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha}$, which is also a symplectic vector of Sp(2($n+6$),$\mathbb{R}$)$\supset$ SL(2,$\mathbb{R}$) $\times$ SO(6,$n$).

Every electric/magnetic split $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha} = (A_\mu^\Lambda, A_{\Lambda\mu})$ such that the symplectic form

$$\mathbb{C}^{\mathcal{M}\mathcal{N}} = \mathbb{C}^{M\alpha N\beta} \equiv \eta^{MN}\epsilon^{\alpha\beta} \tag{8}$$

decomposes as

$$\mathbb{C}^{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \mathbb{C}^{\Lambda\Sigma} & \mathbb{C}^{\Lambda}{}_{\Sigma} \\ \mathbb{C}_{\Lambda}{}^{\Sigma} & \mathbb{C}_{\Lambda\Sigma} \end{pmatrix} = \begin{pmatrix} 0 & \delta^\Lambda_\Sigma \\ -\delta^\Sigma_\Lambda & 0 \end{pmatrix}, \tag{9}$$

defines a symplectic frame and any two symplectic frames are related by a symplectic rotation that is an element of $\mathrm{Sp}(2(n+6),\mathbb{R})$.

# Gauging the Scaling Symmetry

The on-shell global symmetry group of the ungauged $D = 4$, $\mathcal{N} = 4$ supergravity coupled to $n$ vector multiplets is

$$G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, n) \times \mathbb{R}^+ \,, \tag{10}$$

where $\mathbb{R}^+$ denotes the scaling (or trombone) symmetry of the equations of motion, under which the various fields transform as

$$\delta g_{\mu\nu} = 2\lambda g_{\mu\nu} \,, \qquad \delta A_\mu^{\mathcal{M}} = \lambda A_\mu^{\mathcal{M}}, \tag{11}$$

$$\delta\tau = 0 \,, \qquad \delta\phi^{\underline{am}} = 0 \,, \tag{12}$$

$$\delta\psi_\mu^i = \frac{1}{2}\lambda\psi_\mu^i \,, \qquad \delta\chi^i = -\frac{1}{2}\lambda\chi^i, \qquad \delta\lambda^{\underline{a}i} = -\frac{1}{2}\lambda\lambda^{\underline{a}i}, \tag{13}$$

Generators of $G$:

$$t_{\hat{A}} = (t_0, t_A)\,, \tag{14}$$

$t_0$ : generator of $\mathbb{R}^+$,
$t_A$ : generators of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,n)$,

where $A = ([MN], (\alpha\beta))$ is an index labeling the adjoint representation of SL(2,$\mathbb{R}$)$\times$SO(6,$n$).

# Embedding tensor

In the embedding tensor formalism [Nicolai and Samtleben (2001), de Wit, Samtleben and Trigiante (2003,2005,2007)], the generators of the gauge group, $G_g \subset G$, are expressed as

$$X_{\mathcal{M}} = \hat{\Theta}_{\mathcal{M}}{}^{\hat{A}} t_{\hat{A}} = \hat{\Theta}_{\mathcal{M}}{}^{0} t_0 + \hat{\Theta}_{\mathcal{M}}{}^{A} t_A \,, \tag{15}$$

where $\hat{\Theta}_{\mathcal{M}}{}^{\hat{A}}$ is the embedding tensor.

We also introduce vector gauge fields $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha}$, and the gauge covariant exterior derivative

$$\hat{d} = d - gA^{\mathcal{M}} X_{\mathcal{M}} \,, \tag{16}$$

where $g$ is the gauge coupling and $A^{\mathcal{M}} = A_\mu^{\mathcal{M}} dx^\mu$.

Ansatz for embedding tensor [LeDiffon and Samtleben (2009)]:

$$\hat{\Theta}_{\mathcal{M}}{}^{NP} = \Theta_{\mathcal{M}}{}^{NP} + \zeta_1 (t^{NP})_{\mathcal{M}}{}^{\mathcal{Q}} \theta_{\mathcal{Q}}\,, \tag{17}$$

$$\hat{\Theta}_{\mathcal{M}}{}^{\beta\gamma} = \Theta_{\mathcal{M}}{}^{\beta\gamma} + \zeta_2 (t^{\beta\gamma})_{\mathcal{M}}{}^{\mathcal{Q}} \theta_{\mathcal{Q}}\,, \tag{18}$$

$$\hat{\Theta}_{\mathcal{M}}{}^{0} = \theta_{\mathcal{M}}\,, \tag{19}$$

where

- $\Theta_{\mathcal{M}}{}^{A} = (\Theta_{\mathcal{M}}{}^{NP}, \Theta_{\mathcal{M}}{}^{\beta\gamma})$ is the embedding tensor parametrizing the standard gaugings of $D = 4$, $\mathcal{N} = 4$ supergravity, which do not involve the scaling symmetry. It is built out of $f_{\alpha MNP} = f_{\alpha[MNP]}$ and $\xi_{\alpha M}$ [Schön and Weidner (2006)].
- $\zeta_1$ and $\zeta_2$ are real constants.
- $(t_{PQ})_{M\alpha}{}^{N\beta} = \delta^{N}_{[P}\eta_{Q]M}\delta^{\beta}_{\alpha}$ , $(t_{\gamma\delta})_{M\alpha}{}^{N\beta} = \delta^{\beta}_{(\gamma}\epsilon_{\delta)\alpha}\delta^{N}_{M}$.

