Progress in the Construction of AdS U-Folds

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Based on: D.Astesiano, D. Ruggeri, MT, **2401.04209; work in progress….**

Contents

Introduction

- Maximal supergravities in d-dimensions have provided a valuable framework to construct and study Type II/M-theory (flux) backgrounds of the form M_d x M_{int} (M_d maximally symmetric vacuum of the d-dimensional model, e.g. AdS_d). Useful for b.g.s with small residual symmetry.

- *Exceptional Field Theory* (ExFT) provides a direct embedding of (certain) gauged maximal supergravities in Type II string theories or D=11 SUGRA, and allows to compute the KK spectrum.

  [Hohm, Samtleben, **1312.0614, 1312.4542, 1406.3348, 1410.8145;** E. Malek, H. Samtleben, **1911.12640**; **2212.01135**]

- An example is the large class of (supersymmetric) S-fold (J-fold) solutions to Type IIB superstring theory from D=4 maximal supergravity with gauge group

$$\mathcal{G} = [\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes N^{(6,2)}$$

embedded in Type IIB through ExFT [G. Inverso, H. Samtleben, M.T., 1612.05123]

Type IIB S-Folds from D=4 SUGRA

S-fold solutions: non-geometric b.g.s featuring transition functions which involve duality transformation in $SL(2,\mathbf{Z})_{IIB}$ [C.Hull, A. Çatal-Özer, 0308133; C. Hull, 0406102]

- In our case S-folds have topology $\mathrm{AdS}_4 \times \tilde{S}^5 \times S^1$

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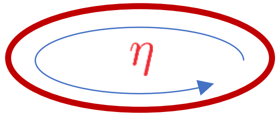

$$\eta \to \eta + T$$

$$\boldsymbol{\Psi} \to \mathfrak{M} \cdot \boldsymbol{\Psi}$$

- The monodromy $\mathfrak{M}$ is a hyperbolic element of SL(2,**Z**)$_{\text{IIB}}$

$$\mathfrak{M} = J_n = -\mathcal{S}\mathcal{T}^n = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}(2,\mathbb{Z})_{\text{IIB}}$$

$T = \text{arccosh}(n/2)$ $(n > 2)$ **"J-fold"**

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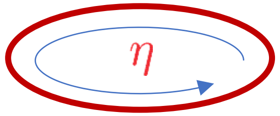

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- Dual to D=3 J-fold SCFT:
  - IR limit of T[U(N)] with U(N) subgroup of U(N)xU(N) gauged by *N*=4 vectors + level–n CS term
  - IR limit of *N*=4 D=4 SYM compactified on a circle with J-monodromy for the complexified c.c.

[D.Gaiotto, E.Witten, 0807.3720;
N=4: B. Assel and A. Tomasiello, 1804.0641;
N=2; N. Bobev, F. Gautason, K. Pilch, M.Suh,J. van Muiden, 2003.09154; E. Beratto, N. Mekareeya, M. Sacchi, 2009.10123; N. Bobev, F. Gautason, J. van Muiden, 2104.00977]

Type IIB S-Folds from D=4 SUGRA

- $\mathcal{N}$=4 with symmetry $SO(4)_R$ J-fold
  [vacuum found in H. Samtleben, A. Gallerati, M.T., 1410.0711; D=10 uplift in: G. Inverso, H. Samtleben, M.T., 1612.05123; Dual SCFT theory studied in: B. Assel and A. Tomasiello, 1804.0641]
- $\mathcal{N}$=2& $SU(2) \times U(1)_R$ J-fold
  [A. Guarino, C. Sterckx, M.T., 2002.03692]
- $\mathcal{N}$=2& $U(1) \times U(1)_R$ J-fold 1-parameter, KK spectrum
  [vacua found in 2002.03692 ; D=10 uplift in: A. Giambrone, E.Malek, H. Samtleben, M.T., 2103.10797]
- $\mathcal{N}$=2& $U(1) \times U(1)_R$ J-fold 2-parameters (D=4 vacuum, SUGRA, KK spectrum, black holes)
  [N. Bobev, F. Gautason, J. van Muiden, 2104.00977; M. Cesaro, G. Larios, O. Varela, 2109.11608; N. Bobev, Nikolay, S. Choi, J. Hong, V. Reys, 2407.13177; **A. Guarino, A. Rudra, C. Sterckx, M.T., 2407.11593**]
- $\mathcal{N}$=0& $U(1) \times U(1)_R$ stable J-fold, 2-parameters (D=4 vacuum and KK spectrum)
  [A.Guarino, C.Sterckx, 2109.06032 ; A. Giambrone, A. Guarino, E.Malek, H. Samtleben, C. Sterckx, M.T., 2112.11966]
- $\mathcal{N}$=0& SO(6) ; $\mathcal{N}$=1&SU(3) J-fold
  [A. Guarino, C. Sterckx, 1907.04177]
- $\mathcal{N}$=0& $U(1)^3$ (3-param.s) ; $\mathcal{N}$=1&$U(1)^2$ (2- param.s) J-folds and DWs
  [vacua found in 2002.03692 ; D=10 uplift in: A. Guarino, C. Sterckx, 2103.12652]

