# Lie-Poisson Electrodynamics

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by V. Kupriyanov, M. K., A. Sharapov
and P. Vitale**

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Outline:

* Lie-Poisson electrodynamics

* Dynamics of the gauge fields

* Charged particle in a gauge background

* Non-commutative Kepler problem

## Lie-Poisson electrodynamics: definition

- The starting point is an $n$-dimensional space $\mathbb{R}^n$ (not space-time!), equipped with a given Poisson bivector,

$$\{x^a, x^b\} = \Theta^{ab}(x) \qquad a, b = 1, ..., n,$$

  Today the time-variable $x^0$ is commutative.

- Poisson electrodynamics (PE) is a field theoretical model, where the infinitesimal gauge transformations close the Poisson gauge algebra and reproduce the standard abelian gauge transformations at the commutative limit ,

$$[\delta_f, \delta_g] A_\mu = \delta_{\{f,g\}} A_\mu, \qquad \lim_{\Theta \to 0} \delta_f A_\mu = \partial_\mu f.$$

- The PE can be seen as a semiclassical limit (or the slowly varying field approximation) of the non-commutative gauge theory [Kupriyanov, Vitale 2020] and [Kupriyanov, Szabo' 2022],

$$[\delta_f, \delta_g] A_\mu \ = \ \delta_{-\mathrm{i}[f,g]_\star} A_\mu, \quad \lim_{\Theta \to 0} \delta_f A_\mu = \partial_\mu f, \quad [f, g]_\star \simeq \mathrm{i}\,\{f, g\}\,.$$

## Lie-Poisson electrodynamics: definition

- Consider a class of Poisson bivectors, which are linear in coordinates,
$$\Theta^{ab} = f^{ab}_c\, x^c\,.$$
The qantity $\Theta$ is a Poisson bivector iff constants $f^{ab}_c$ satisfy the Jacobi identity,
$$f^{kl}_i f^{ja}_l + f^{jl}_i f^{ak}_l + f^{al}_i f^{kj}_l = 0\,.$$
Therefore these constants can be seen as the structure constants of an $n$-dimensional Lie algebra, which we shall address as $\mathfrak{g}$.

- The corresponding $n$-dimensional Lie group, which is unique up to covering, will be addressed as $G$.

- The Poisson electrodynamics of this kind we call the Lie-Poisson electrodynamics.

## Lie-Poisson electrodynamics: basic setup

- I review the relevant elements of the construction, proposed in [Kupriyanov, Sharapov, Szabo' 2024] in the context of the symplectic groupoid approach. Consider the following (trivial) fibre bundle

$$\mathcal{G} = \underbrace{\mathbb{R}^n}_{\text{base space}} \times \underbrace{G}_{\text{fiber}} .$$

- By construction, the phase space of a point-like particle is given by the total space $\mathbb{R}^n \times G$ of this bundle. The group $G$ plays the role of the space of momenta conjugate to the position coordinates $x^a$. We will denote local coordinates on $G$ by $p_i$. The momentum space may well be a manifold with nontrivial topology, e.g. $S_3$, when $\mathfrak{g} = \mathfrak{su}(2)$.

- At any moment of time the spatial part $\vec{A}$ of the gauge field $A$ must be a section of this bundle. Simply speaking, both $\vec{p}$ and $\vec{A}(x)$ live on $G$ (but not on $\mathbb{R}^n$).

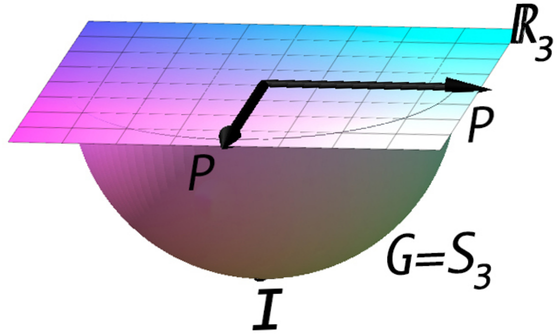
**Lie-Poisson electrodynamics: $\mathfrak{su}(2)$-noncommutativity**

- The Poisson bivector is given by

$$\Theta^{ij}(x) = 2\,\ell\,\varepsilon^{ijk} x^k, \qquad f_k^{ij} = \ell\,\varepsilon^{ijk},$$

so the phase space of the point-like particle coincides with that is $\mathbb{R}^3 \times S_3$. The radius of the momenta 3-sphere equals to $1/\ell$ .