The non-abelian two-form field strengths $H^{\mathcal{M}}$ of the vector gauge fields $A^{\mathcal{M}}$ involve Stueckelberg-type terms of the form [de Wit, Samtleben (2005)]

$$H^{\mathcal{P}} \supset g Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}} B^{\mathcal{M}\mathcal{N}}, \tag{20}$$

where $B^{\mathcal{M}\mathcal{N}} = B^{(\mathcal{M}\mathcal{N})}$ are two-form gauge fields and

$$Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}} \equiv X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{P}}, \tag{21}$$

where

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \equiv \hat{\Theta}_{\mathcal{M}}{}^{\hat{A}}(t_{\hat{A}})_{\mathcal{N}}{}^{\mathcal{P}} = -\theta_{\mathcal{M}}\delta^{\mathcal{P}}_{\mathcal{N}} + \hat{\Theta}_{\mathcal{M}}{}^{A}(t_A)_{\mathcal{N}}{}^{\mathcal{P}}\,. \tag{22}$$

$Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}}$ must project onto the adjoint representation of SL(2,$\mathbb{R}$) $\times$ SO(6,$n$), $(\mathbf{3},\mathbf{1}) + \left(\mathbf{1}, \frac{1}{2}(\mathbf{n}+\mathbf{6})(\mathbf{n}+\mathbf{5})\right)$, in its lower indices, $(\mathcal{M}\mathcal{N})$.

Since the two-fold symmetric tensor product of the fundamental representation of SL(2,$\mathbb{R}$) $\times$ SO(6,$n$), $(\mathbf{2}, \mathbf{n}+\mathbf{6})$, decomposes as

$$\begin{aligned} &((\mathbf{2},\mathbf{n}+\mathbf{6}) \times (\mathbf{2},\mathbf{n}+\mathbf{6}))_{\text{sym.}} \\ &= \left(\mathbf{3}, \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{n}+\mathbf{6})(\mathbf{n}+\mathbf{7}) - \mathbf{1}\right) \\ &+ (\mathbf{3},\mathbf{1}) + \left(\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{n}+\mathbf{6})(\mathbf{n}+\mathbf{5})\right), \end{aligned} \tag{23}$$

the projection of $Z^{\mathcal{P}}{}_{\mathcal{MN}}$ onto the representation $\left(\mathbf{3}, \frac{1}{2}(\mathbf{n}+\mathbf{6})(\mathbf{n}+\mathbf{7})-\mathbf{1}\right)$ must vanish, i.e.

$$Z^{P\gamma}{}_{(M(\alpha|N)\beta)} - \frac{1}{n+6}\eta_{MN}\eta^{RS}Z^{P\gamma}{}_{R(\alpha|S|\beta)} = 0\,, \qquad (24)$$

which is satisfied if

$$\zeta_1 + \zeta_2 = -2\,. \qquad (25)$$

Without loss of generality, we set $\zeta_1 = \zeta_2 = -1$.

Then,

$$\hat{\Theta}_{\alpha M}{}^{NP} = f_{\alpha M}{}^{NP} + \delta_M^{[N}\xi_\alpha^{P]} + \delta_M^{[N}\theta_\alpha^{P]}, \tag{26}$$

$$\hat{\Theta}_{\alpha M}{}^{\beta\gamma} = \delta_\alpha^{(\beta}\xi_M^{\gamma)} - \delta_\alpha^{(\beta}\theta_M^{\gamma)}, \tag{27}$$

$$\hat{\Theta}_{\alpha M}{}^{0} = \theta_{\alpha M}, \tag{28}$$

$$\begin{aligned} X_{M\alpha N\beta}{}^{P\gamma} = & -\delta_\beta^\gamma f_{\alpha MN}{}^P + \frac{1}{2}(\delta_M^P\delta_\beta^\gamma\xi_{\alpha N} - \delta_N^P\delta_\alpha^\gamma\xi_{\beta M} \\ & - \eta_{MN}\delta_\beta^\gamma\xi_\alpha^P + \delta_N^P\epsilon_{\alpha\beta}\xi_M^\gamma) \\ & - \delta_N^P\delta_\beta^\gamma\theta_{\alpha M} + \frac{1}{2}(\delta_M^P\delta_\beta^\gamma\theta_{\alpha N} + \delta_N^P\delta_\alpha^\gamma\theta_{\beta M} \\ & - \eta_{MN}\delta_\beta^\gamma\theta_\alpha^P - \delta_N^P\epsilon_{\alpha\beta}\theta_M^\gamma). \end{aligned} \tag{29}$$

$$Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}} = Z^{\mathcal{P}A}(t_A)_{\mathcal{M}\mathcal{N}}\,, \tag{30}$$

where

$$Z^{M\alpha NP} = -\frac{1}{2}\Theta^{\alpha MNP} + \frac{3}{2}\eta^{M[N|}\theta^{\alpha|P]}, \tag{31}$$

$$Z^{M\alpha\beta\gamma} = \frac{1}{2}\epsilon^{\alpha(\beta}\left(\xi^{\gamma)M} + \theta^{\gamma)M}\right). \tag{32}$$

# Quadratic constraints

The embedding tensor $\hat{\Theta}_{\mathcal{M}}{}^{\hat{A}}$ must be gauge invariant [LeDiffon and Samtleben (2009)]:

$$0 = \hat{\Theta}_{\mathcal{M}}{}^{\hat{A}} t_{\hat{A}} \theta_{\mathcal{N}} = X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \theta_{\mathcal{P}} \,, \qquad (33)$$

$$0 = \hat{\Theta}_{\mathcal{M}}{}^{\hat{A}} t_{\hat{A}} \Theta_{\mathcal{N}}{}^{B} = X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \Theta_{\mathcal{P}}{}^{B} + \hat{\Theta}_{\mathcal{M}}{}^{A} \Theta_{\mathcal{N}}{}^{C} f_{AC}{}^{B} , \qquad (34)$$

$f_{AB}{}^{C}$: the structure constants of the Lie algebra of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, n)$.