Type IIB S-Folds from D=5 SUGRA

- **N=1, N=2&U(2):** Bobev, F. Gautason, K. Pilch, M.Suh,J. van Muiden, 1907.11132, 2003.09154;
- **N=4 and N=2&U(1)2 (1- param.s) J-folds and DWs**: I. Arav, J.Gauntlett, M.Roberts, C.Rosen, 2103.15201

Type IIB S-Folds from D=4 SUGRA

- In all these solutions, as we move around the circle, the axion and the dilaton span a geodesic in their moduli space.
- Solutions can be obtained within a Scherk-Schwarz reduction from the N=8 SO(6)-gauged D=5 theory to D=4, with a hyperbolic twist in $SL(2,R)_{IIB}$, which defines the geodesic.

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- The simplest solution *N*=0& SO(6), of the form $\mathrm{AdS}_4 \times S^1 \times S^5$ with no 2-form field, suggests a general procedure for constructing AdS U-folds:

➢ Consider a $\mathrm{AdS}_d \times S^d \times M$ solution in Type IIB with moduli fields (d odd);
➢ Compactify one direction on the boundary of AdS_d ;
➢ Give the moduli fields a geodesic (in the moduli space) dependence along the compact direction at the boundary;
➢ Backreaction of the evolving scalar fields on spacetime yields a background of the form:

$$\mathrm{AdS}_{d-1} \times S^1 \times S^d$$

with a monodromy along S^1;
➢ The monodromy is the global symmetry element connecting the two end-points of the geodesic

D.Astesiano, D. Ruggeri, MT, **2401.04209**

General Construction

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- Consider a bosonic model in 2d-dimensions (d-odd), describing Einstein gravity coupled to n self-dual and m anti-self-dual (d -1)-form fields and scalar fields, described by a non-linear sigma model with symmetric target space.

$$B^M_{(\text{d-1})}\ ,\quad H^M_{(\text{d})} = \mathrm{d}B^M_{(\text{d-1})} \qquad M = 1,\ldots,n+m$$

$$\phi^s \in \mathscr{M}_{\text{scal}} = \frac{G}{H} \qquad \forall g \in G\ :\ g\cdot \mathbf{L}(\phi) = \mathbf{L}(\phi')\cdot h(\phi,g) \quad \begin{cases} [\mathbf{L}(\phi)] \in \frac{G}{H} \\ h(\phi,g) \in H \end{cases}$$

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- Duality: require G to have a pseudo-orthogonal representation R $\quad G\xrightarrow{R}\mathrm{O}(n,m)$

$$\forall g\in G\ :\ R[g]=(R[g]_M{}^N)\in\mathrm{O}(n,m)\qquad R[g]^T\,\Omega\,R[g]=\Omega\qquad \Omega=\text{diag}(\underbrace{+,\dots,+}_{n},\underbrace{-,\dots,-}_{m})$$

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- Couple the scalar fields to the forms in a G-invariant way by defining the matrix

$$\mathcal{M}(\phi) \equiv R[\mathbf{L}(\phi)]\cdot R[\mathbf{L}(\phi)]^T \in \mathrm{O}(n,m) \qquad \forall g\in G\ :\ \mathcal{M}(\phi') = R[g]\cdot\mathcal{M}\cdot R[g]^T$$

General Construction

- Field equations

(forms) $$d\mathbf{H}_{(\mathrm{d})} = 0\,,\quad {}^*\mathbf{H}_{(\mathrm{d})} = -\Omega\cdot\mathcal{M}(\phi)\cdot\mathbf{H}_{(\mathrm{d})}$$

$$\mathbf{H}_{(\mathrm{d})} \equiv (H^M_{(\mathrm{d})})$$

(scalars) $$D_{\hat\mu}\left(\partial^{\hat\mu}\phi^s\right) = \frac{1}{4d!}\mathscr{G}(\phi)^{st}\,\mathbf{H}^T_{(\mathrm{d})\hat\mu_1\dots\hat\mu_d}\cdot\left(\frac{\partial}{\partial\phi^t}\mathcal{M}\right)\cdot\mathbf{H}_{(\mathrm{d})}^{\hat\mu_1\dots\hat\mu_d}$$