- By identifying the identity element of the group with the southern pole of the 3-sphere, we introduce the local coordinates $p_1$, $p_2$ and $p_3$ in the southern hemisphere in the following way:




- These coordinates obey the restriction $p_1^2 + p_2^2 + p_3^2 < 1/\ell^2$. The commutative limit $\ell \to 0$ for the space coordinates is simultaneously the decompactification limit for the momentum 3-sphere $S_3$.

## Dynamics of the gauge fields: ingredients

- For any Lie algebra type noncommutativity the deformed gauge transformations $\delta_f A$, which obey,

$$[\delta_f, \delta_g] A_\mu = \delta_{\{f,g\}} A_\mu, \qquad \lim_{\Theta \to 0} \delta_f A_\mu = \partial_\mu f,$$

were constructed in [Kupriyanov, Szabo' 2022].

- Apart from that, we need the deformed field strength $\mathcal{F}$, such that

$$\delta_f \mathcal{F}_{\mu\nu} = \{\mathcal{F}_{\mu\nu}, f\}, \qquad \lim_{\Theta \to 0} \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

Finally, we need the deformed gauge covariant derivative $\mathcal{D}$. $\forall \psi(x)$, which transforms in a covariant way, $\delta_f \psi = \{f, \psi\}$,

$$\delta_f (\mathcal{D}_\mu \psi) = \{\mathcal{D}_\mu \psi, f\} \qquad \lim_{\Theta \to 0} \mathcal{D}_\mu \psi = \partial_\mu \psi.$$

- For any Lie algebra type noncommutativity $\mathcal{F}$ and $\mathcal{D}$ were constructed in [Kupriyanov, Kurkov, Vitale' 2022].

## Dynamics of the gauge fields: the action principle

- If the Poisson bivector, defining the non-commutativity, obeys the compatibility condition,

$$\partial_a \Theta^{ab} = 0 \qquad \left(f_a^{ab} = 0\right),$$

the gauge-invariant classical action can be easily constructed,

$$S_g = \int_{\mathbb{R}^n} \mathrm{d}^n x \, \mathcal{L}_g \qquad \mathcal{L}_g = -\frac{1}{4} \mathcal{F}_{\mu\nu} \, \mathcal{F}^{\nu\nu}.$$

- The Euler-Lagrange equations can be rewritten as follows [Kupriyanov, Kurkov, Vitale'2023],

$$\mathcal{D}_\nu \mathcal{F}^{\nu\mu} + \frac{1}{2} \mathcal{F}_{\xi\rho} \, f_\nu^{\xi\rho} \, \mathcal{F}^{\mu\nu} - \mathcal{F}_{\xi\rho} \, f_\nu^{\mu\rho} \, \mathcal{F}^{\xi\nu} = 0.$$

- The corresponding Hamiltonian analysis was performed in [Bascone, Kurkov'2024]. There are two first-class constrains, whilst the second-class constrains are absent.Therefore we have as many physical degrees of freedom as there are present in the Maxwell theory.

## Particle in a gauge background: phase space

- Let $\gamma^a$ and $\bar{\gamma}_b$ be bases of left-invariant vector fields and right-invariant one-forms on $G$ respectively.

$$\bar{\gamma}_b[\gamma^a] = \delta^a_b.$$

- We assume that the local coordinates $p_a$ on $G$ near its identity element are chosen in such a way that

$$\lim_{\Theta\to 0} \gamma^a_b(p) = \delta^a_b, \qquad \lim_{\Theta\to 0} \bar{\gamma}^a_b(p) = \delta^a_b.$$

- In these formulae $\gamma^a_b(p)$ and $\bar{\gamma}^a_b(p)$ denote the components of $\gamma^a$ and $\bar{\gamma}_b$ in the natural bases $\frac{\partial}{\partial p_b}$ and $\mathrm{d}p_a$ ,

$$\gamma^a = \gamma^a_b(p)\frac{\partial}{\partial p_b}, \qquad \bar{\gamma}_b = \bar{\gamma}^a_b(p)\,\mathrm{d}p_a.$$

In our $\mathfrak{su}(2)$-example,

$$\gamma^i_j(p) = \sqrt{1-\ell^2|\vec{p}|^2}\,\delta^i_j + \ell\,\varepsilon^{ijk}p_k.$$

## Particle in a gauge background: phase space

- The Poisson brackets between the phase space variables are defined as follows,

$$\{x^i, x^j\} = \Theta^{ij}(x)\,, \qquad \{x^i, p_j\} = \gamma^i_j(p)\,, \qquad \{p_i, p_j\} = 0\,.$$