The constraints (33) and (34) imply the closure of the gauge algebra:

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}} \,, \qquad (35)$$

and are equivalent to the following quadratic constraints on $f_{\alpha MNP}$, $\xi_{\alpha M}$ and $\theta_{\alpha M}$:

$$\epsilon^{\alpha\beta}\xi_{\alpha(M|}\theta_{\beta|N)} = 0\,, \tag{36}$$

$$\epsilon^{\alpha\beta}\left(\theta_\alpha^P f_{\beta MNP} + \xi_{\alpha[M|}\theta_{\beta|N]} - 3\,\theta_{\alpha M}\theta_{\beta N}\right) = 0\,, \tag{37}$$

$$\theta_{(\alpha}^P f_{\beta)MNP} + \xi_{(\alpha[M}\theta_{\beta)N]} = 0\,, \tag{38}$$

$$\xi_{(\alpha}^M\theta_{\beta)M} + \theta_\alpha^M\theta_{\beta M} = 0\,, \tag{39}$$

$$\xi_{(\alpha}^P f_{\beta)MNP} - \xi_{(\alpha[M}\theta_{\beta)N]} = 0\,, \tag{40}$$

$$\xi_{(\alpha}^M\theta_{\beta)M} + \xi_\alpha^M\xi_{\beta M} = 0\,, \tag{41}$$

$$\epsilon^{\alpha\beta}\left(\xi_\alpha^P f_{\beta MNP} + \xi_{\alpha M}\xi_{\beta N} - 3\xi_{\alpha[M|}\theta_{\beta|N]}\right) = 0\,, \tag{42}$$

$$3f_{\alpha[MN|R}f_{\beta|PQ]}{}^R + 2\xi_{(\alpha[M}f_{\beta)NPQ]} + 2\theta_{(\alpha[M}f_{\beta)NPQ]} = 0\,, \tag{43}$$

$$\epsilon^{\alpha\beta}\theta_{\alpha[M|}f_{\beta|NPQ]} = 0\,, \tag{44}$$

$$\epsilon^{\alpha\beta}\big(f_{\alpha MNR}f_{\beta PQ}{}^{R} - \xi_{\alpha[M|}f_{\beta|N]PQ} + \xi_{\alpha[P|}f_{\beta|Q]MN} + \theta_{\alpha[M|}f_{\beta|N]PQ} - \theta_{\alpha[P|}f_{\beta|Q]MN} + \xi_{\alpha[M|}\xi_{\beta[P}\eta_{Q]N]} - \xi_{\alpha[M|}\theta_{\beta[P}\eta_{Q]N]} + \xi_{\alpha[P|}\theta_{\beta[M}\eta_{N]Q]} - 3\theta_{\alpha[M|}\theta_{\beta[P}\eta_{Q]N]}\big) = 0\,. \tag{45}$$

For $\theta_{\alpha M} = 0$, the quadratic constraints (36)-(45) consistently reduce to those of [Schön and Weidner (2006)].

# Gauge covariant field strengths

Gauge covariant 2-form field strengths of vector gauge fields [de Wit, Samtleben and Trigiante (2005)]:

$$
\begin{aligned}
H^{M\alpha} &= dA^{M\alpha} + \frac{g}{2} X_{N\beta P\gamma}{}^{M\alpha} A^{N\beta} \wedge A^{P\gamma} + g Z^{M\alpha A} B_A \\
&= dA^{M\alpha} + \frac{g}{2} X_{N\beta P\gamma}{}^{M\alpha} A^{N\beta} \wedge A^{P\gamma} \\
&\quad - \frac{g}{2} \Theta^{\alpha M}{}_{NP} B^{NP} + \frac{3}{2} g \theta^{\alpha}_{N} B^{MN} + \frac{g}{2} \left( \xi^{M}_{\beta} + \theta^{M}_{\beta} \right) B^{\alpha\beta},
\end{aligned}
\tag{46}
$$

where $B^{MN} = B^{[MN]}$ and $B^{\alpha\beta} = B^{(\alpha\beta)}$ are 2-form gauge fields in the adjoint representations of SO$(6,n)$ and SL$(2,\mathbb{R})$ respectively.

# Scalar sector

$$\text{gauged SL}(2,\mathbb{R})/\text{SO}(2)\text{ zweibein} : \hat{P} = \frac{i}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha \hat{d}\mathcal{V}_\beta\,, \qquad (47)$$

$$\text{gauged SO}(2)\text{ connection} : \hat{\mathcal{A}} = -\frac{1}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha \hat{d}\mathcal{V}^*_\beta\,, \qquad (48)$$

where

$$\hat{d}\mathcal{V}_\alpha \equiv d\mathcal{V}_\alpha + \frac{1}{2}g\left(\xi_{\alpha M} - \theta_{\alpha M}\right)A^{M\beta}\mathcal{V}_\beta + \frac{1}{2}g\left(\xi^{\beta M} - \theta^{\beta M}\right)A_{M\alpha}\mathcal{V}_\beta\,. \qquad (49)$$

gauged $\mathrm{SO}(6,n)/(\mathrm{SU}(4)\times\mathrm{SO}(n))$

$$\text{vielbein} : \hat{P}_{\underline{a}}{}^{ij} = L^{M}{}_{\underline{a}}\hat{d}L_{M}{}^{ij}, \tag{50}$$

$$\text{gauged SU(4) connection} : \hat{\omega}^{i}{}_{j} = L^{Mik}\hat{d}L_{Mjk}\,, \tag{51}$$

$$\text{gauged SO}(n)\text{ connection} : \hat{\omega}_{\underline{a}}{}^{\underline{b}} = L^{M}{}_{\underline{a}}\hat{d}L_{M}{}^{\underline{b}}, \tag{52}$$

where

$$\hat{d}L_{M}{}^{\underline{M}} \equiv dL_{M}{}^{\underline{M}} + gA^{N\alpha}\hat{\Theta}_{\alpha NM}{}^{P}L_{P}{}^{\underline{M}}. \tag{53}$$