$\mathscr{G}(\phi)_{st}$ metric on G/H

$\hat\mu,\,\hat\nu = 0,\dots,2d-1$

(Einstein) $$R_{\hat\mu\hat\nu} - \frac{1}{2}g_{\hat\mu\hat\nu}R = T^{(s)}_{\hat\mu\hat\nu} + T^{(H)}_{\hat\mu\hat\nu}$$

$$T^{(H)}_{\hat\mu\hat\nu} \equiv \frac{1}{2(d-1)!}\mathbf{H}_{\hat\mu\hat\mu_1\dots\hat\mu_{d-1}}\cdot\mathcal{M}(\phi)\cdot\mathbf{H}_{\hat\nu}{}^{\hat\mu_1\dots\hat\mu_{d-1}}$$

$$T^{(s)}_{\hat\mu\hat\nu} \equiv \frac{1}{2}\mathscr{G}_{st}\left(\partial_{\hat\mu}\phi^s\partial_{\hat\nu}\phi^t - \frac{1}{2}g_{\hat\mu\hat\nu}\,\partial_{\hat\rho}\phi^s\partial^{\hat\rho}\phi^t\right)$$

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- Are invariant under (global) G if $\forall g \in G\ :\ \mathbf{H}_{(\mathrm{d})} \to \mathbf{H}'_{(\mathrm{d})} = R[g]^{-T}\cdot\mathbf{H}_{(\mathrm{d})}$

- Look for solutions of the form: $\underbrace{\mathrm{M_d}} \times \underbrace{S^{\mathrm{d}}}$

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu + g_{ij}\,d\xi^i\,d\xi^j$$

$i,\,j = d,\dots,2d-1$

$\mu,\,\nu = 0,\dots,d-1$

General Construction

- Ansatz for the form field strengths: $$\mathbf{H}_{(\mathrm{d})} = -\Omega\, \mathcal{M}\, \Gamma\, \epsilon_{\mathrm{M}_d} + \Gamma\, \epsilon_{\mathrm{S}^d}$$

$$\epsilon_{\mathrm{M}_d} \equiv \frac{\tilde{e}_d}{d!\, L^d}\, \epsilon_{\mu_1 \ldots \mu_d}\, dx^{\mu_1} \wedge \ldots\, dx^{\mu_d} \qquad L \text{ radius of } S^{\mathrm{d}} \qquad \phi^s = \phi^s(x^\mu)$$

$$\epsilon_{\mathrm{S}^d} \equiv \frac{e_d}{d!\, L^d}\, \epsilon_{i_1 \ldots i_d}\, d\xi^{i_1} \wedge \ldots\, d\xi^{i_d} \qquad \tilde{e}_d = \sqrt{|\mathrm{det}(g_{\mu\nu})|}\,, \quad e_d = \sqrt{\mathrm{det}(g_{ij})}$$

- The charge vector $\Gamma = (\Gamma^M)$ is quantised: $$\Gamma^{\mathrm{M}} \equiv \frac{1}{\mathbb{S}_{\mathrm{S}^d}} \int_{\mathrm{S}^d} H^{\mathrm{M}}_{(\mathrm{d})} \in \boldsymbol{\Gamma}^{n,m}$$

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- The global symmetry group is reduced, by quantum corrections, to $G(\mathbb{Z}) \sim R[G] \cap \mathrm{O}(n,m;\mathbb{Z})$

- Quantum moduli space: $G(\mathbb{Z})\backslash G/H$

- Plugging the ansatz for the form fields in the field equations and *effective black-brane potential* originates:

$$\frac{1}{d!}\,\mathbf{H}^T_{\hat{\mu}_1\dots\hat{\mu}_d}\cdot\frac{\partial}{\partial\phi^s}\mathcal{M}\cdot\mathbf{H}^{\hat{\mu}_1\dots\hat{\mu}_d} = 4\,\frac{\partial}{\partial\phi^s}V(\phi,\Gamma)\,L^{-2d} \qquad V(\phi,\Gamma) \equiv \frac{1}{2}\,\Gamma^{\mathrm{T}}\cdot\mathcal{M}(\phi)\cdot\Gamma$$