- The infinitesimal gauge transformations of $x(t)$, $p(t)$ read [Kupriyanov, Sharapov, Szabo' 2024],

$$\begin{aligned} \delta_f x^i &= \{f, x^i\} = -\Theta^{il}(x)\,\partial_l f(x)\,, \\ \delta_f p_i &= \{f, p_i\} = \gamma^l_i(p)\,\partial_l f(x)\,, \end{aligned}$$

whilst the gauge field $A$ transforms as follows,

$$\begin{aligned} \delta_f A_i &= \gamma^k_i(A)\,\partial_k f + \{A_i, f\}\,, \\ \delta_f A_0 &= \partial_0 f + \{A_0, f\}\,. \end{aligned}$$

- These transformations close the Lie-Poisson gauge algebra

$$[\delta_f, \delta_g] = \delta_{\{f,g\}},$$

and exhibit correct commutative limits.

## Particle in a gauge background: gauge-invariant momenta

- The gauge-invariant momenta $\pi_i$,

$$\delta_f \, \pi_i = 0$$

can be constructed as follows [Basilio, Kurkov, Kupriyanov'2024].

- One has to find a solution of the equation,

$$\gamma_k^m(p)\, \partial_p^k \pi_i(p,A) + \gamma_k^m(A)\, \partial_A^k \pi_i(p,A) = 0\,,$$

which exhibits the correct commutative limit,

$$\lim_{\Theta \to 0} \pi_i(p,A) = p_i - A_i.$$

- In our $\mathfrak{su}(2)$-example the gauge-invariant momenta can be chosen as follows,

$$\vec{\pi}(p,A) = \vec{p}\sqrt{1-\ell^2|\vec{A}|^2} - \vec{A}\sqrt{1-\ell^2|\vec{p}|^2} + \ell\,\vec{p}\times\vec{A}.$$

## Particle in a gauge background: dynamics

- By using the gauge-invariant momenta $\pi_i\Big(p, A(x)\Big)$, we can write down the Hamiltonian [Basilio, Kurkov, Kupriyanov'2024],

$$H(x,p) = J(\pi) + A_0(x)\,, \qquad \forall J.$$

- The gauge-invariant first-order action reads,

$$S_c = -\int dt\,\Big[\dot{p}_l\,\bar{\gamma}^l_i(p)\,x^i + H(x,p)\Big]\,.$$

- The corresponding Hamiltonian equations of motion,

$$\dot{x}^i = \{x^i, H\}\,, \qquad \text{and} \qquad \dot{p}_i = \{p_i, H\}\,,$$

respect the Lie-Poisson gauge symmetry.

## Particle in a gauge background: dynamics

- Various choices of the form factor $J$ lead to models with the desired commutative limits, e.g.

$$H = \sqrt{m^2 + |\vec{\pi}|^2} + A_0(x)\,, \quad \text{(relativistic commutative limit)},$$

or

$$H = \frac{|\vec{\pi}|^2}{2m} + A_0(x), \quad \text{(non-relativistic commutative limit)}.$$

- In our $\mathfrak{su}(2)$-example the Hamiltonian

$$H = -\frac{\sqrt{1 - \ell^2|\vec{\pi}|^2}}{\ell^2 m} + \frac{1}{\ell^2 m} + A_0\,,$$

is of a special interest. The outcoming non-commutative Kepler problem is super-integarble.

- The commutative limit is the non-relativistic one,

$$\lim_{\ell\to 0} H = \frac{\left|\vec{p} - \vec{A}\right|^2}{2m} + A_0.$$

## Intermediate summary: general setting

- The starting point is a Poisson bivector $\Theta^{ab} = f^{ab}_c x^c$ of the Lie algebra type. By definition, the infinitesimal gauge transformations close the Lie-Poisson gauge algebra

$$[\delta_f, \delta_g] = \delta_{\{f,g\}}.$$

- By construction the phase space of the point-like particle is $\mathbb{R}^n \times G$. The corresponding Poisson brackets read,

$$\{x^i, x^j\} = \Theta^{ij}(x)\,, \qquad \{x^i, p_j\} = \gamma^i_j(p)\,, \qquad \{p_i, p_j\} = 0\,.$$

- The Lie-Poisson gauge transformations act on $x$, $p$ and $A$. The gauge-invariant classical action reads,

$$S[x,p,A] = S_g + S_c,$$

where

$$S_g = \int_{\mathbb{R}^n} \mathrm{d}^n x \left(-\frac{1}{4}\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\nu\nu}\right), \quad S_c = -\int dt\,\left[\dot{p}_l\,\bar{\gamma}^l_i(p)\,x^i + H(x,p)\right].$$

## Intermediate summary: $\mathfrak{su}(2)$-example

- The Poisson bivector is given by

$$\Theta^{ij}(x) = 2\,\ell\,\varepsilon^{ijk}x^k,$$

so the phase space of the point-like particle coincides with that is $\mathbb{R}^3 \times S_3$.