# Supersymmetry Transfromation Rules

The $\mathcal{N} = 4$ local supersymmetry transformations of the bosonic fields $e^a_\mu$, $\mathcal{V}_\alpha$, $L_{Mij}$, $L_{M\underline{a}}$ and $A^{M\alpha}_\mu$ are the same as in the ungauged theory [Dall'Agata, Liatsos, Noris and Trigiante (2023)]:

$$\delta_\epsilon e^a_\mu = \bar{\epsilon}^i \gamma^a \psi_{i\mu} + \bar{\epsilon}_i \gamma^a \psi^i_\mu \,, \tag{54}$$

$$\delta_\epsilon \mathcal{V}_\alpha = \mathcal{V}^*_\alpha \bar{\epsilon}_i \chi^i , \tag{55}$$

$$\delta_\epsilon L_{Mij} = L_{M\underline{a}} (2\bar{\epsilon}_{[i} \lambda^{\underline{a}}_{j]} + \epsilon_{ijkl} \bar{\epsilon}^k \lambda^{\underline{a}l}) \,, \tag{56}$$

$$\delta_\epsilon L_M{}^{\underline{a}} = 2 L_M{}^{ij} \bar{\epsilon}_i \lambda^{\underline{a}}_j + c.c. \,, \tag{57}$$

$$\begin{aligned} \delta_\epsilon A^{M\alpha}_\mu = & (\mathcal{V}^\alpha)^* L^M{}_{ij} \bar{\epsilon}^i \gamma_\mu \chi^j - \mathcal{V}^\alpha L^{M\underline{a}} \bar{\epsilon}^i \gamma_\mu \lambda_{\underline{a}i} \\ & + 2 \mathcal{V}^\alpha L^M{}_{ij} \bar{\epsilon}^i \psi^j_\mu + c.c. \,, \end{aligned} \tag{58}$$

while the corresponding transformations of the linear combinations

$$B^{M\alpha}_{\mu\nu} \equiv -\frac{1}{2}\Theta^{\alpha M}{}_{NP} B^{NP}_{\mu\nu} + \frac{3}{2}\theta^{\alpha}_{N} B^{MN}_{\mu\nu} + \frac{1}{2}\left(\xi^{M}_{\beta} + \theta^{M}_{\beta}\right) B^{\alpha\beta}_{\mu\nu} \quad (59)$$

of the antisymmetric tensor gauge fields read

$$
\begin{aligned}
\delta_\epsilon B_{\mu\nu}^{M\alpha} = & -4iZ^{M\alpha NP} L_N{}^{\underline{a}} L_P{}^{ij} \bar{\epsilon}_i \gamma_{\mu\nu} \lambda_{\underline{a}j} \\
& + \frac{1}{2}\left(\xi_\beta^M + \theta_\beta^M\right) (\mathcal{V}^\alpha)^* (\mathcal{V}^\beta)^* \bar{\epsilon}_i \gamma_{\mu\nu} \chi^i \\
& + 4iZ^{M\alpha NP} L_N{}^{\underline{a}} L_{Pij} \bar{\epsilon}^i \gamma_{\mu\nu} \lambda_{\underline{a}}^j \\
& + \frac{1}{2}\left(\xi_\beta^M + \theta_\beta^M\right) \mathcal{V}^\alpha \mathcal{V}^\beta \bar{\epsilon}^i \gamma_{\mu\nu} \chi_i \\
& + 8iZ^{M\alpha NP} L_N{}^{ik} L_{Pjk} \left(\bar{\epsilon}^j \gamma_{[\mu|} \psi_{i|\nu]} + \bar{\epsilon}_i \gamma_{[\mu} \psi_{\nu]}^j\right) \\
& + \left(\xi_\beta^M + \theta_\beta^M\right) M^{\alpha\beta} \left(\bar{\epsilon}^i \gamma_{[\mu|} \psi_{i|\nu]} + \bar{\epsilon}_i \gamma_{[\mu} \psi_{\nu]}^i\right) \\
& + 2Z^{M\alpha}{}_{NP} \epsilon_{\beta\gamma} A_{[\mu}^{N\beta} \delta_\epsilon A_{\nu]}^{P\gamma} - \left(\xi_\beta^M + \theta_\beta^M\right) \eta_{NP} A_{[\mu}^{N(\alpha|} \delta_\epsilon A_{\nu]}^{P|\beta)} .
\end{aligned}
\tag{60}
$$

To write the local supersymmetry transformation rules for the fermionic fields in a manifestly $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)$-covariant form, we introduce the symplectic vector $\mathcal{G}^{M\alpha}_{\mu\nu}=(H^{\Lambda}_{\mu\nu},\mathcal{G}_{\Lambda\mu\nu})$, where

$$\mathcal{G}_{\Lambda\mu\nu}\equiv\mathcal{R}_{\Lambda\Sigma}H^{\Sigma}_{\mu\nu}-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{I}_{\Lambda\Sigma}H^{\Sigma\rho\sigma}+\text{fermions}\,, \tag{61}$$

where $\mathcal{R}_{\Lambda\Sigma}$ and $\mathcal{I}_{\Lambda\Sigma}$ are real symmetric matrices that depend on the choice of symplectic frame and are defined by

$$\begin{aligned}\mathcal{M}_{\mathcal{M}\mathcal{N}}\equiv M_{MN}M_{\alpha\beta}&=\begin{pmatrix}\mathcal{M}_{\Lambda\Sigma} & \mathcal{M}_{\Lambda}{}^{\Sigma}\\ \mathcal{M}^{\Lambda}{}_{\Sigma} & \mathcal{M}^{\Lambda\Sigma}\end{pmatrix}\\ &=\begin{pmatrix}-(\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})_{\Lambda}{}^{\Sigma}\\ (\mathcal{I}^{-1}\mathcal{R})^{\Lambda}{}_{\Sigma} & -(\mathcal{I}^{-1})^{\Lambda\Sigma}\end{pmatrix}.\end{aligned} \tag{62}$$

