The $\mathfrak{so}(2,2)$ Poisson Sigma Model

To appear : arXiv24**.*****, G. Chirco, P. Vitale, L. Vacchiano

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Corfù Summer Institute 2024

September 22, 2024

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Extended SYK/JT correspondence

Schwarzian

JT Gravity ⟷ SYK

$diff(S^1)$

+ Gauge Fields

+ Additional Symmetries

JT – Yang Mills

SYK-like tensor Models

$diff(S^1) \ltimes \hat{g}$

Schwarzian + Particle on a group manifold

The JT and JT-YM formulation via PSM

The bulk JT-gravity theory can be written as a BF theory over the $\mathfrak{sl}(2,\mathbb{R})$ algebra with J_i generators. Given $A = A^i_\mu J_i$, $B = B^i J_i$ and $F = \mathcal{D}_A A$, one has :

$$S_{JT} = \int d^2x\sqrt{-g}\Phi(R-2) \sim S_{BF} = \int \mathrm{Tr}(BF)$$

Moreover, given a Linear Poisson Sigma Model with the same $A = A^i_\mu J_i$:

$$S_{PSM} = \int_\Sigma A_i \wedge dX^i + \frac{1}{2}\Pi^{ij}(X) A_i \wedge A_j, \qquad \Pi^{ij}(X) = f^{ij}_k X^k$$

if f^k_{ij} are chosen to be the $\mathfrak{sl}(2,\mathbb{R})$ structure constants, then

$$S_{PSM} = S_{BF} - \int_{S^1} X^i A_i.$$

Recovering the boundary action

The dynamical boundary action of JT gravity can be then recovered by introducing a boundary **Casimir function**.

$$S_{TOT} = S_{PSM} + \int_{S^1} X^i X_i d\tau$$

The equations of motion are $\mathcal{D}_A A = 0$ and $\delta_X A = 0$
With the **boundary conditions** $X_i|_{S^1} d\tau = A_i|_{S^1}$, the **on-shell** boundary action is

$$S_{PSM}|_{S^1, On-Shell} = \int_{S^1} \text{Tr}(g^{-1} g')^2 d\tau.$$

which reduces to

$$S_{PSM}|_{S^1, On-shell} = \int_{S^1} d\tau \{ \frac{1}{2} \phi'^2 + S(\phi) \}$$

Yang-Mills extension

By promoting $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}$ one gets the **Yang-Mills** extension of the JT model. We fix $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$. Within $\mathfrak{so}(2,2) \simeq \mathfrak{sl}_L(2,\mathbb{R}) \oplus \mathfrak{sl}_R(2,\mathbb{R})$ we can introduce the basis

$$[L_i, L_j] = f_{ij}^k L_k, \qquad [R_i, R_j] = f_{ij}^k R_k \qquad [L_i, R_j] = 0.$$

and then rotate it into

$$[L_i, L_j] = f_{ij}^k L_k, \qquad [J_i, J_j] = J_{ij}^k R_k, \qquad [J_i, L_j] = f_{ij}^k L_k.$$

with $J_i = L_i + R_i$ and f_{ij}^k structure constants of $\mathfrak{sl}(2,\mathbb{R})$. Let Ω be the Lie algebra-valued 1-form and $\mathfrak{Z}$ the embedding maps

$$\Omega = A^i J_i + B^i L_i \qquad \mathfrak{Z} = X_i J^i + Y_i L^i$$

The $\mathfrak{so}(2,2)$ Poisson Sigma Model

The $\mathfrak{so}(2,2)$-Poisson Sigma Model bulk action is

$$S_{PSM} = \int_{\Sigma} d\Omega_i \wedge \mathfrak{Z}^i + \frac{1}{2}\Pi^{ij}(\mathfrak{Z})\,\Omega_i \wedge \Omega_j + \int_{S^1} \mathfrak{Z}^{\mathfrak{i}}\mathfrak{Z}_{\mathfrak{i}} d\tau$$

where the additional **boundary** term is again written as a Casimir function.
Boundary conditions are chosen such that :

$$\mathfrak{Z}_{\mathfrak{i}}|_{S^1} d\tau = \Omega_i|_{S^1}.$$

Given $\delta_x = dX + [X, \,]$, the equations of motion for the **gravitational sector** are :

$$\mathfrak{D}_A A = 0, \qquad \delta_X A = 0$$

while for the **YM sector** one has

$$\mathfrak{D}_\Omega B = \frac{1}{2}[A, B], \qquad \delta_{\mathfrak{Y}}\Omega = -[X, B].$$