General Construction

- Scalar field equation:

$$D_{\hat{\mu}}(\partial^{\hat{\mu}}\phi^s) = \nabla_{\hat{\mu}}(\partial^{\hat{\mu}}\phi^s) + \tilde{\Gamma}^s_{uv}\partial_{\hat{\mu}}\phi^u\partial^{\hat{\mu}}\phi^v = \mathcal{G}^{st}\frac{\partial}{\partial\phi^t}V\, L^{\text{-2d}}$$

- Ansatz for the scalar fields: $\phi^s \to \varphi^a,\ g^k \ :\ \mathbf{L}(\phi) = \mathbf{L}(\varphi)\,\mathbf{L}(g) = \mathbf{L}(g)\,\mathbf{L}(\varphi)$

do not enter the potential $\quad \mathbf{L}(\varphi) \in \frac{G_0}{H_0}\ ,\quad R[G_0]\cdot\Gamma = R[G_0]^{\mathsf{T}}\cdot\Gamma = \Gamma$

$$V(\varphi, g) = \frac{1}{2}\Gamma^{\mathsf{T}}\cdot R[\mathbf{L}(\varphi)]\cdot R[\mathbf{L}(g)]\cdot R[\mathbf{L}(g)]^{\mathsf{T}}\cdot R[\mathbf{L}(\varphi)]^{\mathsf{T}}\cdot\Gamma = \frac{1}{2}\Gamma^{\mathsf{T}}\cdot R[\mathbf{L}(g)]\cdot R[\mathbf{L}(g)]^{\mathsf{T}}\cdot\Gamma = V(g)$$

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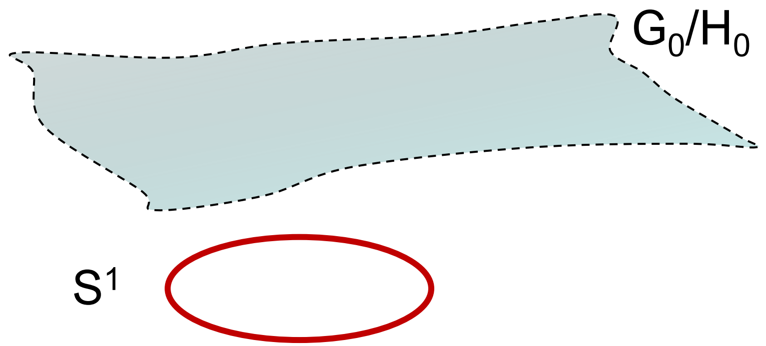
- Fix $g^k = g^k_*\ ,\ \left.\frac{\partial V}{\partial g^k}\right|_{g=g_*} = 0$ and let $V_* \equiv V(g_*) > 0$

- Cases:

$$\begin{cases} \mathrm{M_d} = \mathrm{AdS_d}\ ,\quad \varphi^a = \text{cost.} & \text{(moduli fields of AdS}_d\text{ x S}^d\text{)} \\ \mathrm{M_d} = \mathrm{AdS_{d\text{-}1}}\times \mathrm{S}^1\ ,\quad \varphi^a = \varphi^a(\eta) & \text{(geodesic on G}_0/\text{H}_0\text{)} \end{cases}$$

$$x^\mu = x^\alpha\ ,\ \eta$$

$$\kappa^2 = \frac{1}{2}\mathcal{G}_{ab}\,\dot{\varphi}^a\,\dot{\varphi}^b$$



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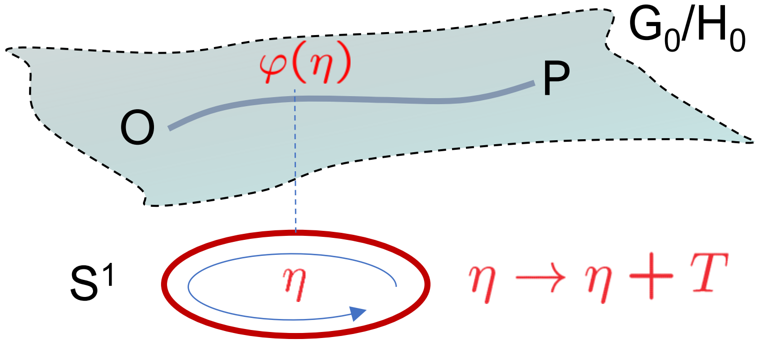