Up to terms quadratic in the fermions, we have

$$\delta_\epsilon \psi_{i\mu} = \hat{D}_\mu \epsilon_i - \frac{i}{8}\mathcal{V}_\alpha L_{Mij} \mathcal{G}^{M\alpha}_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu \epsilon^j - \frac{1}{3} g \bar{A}_{1ij}\gamma_\mu \epsilon^j + \frac{g}{2}\epsilon_{ijkl} B^{kl}\gamma_\mu \epsilon^j, \tag{63}$$

$$\delta_\epsilon \chi_i = -\frac{i}{4}\mathcal{V}^*_\alpha L_{Mij}\mathcal{G}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + \hat{P}^*_\mu \gamma^\mu \epsilon_i + \frac{2}{3} g \bar{A}_{2ij}\epsilon^j - g\bar{B}_{ij}\epsilon^j, \tag{64}$$

$$\delta_\epsilon \lambda_{\underline{a}i} = \frac{i}{8}\mathcal{V}^*_\alpha L_{M\underline{a}}\mathcal{G}^{M\alpha}_{\mu\nu}\gamma^{\mu\nu}\epsilon_i - \hat{P}_{\underline{a}ij\mu}\gamma^\mu \epsilon^j + g\bar{A}_{2\underline{a}}{}^j{}_i \epsilon_j - \frac{1}{4} g \bar{B}_{\underline{a}}\epsilon_i\,, \tag{65}$$

where the fermion shift tensors are defined by

$$A_2^{ij} = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{kl} L^{Nik} L^{Pjl} + \frac{3}{2}\xi_{\alpha M}\mathcal{V}^\alpha L^{Mij}, \tag{66}$$

$$A_{2\underline{a}i}{}^j = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{\underline{a}} L^N{}_{ik} L^{Pjk} - \frac{1}{4}\delta_i^j \xi_{\alpha M}\mathcal{V}^\alpha L^M{}_{\underline{a}}, \tag{67}$$

$$A_1^{ij} = f_{\alpha MNP}(\mathcal{V}^\alpha)^* L^M{}_{kl} L^{Nik} L^{Pjl}, \tag{68}$$

$$B^{ij} = \theta_{\alpha M}\mathcal{V}^\alpha L^{Mij}, \tag{69}$$

$$B^{\underline{a}} = \theta_{\alpha M}\mathcal{V}^\alpha L^{M\underline{a}}, \tag{70}$$

and

$$\hat{D}_\mu \epsilon_i \equiv \partial_\mu \epsilon_i + \frac{1}{4}\omega_{\mu ab}(e, A, \psi)\gamma^{ab}\epsilon_i - \frac{i}{2}\hat{A}_\mu \epsilon_i - \hat{\omega}_i{}^j{}_\mu \epsilon_j - \frac{g}{2}\theta_{\alpha M} A_\mu^{M\alpha}\epsilon_i \,, \tag{71}$$

where

$$\omega_\mu{}^{ab}(e, A, \psi) = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{b]} - e^{\nu[a}e^{b]\rho}e_{c\mu}\partial_\nu e_\rho^c + \bar{\psi}_\mu^i \gamma^{[a}\psi_i^{b]} + \bar{\psi}^{i[a}\gamma^{b]}\psi_{i\mu} + \bar{\psi}^{i[a}\gamma_\mu \psi_i^{b]} - 2g e_\mu^{[a} e^{b]\nu}\theta_{\alpha M} A_\nu^{M\alpha}. \tag{72}$$

# Equations of Motion

Since four-dimensional $\mathcal{N} = 4$ matter-coupled supergravity with local scaling symmetry does not admit an action, it must be constructed directly on the level of the equations of motion.

# Fermionic field equations

E.o.m. for the dilatini:

$$
\begin{aligned}
(\mathcal{E}_\chi)_i \equiv & - \gamma^\mu \hat{D}_\mu \chi_i + \gamma^\mu \gamma^\nu \psi_{i\mu} \hat{P}^*_\nu \\
& - \frac{i}{4} \mathcal{V}^*_\alpha L_{Mij} \mathcal{G}^{M\alpha}_{\nu\rho} \gamma^\mu \gamma^{\nu\rho} \psi^j_\mu + \frac{i}{4} \mathcal{V}^*_\alpha L_{M\underline{a}} \mathcal{G}^{M\alpha}_{\mu\nu} \gamma^{\mu\nu} \lambda^{\underline{a}}_i \\
& + \frac{2}{3} g \bar{A}_{2ij} \gamma^\mu \psi^j_\mu - 2 g \bar{A}_2{}^{\underline{a}j}{}_i \lambda_{\underline{a}j} + 2 g \bar{A}_2{}^{\underline{a}j}{}_j \lambda_{\underline{a}i} \\
& - g \bar{B}_{ij} \gamma^\mu \psi^j_\mu + \frac{5}{2} g \bar{B}^{\underline{a}} \lambda_{\underline{a}i} = 0 \,,
\end{aligned} \tag{73}
$$

where

$$
\begin{aligned}
\hat{D}_\mu \chi_i \equiv & \, \partial_\mu \chi_i + \frac{1}{4} \omega_\mu{}^{ab}(e, A, \psi) \gamma_{ab} \chi_i + \frac{3i}{2} \hat{A}_\mu \chi_i - \hat{\omega}_i{}^j{}_\mu \chi_j \\
& + \frac{g}{2} \theta_{\alpha M} A^{M\alpha}_\mu \chi_i \,.
\end{aligned} \tag{74}
$$