- The Poisson brackets of the phase-space coordinates read,

$$\begin{aligned} \{x^i, x^j\} &= 2\,\ell\,\varepsilon^{ijk}x^k\,, \\ \{x^i, p_j\} &= \sqrt{1-\ell^2|\vec{p}|^2}\,\delta^i_j + \ell\,\varepsilon^{ijk}p_k\,, \\ \{p_i, p_j\} &= 0\,. \end{aligned}$$

- We shall work with the Hamiltonian,

$$\begin{aligned} H(x,p) &= -\frac{\sqrt{1-\ell^2|\vec{\pi}|^2}}{\ell^2 m} + \frac{1}{\ell^2 m} + A_0(x)\,, \\ \vec{\pi}(p, A(x)) &:= \vec{p}\sqrt{1-\ell^2|\vec{A}|^2} - \vec{A}\sqrt{1-\ell^2|\vec{p}|^2} + \ell\,\vec{p}\times\vec{A}. \end{aligned}$$

## Non-commutative Kepler problem: the setup

- The standard Coulomb potential, i.e., the field configuration $\vec{A} = 0$ and $A_0 = C/|\vec{x}|$, solves the field equations of Poisson electrodynamics with $\mathfrak{su}(2)$ non-commutativity in the whole space except the origin.
- In this case the following Hamiltonian describes the dynamics of the test particle,

$$H(x,p) = -\frac{\sqrt{1-\ell^2|\vec{p}|^2}}{\ell^2 m} + \frac{1}{\ell^2 m} + \frac{C}{|\mathbf{x}|}.$$

- The Hamiltonian dynamics,

$$\dot{x}^i = \{x^i, H\}\,, \qquad \text{and} \qquad \dot{p}_i = \{p_i, H\}\,,$$

is affected by the compactness of the momenta space, which yields the boundedness of the kinetic energy. From now on I follow the recent article [Kupriyanov, Kurkov, Sharapov' 2024]. The maximal or "critical" value of the kinetic energy, $E_c = 2/\ell^2 m$ is achieved at the northern pole of the momenta 3-sphere.

## Non-commutative Kepler problem: conserved quantities

- The rotational invariance of the system gives rise to one more conserved quantity, namely, the deformed angular momentum vector

$$\vec{L} = (\vec{x} \times \vec{p})\sqrt{1 - \ell^2|\vec{p}|^2} + \ell\,\vec{p} \times (\vec{x} \times \vec{p})\,.$$

- Besides the angular momentum, the system admits an additional conserved vector

$$\vec{Q} = \vec{p} \times \vec{L} + \frac{mC}{|\vec{x}|}\left(\vec{x} - \ell\,\vec{L}\right).$$

  The square of this deformed Laplace–Runge–Lenz vector reads,

$$Q^2 = C^2m^2 + 2m\,L^2\,H\left(1 - H/E_c\right).$$

- The conserved quantities close the following algebra w.r.t. the Poisson brackets :

$$\{L_i, L_j\} = \varepsilon_{ijk}L_k\,, \qquad \{Q_i, L_j\} = \varepsilon_{ijk}Q_k\,,$$

$$\{Q_i, Q_j\} = -2mH\left(1 - H/E_c\right)\varepsilon_{ijk}L_k\,.$$

## Non-commutative Kepler problem: trajectories

- All space trajectories are plane curves,

$$\vec{L} \cdot \vec{x} = \ell L^2 .$$

The corresponding planes are orthogonal to the deformed angular momentum $\vec{L}$. The distance between the origin and the trajectory plane is equal to $\ell\, L$.