General Construction

The Einstein equation: $R_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\mathscr{G}_{st}\,\partial_{\hat{\mu}}\phi^s\partial_{\hat{\nu}}\phi^t + T^{(H)}_{\hat{\mu}\hat{\nu}}$

- $\mathrm{AdS}_d \times S^d$ metric: $ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu + g_{ij}\,d\xi^i\,d\xi^j = v_1^2\,ds^2_{\mathrm{AdS}_d} + L^2\,ds^2_{S^d}$

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- AdS_{d-1} x S^1 x S^d metric: $$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu + g_{ij}\,d\xi^i\,d\xi^j = v_1^2\,ds^2_{\mathrm{AdS}_{d-1}} + v_2^2\,d\eta^2 + L^2\,ds^2_{S^d}$$

$$v_1 = \sqrt{\frac{d-2}{d-1}}\,L\,,\quad v_2 = \frac{\kappa}{\sqrt{d-1}}\,L \qquad L = \left[\frac{V_*}{2(d-1)}\right]^{\frac{1}{2(d-1)}}$$

"velocity" of the geodesic

U-fold structure: AdS_{d-1} x S^1 x S^d background can be a consistent solution of the quantum theory provided the ending points of the geodesic are identified in the quantum moduli space:

$$O \equiv (\varphi^a(0)) \longrightarrow P \equiv (\varphi^a(T)) \qquad \exists\mathfrak{M} \in G_0(\mathbb{Z}) \;:\; \mathfrak{M}\cdot O = P$$

Solutions defined by conjugacy classes of $\mathfrak{M}$ in $G_0(Z)$

AdS$_2$ U-Folds

Applications of the construction: Type IIB in D=10=2d, d=5

$$G = \mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}\ ,\quad G(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})_{\mathrm{IIB}}$$

- Bosonic section consists of the metric and $\left(\begin{array}{c} \rho = C_{(0)} + i\,e^{-\phi} \in \frac{G}{H} = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \\ B^{\alpha}_{(2)}\ ,\ (\alpha = 1,2) \\ B^{\mathrm{M}}_{(4)} = C_{(4)}\ ,\ \hat{F}_{(5)} = {}^*\hat{F}_{(5)}\ ,\ R = 1 \end{array}\right)$
  - ➢ $AdS_5 \times S^5$ near horizon geometry of stack of D3 branes

AdS$_2$ U-Folds

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$$\mathfrak{M} = J_{\mathrm{n}} \in \mathrm{SL}(2,\mathbb{Z})$$
$$\mathrm{n} > 0$$

$$e^{\phi(\eta)} = \frac{\mathrm{n}\sinh\left(\sqrt{2}\kappa\eta\right)}{\sqrt{\mathrm{n}^2+4}} + \cosh\left(\sqrt{2}\kappa\eta\right)\ ,\quad C_{(0)}(\eta) = -\frac{2\sinh\left(\sqrt{2}\kappa\eta\right)}{\sqrt{\mathrm{n}^2+4}\cosh\left(\sqrt{2}\kappa\eta\right) + \mathrm{n}\sinh\left(\sqrt{2}\kappa\eta\right)}$$

$$T = \frac{1}{\sqrt{2}\kappa}\cosh^{-1}\left(\frac{\mathrm{n}^2}{2}+1\right)$$

$$\left(v_1 = \sqrt{3}L/2,\ \Gamma = Q = 4\,L^4\right)$$

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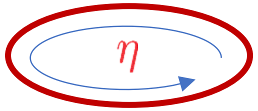

$$\eta \to \eta + T \quad \Rightarrow \quad \rho(0) \to \rho(T) = -\frac{1}{\rho(0)+\mathrm{n}}$$

[D.Astesiano, D. Ruggeri, MT, **2401.04209**]

AdS$_2$ U-Folds

Applications of the construction: Type IIB on CY$_2$ = T^4 , K3 (D=2d=6, d=3)

$$G = \mathrm{O}(5,m)\ ,\ \ G(\mathbb{Z}) = \mathrm{O}(5,m;\mathbb{Z})\ \rightarrow\ \boldsymbol{\Gamma}^{5,m}$$

- Type IIB on T^4: m=5. N=(2,2) maximal surga in D=2d=6, d=3;

$$\left(\begin{array}{c} \phi^{s} \in \frac{G}{H} = \frac{\mathrm{O}(5,m)}{\mathrm{O}(5)\times\mathrm{O}(m)} \\ A^{\alpha}_{(1)}\ ,\ \ (m=5) \\ B^{\mathrm{M}}_{(2)}\ (\mathrm{M}=1,\dots,5+m),\ R = \mathbf{5+m} \end{array}\right)$$