E.o.m. for the gaugini:

$$
\begin{aligned}
(\mathcal{E}_\lambda)_{\underline{a}i} \equiv & -\gamma^\mu \hat{D}_\mu \lambda_{\underline{a}i} - \gamma^\mu \gamma^\nu \psi^j_\mu \hat{P}_{\underline{a}ij\nu} + \frac{i}{8}\mathcal{V}^*_\alpha L_{M\underline{a}} \mathcal{G}^{M\alpha}_{\nu\rho} \gamma^\mu \gamma^{\nu\rho} \psi_{i\mu} \\
& + \frac{i}{4}\mathcal{V}^*_\alpha L_{Mij} \mathcal{G}^{M\alpha}_{\mu\nu} \gamma^{\mu\nu} \lambda^j_{\underline{a}} + \frac{i}{8}\mathcal{V}_\alpha L_{M\underline{a}} \mathcal{G}^{M\alpha}_{\mu\nu} \gamma^{\mu\nu} \chi_i \\
& + g\bar{A}_{2\underline{a}}{}^j{}_i \gamma^\mu \psi_{j\mu} - gA_{2\underline{a}i}{}^j \chi_j + gA_{2\underline{a}j}{}^j \chi_i \\
& + 2g\bar{A}_{\underline{ab}ij} \lambda^{\underline{b}j} + \frac{2}{3} g\bar{A}_{2(ij)} \lambda^j_{\underline{a}} \\
& - \frac{g}{4}\bar{B}_{\underline{a}} \gamma^\mu \psi_{i\mu} - 2g\bar{B}_{ij}\lambda^j_{\underline{a}} - \frac{3}{4} g B_{\underline{a}} \chi_i = 0\,,
\end{aligned}
\tag{75}
$$

where

$$
A_{\underline{ab}}{}^{ij} \equiv f_{\alpha MNP} \mathcal{V}^\alpha L^M{}_{\underline{a}} L^N{}_{\underline{b}} L^{Pij}\,, \tag{76}
$$

and

$$\hat{D}_\mu \lambda_{\underline{a}i} \equiv \partial_\mu \lambda_{\underline{a}i} + \frac{1}{4}\omega_\mu{}^{ab}(e, A, \psi)\gamma_{ab}\lambda_{\underline{a}i} + \frac{i}{2}\hat{\mathcal{A}}_\mu \lambda_{\underline{a}i} - \hat{\omega}_i{}^j{}_\mu \lambda_{\underline{a}j} + \hat{\omega}_{\underline{a}}{}^{\underline{b}}{}_\mu \lambda_{\underline{b}i} + \frac{g}{2}\theta_{\alpha M} A_\mu^{M\alpha}\lambda_{\underline{a}i}\,. \quad (77)$$

E.o.m. for the gravitini:

$$\begin{aligned}(\mathcal{E}_\psi)_{i\nu} \equiv & -\gamma^\mu \hat{\rho}_{i\mu\nu} + \hat{P}_\nu \chi_i + 2\hat{P}_{\underline{a}ij\nu}\lambda^{\underline{a}j} - \frac{i}{8}\mathcal{V}_\alpha L_{Mij}\mathcal{G}^{M\alpha}_{\rho\sigma}\gamma^\mu\gamma^{\rho\sigma}\gamma_\nu\psi^j_\mu \\ & - \frac{i}{8}\mathcal{V}_\alpha L_{M\underline{a}}\mathcal{G}^{M\alpha}_{\mu\rho}\gamma^{\mu\rho}\gamma_\nu\lambda^{\underline{a}}_i + \frac{i}{8}\mathcal{V}^*_\alpha L_{Mij}\mathcal{G}^{M\alpha}_{\mu\rho}\gamma^{\mu\rho}\gamma_\nu\chi^j \\ & + g\bar{A}_{1ij}\psi^j_\nu - \frac{g}{3}\bar{A}_{1ij}\gamma_{\mu\nu}\psi^{j\mu} + \frac{g}{3}\bar{A}_{2ji}\gamma_\nu\chi^j + gA_{2\underline{a}i}{}^j\gamma_\nu\lambda^{\underline{a}}_j \\ & - \frac{3}{2}g\epsilon_{ijkl}B^{kl}\psi^j_\nu + \frac{g}{2}\epsilon_{ijkl}B^{kl}\gamma_{\mu\nu}\psi^{j\mu} \\ & - \frac{3}{2}g\bar{B}_{ij}\gamma_\nu\chi^j + \frac{7}{4}gB^{\underline{a}}\gamma_\nu\lambda_{\underline{a}i} = 0\,,\end{aligned} \quad (78)$$

where

$$\hat{\rho}_{i\mu\nu} \equiv 2\partial_{[\mu|}\psi_{i|\nu]} + \frac{1}{2}\omega_{[\mu|}{}^{ab}(e, A, \psi)\gamma_{ab}\psi_{i|\nu]} - i\hat{\mathcal{A}}_{[\mu|}\psi_{i|\nu]} - 2\hat{\omega}_i{}^j{}_{[\mu|}\psi_{j|\nu]} - g\theta_{\alpha M}A^{M\alpha}_{[\mu|}\psi_{i|\nu]}. \tag{79}$$

In the presence of a gauging of the scaling symmetry ($\theta_{\alpha M} \neq 0$), there is no action that reproduces the above equations of motion via the variational principle.