- By introducing the Cartesian coordinates $X$ and $Y$ in the plane of the orbit, we see that our trajectiories are conic sections,

$$\omega\,(X - X_0)^2 + \xi\, Y^2 = 1 .$$

- The parameters of the orbits depend on the energy in a non-trivial way,

$$\omega = \frac{4E^2(1 - E/E_c)^2}{C^2}, \qquad \xi = -\frac{2m\,E\,(1 - E/E_c)}{L^2},$$

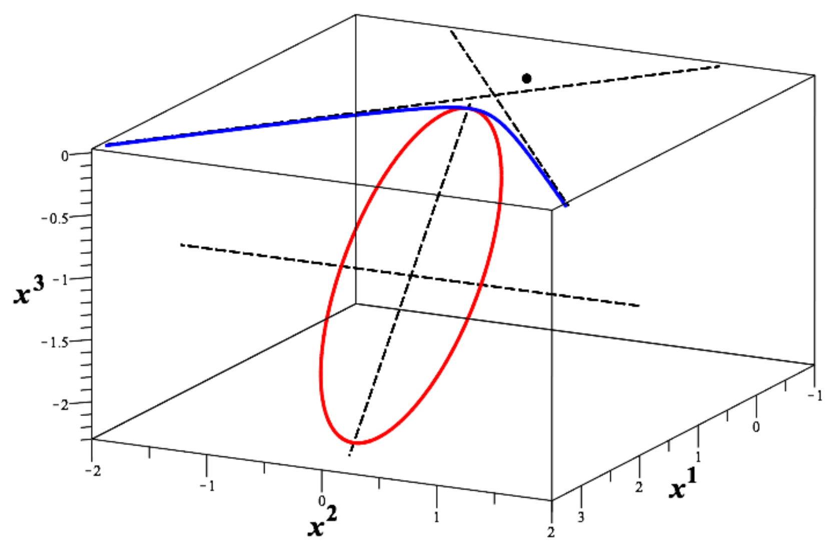
$$X_0 = \frac{(1 - 2E/E_c)A}{2m\,E\,(1 - E/E_c)}.$$

**Non-commutative Kepler problem: repulsive potential**

- The first novelty compared to the commutative case is the possibility of bounded (elliptic or radial) motion for the *repulsive* potential ($C > 0$), when $E > E_c$.
- The energy region $E > E_c$ is perfectly accessible when the particle is sufficiently close to the centre. Any trajectory that passes through the 'trapping region'

$$\mathcal{C} = \{\vec{x} \in \mathbb{R}^3 \mid |\vec{x}| < C/E_c \sim \ell\}\,, \qquad E_c^{-1} \sim \ell,$$

  is bounded.
- Numerical illustration of this effect:

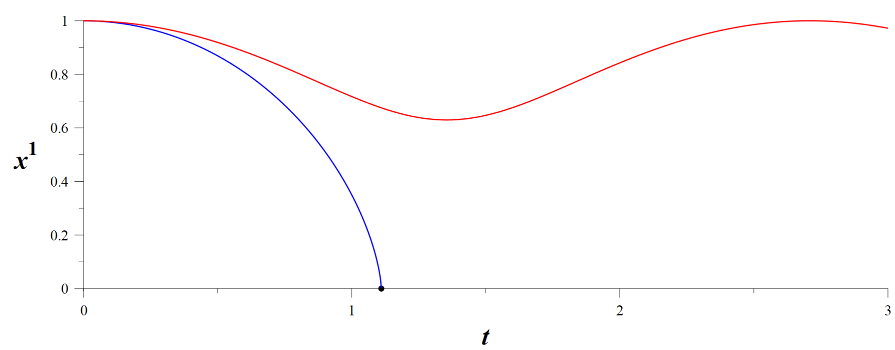


## Non-commutative Kepler problem: attractive potential

- Consider a radial motion, that is, $L = 0$. Another important novelty: even though the potential is attractive, the particle cannot fall to the centre!

- When $|\vec{x}| \to 0$, the potential energy $U(|\vec{x}|) \to -\infty$. Since the total energy is constant, the kinetic energy $T \to +\infty$. But this is impossible, since $T \leq 2/\ell^2 m$.

- Numerical illustration of this effect:



## Summary

- The Lie-Poisson electrodynamics describes the semi-classical approximation of the non-commutative $U(1)$ gauge theory with the Lie-algebra type non-commutativity.

- We presented the classical action and the equations of motion for both the gauge field and the charged particle. A compact momenta space of the particle naturally arises in this formalism.

- We discussed a super-integrable non-commutative Kepler problem for the $\mathfrak{su}(2)$ non-commutativity. The compactness of the momenta-space yields rather unexpected physical phenomena such as bounded motion for repulsive central force, and no-fall-into-the-centre for attractive Coulomb potential.