AdS_2 U-Folds

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- ➢ AdS$_3$ x S^3 x CY$_2$ near horizon geometry of D1-D5 system and duals (e.g. F1-NS5); $Q_1,\, Q_5 \in \boldsymbol{\Gamma}^{1,1} \subset \boldsymbol{\Gamma}^{5,m}$

$$\boldsymbol{\Gamma}^{5,m} = \boldsymbol{\Gamma}^{1,1} \oplus \boldsymbol{\Gamma}^{4,m-1}$$

$$G_0(\mathbb{Z}) \subset \mathrm{O}(4,m-1;\mathbb{Z})\ ,\ \ \varphi^a \in \frac{G_0}{H_0} \subset \frac{\mathrm{O}(4,m-1)}{\mathrm{O}(4)\times\mathrm{O}(m-1)}$$

- The relevant scalar fields:

$$\mathrm{O}(1,1)[g] \times \frac{G_0}{H_0}[\varphi^a] \subset \mathrm{O}(1,1)[g] \times \frac{\mathrm{O}(4,m-1)}{\mathrm{O}(4)\times\mathrm{O}(m-1)} \subset \frac{\mathrm{O}(5,m)}{\mathrm{O}(5)\times\mathrm{O}(m)}$$

AdS_2 U-Folds

An explicit solution Type IIB on T^4 with D1-D5 charges $\Gamma^M = 2\,(Q_5,\, Q_1;\, 0, \dots, 0)$

$$e^g = e^\phi\, \det(G_{ij})^{\frac{1}{2}} = e^{g_*} = \frac{Q_1}{Q_5} \qquad \varphi^a \in \frac{G_0}{H_0} \subset \frac{\mathrm{O}(4,4)}{\mathrm{O}(4)\times\mathrm{O}(4)} \left[\tilde{G}_{ij} = e^{-\frac{\phi}{2}}\, G_{ij},\, C_{ij}\right]$$

AdS$_2$ U-Folds

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- Geodesic evolution of moduli fields with monodromy $\mathfrak{M} \in \mathrm{O}(4,4;\mathbb{Z})$

Choose e.g.:
$$\begin{cases} (\varphi^a) = (\rho_1,\, \rho_2) = (C_{12} + i\,\tilde{G}_{11},\, C_{34} + i\,\tilde{G}_{33}) \in \frac{G_0}{H_0} = \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right)^2 \subset \frac{\mathrm{O}(4,4)}{\mathrm{O}(4)\times\mathrm{O}(4)} & \begin{matrix}\tilde{G}_{22} = \tilde{G}_{11} \\ \tilde{G}_{44} = \tilde{G}_{33}\end{matrix} \\ \mathfrak{M} = \mathfrak{M}_1 \cdot \mathfrak{M}_2 = J_{\mathtt{n}_1} \cdot J_{\mathtt{n}_2} \in \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{O}(4,4;\mathbb{Z})\;, & \mathtt{n}_\ell > 0 \end{cases}$$

AdS$_2$ U-Folds

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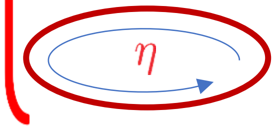

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AdS$_2$ U-Folds

Explicit geodesic evolution:

$$\tilde{G}_{22}(\eta)^{-1} = \tilde{G}_{11}(\eta)^{-1} = \frac{\sqrt{n_1^2+4}\cosh\left(\frac{\eta\cosh^{-1}\left(\frac{1}{2}(n_1^2+2)\right)}{T}\right) + n_1\sinh\left(\frac{\eta\cosh^{-1}\left(\frac{1}{2}(n_1^2+2)\right)}{T}\right)}{\sqrt{n_1^2+4}}$$

$$C_{12}(\eta) = -\frac{2\sinh\left(\frac{\eta\cosh^{-1}\left(\frac{1}{2}(n_1^2+2)\right)}{T}\right)}{\sqrt{n_1^2+4}\cosh\left(\frac{\eta\cosh^{-1}\left(\frac{1}{2}(n_1^2+2)\right)}{T}\right) + n_1\sinh\left(\frac{\eta\cosh^{-1}\left(\frac{1}{2}(n_1^2+2)\right)}{T}\right)}$$