Indeed, the fermion mass matrices that can be read off from the fermionic field equations are not symmetric:

$$(\mathcal{M}_{\frac{3}{2}})_{ij} = -\frac{2}{3}g\left(\bar{A}_{1ij} - \frac{3}{2}\epsilon_{ijkl}B^{kl}\right), \tag{80}$$

$$(\mathcal{M}_{\frac{1}{2}})_{ij} = 0\,, \tag{81}$$

$$(\mathcal{M}_{\frac{1}{2}})_i{}^{\underline{a}j} = -\sqrt{2}g\bar{A}_2{}^{\underline{a}j}{}_i + \sqrt{2}g\delta_i^j\bar{A}_2{}^{\underline{a}k}{}_k + \frac{5\sqrt{2}}{4}g\delta_i^j\bar{B}^{\underline{a}}, \tag{82}$$

$$(\mathcal{M}_{\frac{1}{2}})^{\underline{a}i}{}_j = -\sqrt{2}g\bar{A}_2{}^{\underline{a}i}{}_j + \sqrt{2}g\delta_j^i\bar{A}_2{}^{\underline{a}k}{}_k - \frac{3\sqrt{2}}{4}g\delta_j^i\bar{B}^{\underline{a}}, \tag{83}$$

$$(\mathcal{M}_{\frac{1}{2}})^{\underline{a}i,\underline{b}j} = 2gA^{\underline{ab}ij} + \frac{2}{3}g\delta^{\underline{ab}}A_2^{(ij)} - 2gB^{ij}\delta^{\underline{ab}}. \tag{84}$$

# Bosonic field equations

The equations of motion for the bosonic fields follow from the requirement that the fermionic field equations be invariant under local supersymmetry transformations:

$$\delta_\epsilon(\mathcal{E}_\chi)_i = \delta_\epsilon(\mathcal{E}_\lambda)_{\underline{a}i} = \delta_\epsilon\left(\mathcal{E}_\psi\right)_{i\nu} = 0\,. \qquad (85)$$

E.o.m. for the complex scalar of the $\mathcal{N}=4$ supergravity multiplet:

$$\begin{aligned}\mathcal{E} \equiv & - e^{-1}\hat{D}_\mu\left(e(\hat{P}^\mu)^*\right) + \frac{1}{8}\mathcal{V}_\alpha^*\mathcal{V}_\beta^* M_{MN}\mathcal{G}_{\mu\nu}^{M\alpha}\mathcal{G}^{N\beta\mu\nu} \\ & + g^2\Bigg(-\frac{2}{9}A_1^{ij}\bar{A}_{2ij} + \frac{1}{9}\epsilon^{ijkl}\bar{A}_{2ij}\bar{A}_{2kl} - \frac{1}{2}\bar{A}_{2}{}^{\underline{a}i}{}_j\bar{A}_{2\underline{a}}{}^j{}_i \\ & - \frac{3}{8}\epsilon^{ijkl}\bar{A}_{2ij}\bar{B}_{kl} + \frac{1}{8}\bar{A}_{2\underline{a}}{}^i{}_i\bar{B}^{\underline{a}} + \frac{3}{16}\epsilon^{ijkl}\bar{B}_{ij}\bar{B}_{kl}\Bigg) = 0\,,\end{aligned} \tag{86}$$

where

$$\begin{aligned}\hat{D}_\mu\left(e(\hat{P}^\mu)^*\right) \equiv & \, \partial_\mu\left(e(\hat{P}^\mu)^*\right) + 2ie\hat{\mathcal{A}}_\mu(\hat{P}^\mu)^* \\ & - 2ge\theta_{\alpha M}A_\mu^{M\alpha}(\hat{P}^\mu)^*.\end{aligned} \tag{87}$$

E.o.m. for the scalars of the vector multiplets:

$$\mathcal{E}_{\underline{a}ij} \equiv e^{-1}\hat{D}_\mu\left(e\hat{P}_{\underline{a}ij}{}^{\mu}\right) - \frac{1}{2}M_{\alpha\beta}L_{M\underline{a}}L_{Nij}\mathcal{G}^{M\alpha}_{\mu\nu}\mathcal{G}^{N\beta\mu\nu} + g^2\left(\mathcal{C}_{\underline{a}ij} + \frac{1}{2}\epsilon_{ijkl}\bar{\mathcal{C}}_{\underline{a}}{}^{kl}\right) = 0\,, \tag{88}$$

where

$$\hat{D}_\mu\left(e\hat{P}_{\underline{a}ij}{}^{\mu}\right) \equiv \partial_\mu\left(e\hat{P}_{\underline{a}ij}{}^{\mu}\right) + e\hat{\omega}_{\underline{a}}{}^{\underline{b}}{}_{\mu}\hat{P}_{\underline{b}ij}{}^{\mu} + 2e\hat{\omega}_{[i|}{}^{k}{}_{\mu}\hat{P}_{\underline{a}|j]k}{}^{\mu} - 2ge\theta_{\alpha M}A^{M\alpha}_{\mu}\hat{P}_{\underline{a}ij}{}^{\mu}, \tag{89}$$

$$\mathcal{C}_{\underline{a}ij} = -\frac{2}{3}\bar{A}_{2\underline{a}}{}^{k}{}_{[i}\bar{A}_{1j]k} - \frac{1}{6}A_{2\underline{a}[i}{}^{k}\bar{A}_{2j]k} - \frac{1}{2}A_{2\underline{a}[i|}{}^{k}\bar{A}_{2k|j]} + \bar{A}_{\underline{ab}[i|k}A_{2}{}^{\underline{b}}{}_{|j]}{}^{k} + \frac{1}{3}A_{2\underline{a}k}{}^{k}\bar{A}_{2[ij]} + \frac{5}{2}A_{2\underline{a}[i}{}^{k}\bar{B}_{j]k} + \frac{1}{2}A_{2\underline{a}k}{}^{k}\bar{B}_{ij} - \frac{1}{4}\bar{A}_{\underline{ab}ij}B^{\underline{b}} - \frac{1}{4}\bar{A}_{2[ij]}B_{\underline{a}} + \frac{1}{8}\bar{B}_{ij}B_{\underline{a}}\,. \tag{90}$$