Asymptotic Symmetries

In a basis independent fashion, the equations of motion read :

$$\mathcal{D}_\Omega \Omega = 0, \qquad \delta_{\mathfrak{Z}} \Omega := d\mathfrak{Z} + [\mathfrak{Z}, \Omega] = 0.$$

By virtue of the boundary conditions $\mathfrak{Z}_{\mathrm{i}}|_{S^1} d\tau = \Omega_i|_{S^1}$, we get that the **on-shell** boundary action is again given by the particle on a group manifold action :

$$S|_{S^1,\ on-shell} = \int \mathrm{Tr}(g^{-1}g')^2 d\tau$$

The allowed infinitesimal gauge transformations at the boundary are those which stabilize the connection

$$\delta_\Lambda \Omega|_{S^1} = 0.$$

The $\mathfrak{Z}$ fields, and therefore the $\mathfrak{Z}|_{S^1}$ fields, induce infinitesimal gauge transformations that stabilize the connection.

Asymptotic Symmetries

We can get a residual gauge symmetry at boundary by further imposing $X_i|_{S^1} = -Y_i|_{S^1}$, which implies

$$\mathfrak{Z} = (X_i + Y_i)J^i + Y_i(L^i - J^i) \xrightarrow{X_i = -Y_i|_{S1}} \mathfrak{Z}|_{S^1} = -Y_i R^i \in \widehat{\mathfrak{sl}_R(2,\mathbb{R})}$$

- **Gauge** : With this condition, the stabilizer is no longer $\mathfrak{so}(2,2)$-valued at the boundary. An $\widehat{\mathfrak{sl}(2,\mathbb{R})}$ (or $\mathcal{LG}$) residual gauge symmetry is established at the boundary.
- **Coordinates**: The Diff(S^1) reparametrisation invariance is explicitly broken by the boundary term. For the Schwarzian case it is broken into $H = SL(2,R)$.

Coadjoint Orbits

- **Boundary Symmetry Breaking** $\mathcal{K}_{S^1} \to \mathcal{H}_{S^1}$

  The reduction of the **on-shell** boundary action on the actual coset $\mathcal{K}_{S^1}/\mathcal{H}_{S^1}$ can be performed via the (Kirillov) coadjoint orbit method.

- **Our case** : $\mathrm{Diff}(S^1) \to \mathrm{SL}(2,\mathbb{R})$ and $\widehat{\mathrm{SO}(2,2)} \to \widehat{\mathrm{SL}_R(2,\mathbb{R})}$ (or in general $\mathcal{LG}$).

  We need to compute coadjoint orbits for $\mathrm{Diff}(S^1) \ltimes \mathcal{LG}$.

Coadjoint Orbits

From the Kirillov's method it follows that Coadjoint Orbits for Diff(S^1) and Loop Groups are given by :

- **Diff**(S^1) : $\widetilde{Ad}^*_{\phi^{-1}}(u, \xi) = (u \circ \phi + \xi S(\phi), \xi)$
- $\mathcal{LG}$: $\widetilde{Ad}^*_{g^{-1}}(a, \beta) = (Ad^*_{g^{-1}} a + \beta g^{-1} dg, \beta)$

This is sufficient to compute the full coadjoint orbit (Zuevsky 2018) :

$$S|_{S^1, on-shell} = \int d\tau \Big\{ u\phi'^2 + \xi S(\phi) + \mathrm{Tr}\left(g^{-1}g', a\phi'^2 + \frac{1}{2}\beta g^{-1}g'\right) \Big\}$$

3D Topological Field Theory

The $\mathfrak{so}(2,2)$-PSM, boundary action included, can be recovered through the dimensional reduction of a Chern-Simons theory with WZW boundary term :

$$S = \frac{k}{4\pi}\int_{\Sigma^3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) - \frac{k}{8\pi}\int_{\partial\Sigma^3} d^2x \, \mathrm{Tr}\left(g^{-1}\partial^{\mu} g, g^{-1}\partial_{\mu} g\right)$$

The Kac-Moody modes in SYK-tensor models can be also interpreted as K.K. modes form the 3D perspective.

The End