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$$\kappa^2 = \frac{1}{2T^2}\left(\cosh^{-1}\left(\frac{n_1^2}{2}+1\right) + \cosh^{-1}\left(\frac{n_2^2}{2}+1\right)\right)$$

AdS_2 U-Folds

Explicit geodesic evolution:

$$e^{\phi(0)} = \left(\frac{Q_1}{Q_5}\right)^{\frac{1}{2}}$$

↓

$$e^{\phi(T)} = \left(\frac{Q_1}{Q_5}\right)^{\frac{1}{2}} \sqrt{(\mathtt{n}_1+1)(\mathtt{n}_2+1)}$$

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Conclusions

- Discussed how to construct a $AdS_{d-1} \times S^1 \times S^d$ U-fold from $AdS_d \times S^d$ with moduli by giving the moduli a geodesic (in the moduli space) dependence along a compact direction on the boundary of AdS_d . Constructed explicit examples in Type IIB superstring theory

- Effect of the geodesic evolution of the moduli is equivalently described by a Scherk-Schwarz (SS) reduction on S^1 with non-compact twist $\mathcal{A}(\eta) = L(\varphi^a(\eta))$

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Open issues:

- Construct more general, supersymmetric $\mathrm{AdS}_2 \times S^1 \times \tilde{S}^3$ backgrounds in which the vector fields in D=2d=6 play the role of the 2-forms in the supersymmetric $\mathrm{AdS}_4 \times S^1 \times \tilde{S}^5$

- Partial results obtained by performing a SS compactification to D=2 from half-maximal D=3 gauged supergravity featuring supersymmetric AdS_3 vacua [Adolfo's talk], with SS-twist in the isometry group of the moduli space of the AdS_3 vacua.
  [D.Astesiano, S. Maurelli, M. Oyarzo,H. Samtleben, MT, **work in progress…]**

- Starting from an N=2 truncation (n_v=3,n_h=4) of the N=8, D=4 gauged sugra, describing the N=4 S-fold and its N=2 marginal deformations, constructed a susy $AdS_2 \times \Sigma_2 \times S^1 \times S^5$ J-fold with parametrically controlled scale separation between AdS_2 and Σ_2
  [A. Guarino, A. Rudra, C. Sterckx, MT, **2407.11593]**

Thank You!

Einstein equations

- AdS_{d-1} x S^1 x S^d metric: $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + g_{ij}\, d\xi^i\, d\xi^j = v_1^2\, ds^2_{\mathrm{AdS}_{d-1}} + v_2^2\, d\eta^2 + L^2\, ds^2_{S^d}$

$$R_{\hat\mu\hat\nu} = \tfrac{1}{2}\mathscr{G}_{st}\,\partial_{\hat\mu}\phi^s\partial_{\hat\nu}\phi^t + T^{(H)}_{\hat\mu\hat\nu}$$

LHS

$$R_{\alpha\beta\gamma\delta} = -v_1^{-2}\,(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \;\Rightarrow\; R_{\alpha\beta} = R_{\alpha\gamma\beta}{}^{\gamma} = -\frac{(d-2)}{v_1^2}\,g_{\alpha\beta}$$

$$R_{ijkl} = L^{-2}\,(g_{ik}g_{jl} - g_{il}g_{jk}) \;\Rightarrow\; R_{ij} = R_{ikj}{}^{k} = \frac{(d-1)}{L^2}\,g_{ij}$$

$$R_{\eta\eta} = 0$$

RHS

$$T^{(H)}_{\alpha\beta} = -\frac{1}{2}V_*\,L^{-2d}\,g_{\alpha\beta}$$

$$T^{(H)}_{\eta\eta} = -\frac{1}{2}V_*\,v_2^2\,L^{-2d}$$

$$T^{(H)}_{ij} = \frac{1}{2}V_*\,L^{-2d}\,g_{ij}$$

$$\frac{1}{2}\mathscr{G}_{ab}\,\partial_\eta\varphi^a\partial_\eta\varphi^b = \kappa^2$$

$$v_1 = \sqrt{\frac{d-2}{d-1}}\,L\,,\quad v_2 = \frac{\kappa}{\sqrt{d-1}}\,L \qquad L = \left[\frac{V_*}{2(d-1)}\right]^{\frac{1}{2(d-1)}}$$

Geodesics

- Description of the geodesic originating in $\varphi(0) = (\varphi^a(0))$ with «velocity» $\mathbb{Q} \in T_{\varphi(0)}\mathscr{M}_{\text{scal}}$.