Einstein equations:

$$\begin{aligned}(\mathcal{E}_{\text{Einstein}})_{\mu\nu} \equiv \hat{R}_{(\mu\nu)} - 2\hat{P}_{(\mu}\hat{P}^{*}_{\nu)} - \hat{P}_{\underline{a}ij\mu}\hat{P}^{\underline{a}ij}{}_{\nu} - \frac{1}{2}M_{MN}M_{\alpha\beta}\mathcal{G}^{M\alpha}_{\mu\rho}\mathcal{G}^{N\beta}{}_{\nu}{}^{\rho} \\ + g^2\Big(\frac{1}{3}A_1^{ij}\bar{A}_{1ij} - \frac{1}{9}A_2^{ij}\bar{A}_{2ij} - \frac{1}{2}A_{2\underline{a}i}{}^{j}\bar{A}_2{}^{\underline{a}i}{}_{j} \\ + \frac{1}{6}A_2^{ij}\bar{B}_{ij} - \frac{3}{2}B^{ij}\bar{B}_{ij} + \frac{1}{8}B^{\underline{a}}\bar{B}_{\underline{a}}\Big)g_{\mu\nu} = 0\,,\end{aligned} \tag{91}$$

where

$$\hat{R}_{\mu\nu} = 2e_{a\nu}e_b^{\rho}(\partial_{[\mu}\omega_{\rho]}{}^{ab}(e,A,\psi) + \omega_{[\mu}{}^{ac}(e,A,\psi)\omega_{\rho]c}{}^{b}(e,A,\psi))\,.$$

Effective cosmological constant:

$$\begin{aligned}\Lambda = & g^2\Big(-\frac{1}{3}A_1^{ij}\bar{A}_{1ij} + \frac{1}{9}A_2^{ij}\bar{A}_{2ij} + \frac{1}{2}A_{2\underline{a}i}{}^{j}\bar{A}_2{}^{\underline{a}i}{}_{j} \\ & - \frac{1}{6}A_2^{ij}\bar{B}_{ij} + \frac{3}{2}B^{ij}\bar{B}_{ij} - \frac{1}{8}B^{\underline{a}}\bar{B}_{\underline{a}}\Big)\,.\end{aligned} \tag{92}$$

E.o.m. for the vector fields:

$$(\mathcal{E}_{\text{vector}})^{M\alpha\mu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{D}_\nu \mathcal{G}^{M\alpha}_{\rho\sigma} + 2gZ^{M\alpha NP} L_{N\underline{a}} L_{Pij} \hat{P}^{\underline{a}ij\mu} - \frac{i}{2}g\left(\xi^M_\beta + \theta^M_\beta\right)\mathcal{V}^\alpha\mathcal{V}^\beta(\hat{P}^\mu)^* + \frac{i}{2}g\left(\xi^M_\beta + \theta^M_\beta\right)(\mathcal{V}^\alpha)^*(\mathcal{V}^\beta)^*\hat{P}^\mu = 0\,, \tag{93}$$

where $\hat{D}_\mu \mathcal{G}^{M\alpha}_{\nu\rho} \equiv \partial_\mu \mathcal{G}^{M\alpha}_{\nu\rho} + gX_{N\beta P\gamma}{}^{M\alpha} A^{N\beta}_\mu \mathcal{G}^{P\gamma}_{\nu\rho}$.

# Maximally Symmetric Solutions

A solution to the field equations with constant scalar and vanishing vector, two-form and fermionic fields satisfies the following two conditions:

$$
\begin{aligned}
&-\frac{2}{9}A_1^{ij}\bar{A}_{2ij}+\frac{1}{9}\epsilon^{ijkl}\bar{A}_{2ij}\bar{A}_{2kl}-\frac{1}{2}\bar{A}_2{}^{\underline{a}i}{}_j\bar{A}_{2\underline{a}}{}^j{}_i \\
&-\frac{3}{8}\epsilon^{ijkl}\bar{A}_{2ij}\bar{B}_{kl}+\frac{1}{8}\bar{A}_{2\underline{a}}{}^i{}_i\bar{B}^{\underline{a}}+\frac{3}{16}\epsilon^{ijkl}\bar{B}_{ij}\bar{B}_{kl}=0\,, \qquad (94)
\end{aligned}
$$

and

$$
\mathcal{C}_{\underline{a}ij}+\frac{1}{2}\epsilon_{ijkl}\bar{\mathcal{C}}_{\underline{a}}{}^{kl}=0\,. \qquad (95)
$$

For the standard gaugings, for which $\theta_{\alpha M}=0$, these conditions reproduce the extremization conditions of the scalar potential [Dall'Agata, Liatsos, Noris and Trigiante (2023)].

The squared mass matrix of the fluctuations of the scalar fields around maximally symmetric solutions of the field equations is not symmetric $\implies$ it cannot arise from a scalar potential.

# Conclusion

- Determination of the algebraic stucture of the embedding tensor that parametrizes the gaugings of $D = 4$, $\mathcal{N} = 4$ supergravity that involve the scaling symmetry and of the quadratic consistency constraints on its irreducible components.
- Explicit derivation of the equations of motion of the most general half-maximal supergravity with local scaling symmetry in four dimensions, which cannot be obtained from an action.

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