$$\mathcal{M}(\varphi(\eta)) = \mathcal{M}(\varphi(0)) \cdot e^{\mathbb{Q}^T \eta} = R[\mathbf{L}(\varphi(0))] \cdot e^{\mathbb{Q}_0 \eta} \cdot R[\mathbf{L}(\varphi(0))]^T$$

$$\mathbb{Q}_0 = R[\mathbf{L}(\varphi(0))]^{-1} \cdot \mathbb{Q} \cdot R[\mathbf{L}(\varphi(0))] \in T_O\mathscr{M}_{\text{scal}}.$$

- If $\mathfrak{M} \cdot \varphi(0) = \varphi(T)\,,\ \ \mathfrak{M} \in G_0(\mathbb{Z})$

$$R[\mathbf{L}(\varphi(0))] \cdot e^{\mathbb{Q}_0 T} \cdot R[\mathbf{L}(\varphi(0))]^T = \mathfrak{M}\, R[\mathbf{L}(\varphi(0))] \cdot R[\mathbf{L}(\varphi(0))]^T \cdot \mathfrak{M}^T$$

$$\Downarrow$$

To be solved in $\mathbb{Q}_0\, T$

D1-D5 System

Precise correspondence between sugra charge parameters Q_1, Q_5 and string flux parameters $\mathbf{Q}_1$, $\mathbf{Q}_5$

$$Q_1 = r_1^2 = g_s\, \mathbf{Q}_1 (\alpha')^3 \frac{(2\pi)^4}{V_4} \qquad Q_5 = r_5^2 = g_s\, \mathbf{Q}_5 \alpha' \qquad L = (Q_1 Q_5)^{\frac{1}{4}} = \sqrt{\alpha'}\, [g_6^2\, \mathbf{Q}_1\, \mathbf{Q}_5]^{\frac{1}{4}}$$

$$V_4/(2\pi)^4 = (\det(G_{ij}^{(s)}))^{\frac{1}{2}} \qquad g_6^2 = g_s^2/(\det(G_{ij}^{(s)}))^{\frac{1}{2}} = g_s/(\det(G_{ij}^{(E)}))^{\frac{1}{2}}$$

F1-NS5 System

An explicit solution Type IIB on T⁴ with F1-NS5 charges $\Gamma^M = 2\,(Q_5, Q_1; 0, \ldots, 0)$

$$e^g = e^{-2\phi_6} = e^{-\phi} \det(G_{ij})^{\frac{1}{2}} = e^{g_*} = \frac{Q_1}{Q_5} \qquad \varphi^a \in \frac{G_0}{H_0} \subset \frac{O(4,4)}{O(4)\times O(4)} \left[G_{ij}^{(s)} = e^{\frac{\phi}{2}} G_{ij},\, B_{ij}\right]$$

$$O(4,4) = \text{T-duality along } T^6$$

D=3 Approach

Start from N=(1,1), D=6 coupled to n=4 vector multiplets (1 tensor, 8 vectors, 17 scalars)

$$\mathscr{M}^{(D=6)}_{\rm scal} = \mathrm{SO}(1,1) \times \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4)\times\mathrm{SO}(4)}$$

From Type IIB on $T^4/\mathbf{Z}_2$
[O(5)-orientifold]
(see Adolfo's talk)

Compactified to D=3 on a 3-sphere. Gauged, half-maximal D=3 sugra

$$\mathscr{M}^{(D=3)}_{\rm scal} = \frac{\mathrm{SO}(8,8)}{\mathrm{SO}(8)\times\mathrm{SO}(8)} = \left[\frac{\mathrm{SO}(3,3)}{\mathrm{SO}(3)\times\mathrm{SO}(3)} \times \mathrm{SO}(1,1) \times \underset{\displaystyle \varphi^a}{\underset{\Psi}{\frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4)\times\mathrm{SO}(4)}}}\right] \times \exp\left((\mathbf{4},\mathbf{8})_{+1} \oplus (\mathbf{6},\mathbf{1})_{+2}\right)$$

And gauge group:

$$G_{\rm gauge} = (T^4)^8 \times [SO(4) \ltimes (T^3 \times T^3)]$$

Perform a SS reduction to D=2 with non-compact twist $\mathcal{A}(\eta) = \mathbf{L}(\varphi(\eta)) \in \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4)\times\mathrm{SO}(4)}$

describing a geodesic motion of the moduli fields. Look for AdS_2 extrema of the D=2 potential

Recovered the backgrounds described earlier and deformations thereof (possibly susy)