# Renormalization of Composite Operators & RG flow
## *(In progress ...)*

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# Outline

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# Motivation

- Spectral index of the CMB $n_s^{(obs)} = 0.964$[1]
- Ising critical exponent $\eta^{(lattice)} = 0.036$
- $n_s^{(obs)} = 1 - \eta^{(lattice)} \longrightarrow$ Ising Cosmology[2]

$$n_s = 1 + \frac{\partial}{\partial \ln \mu} \langle \Theta\Theta \rangle \quad \text{[See Kalogirou's talk]} \tag{1}$$

- Diagrammatic calculation of $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ and consequently of $\langle \Theta\Theta \rangle$ in the context of $\lambda\phi^4$ theory for $d = 4 - \epsilon$
- Check of the CS equation of $\langle \Theta\Theta \rangle$

$$\begin{aligned} &\left[\frac{\partial}{\partial \ln \mu} + \beta_\lambda \frac{\partial}{\partial \lambda}\right] \langle \Theta\Theta \rangle = 0 \\ &\Rightarrow \frac{\partial}{\partial \ln \mu} \langle \Theta\Theta \rangle = -\beta_\lambda \frac{\partial}{\partial \lambda} \langle \Theta\Theta \rangle \underbrace{=}_{?} -\eta \langle \Theta\Theta \rangle \end{aligned} \tag{2}$$

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1 Planck Collaboration et al. *Planck 2018 results. IX. Constraints on primordial non-Gaussianity*. 2019. arXiv: 1905.05697 [astro-ph.CO]. URL: https://arxiv.org/abs/1905.05697.

2 Nikos Irges, Antonis Kalogirou, and Fotis Koutroulis. "Ising Cosmology". In: *The European Physical Journal C* 83.5 (May 2023). DOI: 10.1140/epjc/s10052-023-11622-8. URL: https://doi.org/10.1140%2Fepjc%2Fs10052-023-11622-8.

# Motivation

## Questions

1. Which is the appropriate definition of Energy-Momentum Tensor (EMT) as an QFT operator? (There exist Improvements[3] )

$$T^{(0)}_{\mu\nu} = \partial_\mu \phi_0 \partial_\nu \phi_0 + \frac{\lambda_0}{4!} g_{\mu\nu} \phi_0^4 - \frac{1}{2} g_{\mu\nu} \left(\partial \phi_0\right)^2 \tag{3}$$

$$\Theta^{(0)} = \left(1 - \frac{d}{2}\right) \left(\partial \phi_0\right)^2 + \frac{\lambda_0 d}{4!} \phi_0^4 \tag{4}$$

$$\begin{aligned} T^{(0)}_{\mu\nu} &\to T^{(0)}_{\mu\nu} + \# \left(\partial_\mu \partial_\nu - g_{\mu\nu} \Box\right) \phi_0^2 \\ \Theta^{(0)} &\to \Theta + (d-1) \# \Box \phi_0^2 \end{aligned} \tag{5}$$

2. Can we define a bare $\Theta_0$ which, after the renormalization procedure, takes the form $\Theta = Z_\Theta^{-1} \Theta_0 = \beta_i \mathcal{O}^i$?
3. Is the eigenvalue of $\Theta$ affected by how we define $\Theta$?

[3] Curtis G Callan, Sidney Coleman, and Roman Jackiw. "A new improved energy-momentum tensor". In: *Annals of Physics* 59.1 (1970), pp. 42–73. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(70)90394-5. URL: https://www.sciencedirect.com/science/article/pii/0003491670903945.

# Outline

# Renormalization (of a theory) Algorithm

**Steps:**

1. We define the renormalized field $\phi$ and the renormalized coupling constant $\lambda$ by inserting the counterterms, and we write down the renormalized Lagrangian.
2. We apply Wick's theorem to the renormalized Lagrangian and evaluate all the corresponding loop diagrams for $d = 4 - \epsilon$, keeping the finite terms.
3. We impose the renormalization conditions at a certain energy scale. Using these conditions, we evaluate the expressions of the counterterms.
4. We substitute the counterterms and obtain the renormalized expressions, which give the dependence of the correlation functions on the energy scale.
5. With the use of the Callan-Symanzik equation, we calculate the RG-functions.

# Renormalization of $\lambda\phi^4$-theory (Warm up )

The bare theory is described by the following action:

$$S^{(0)}[\phi_0; \lambda_0] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi_0)^2 - \frac{\lambda_0}{4!}\phi_0^4\right] \tag{6}$$

After introducing the counterterms the renormalized action is given by:

$$\begin{aligned} S &= \int \mathrm{d}^d x \left[\frac{Z_\phi}{2}(\partial_\mu\phi)^2 - \frac{Z_\lambda}{4!}\lambda\phi^4\right] \\ &= \int \mathrm{d}^d x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{\delta_\phi}{2}(\partial_\mu\phi)^2 - \frac{\delta_\lambda\lambda}{4!}\phi^4\right] \end{aligned} \tag{7}$$

with

$$\begin{aligned} \phi_0 &= Z_\phi^{1/2}\phi \\ \lambda_0 &= Z_\lambda Z_\phi^{-2}\lambda \end{aligned} \tag{8}$$

where $Z_\phi = 1 + \delta_\phi$ and $Z_\lambda = 1 + \delta_\lambda$

# (Off-Shell)Renormalization conditions - Propagator

- The interacting propagator is written as:

$$\langle \phi(p)\phi(-p)\rangle = \frac{ig(p^2)}{p^2 + \Pi(p^2)} \tag{9}$$

where $\Pi(p^2)$ is the sum of all 1 P.I. correction of the propagator.

- We introduce an energy scale $\mu^2$ where we imply the following renormalization conditions for the propagator:

$$\begin{aligned} \Pi(p^2)\Big|_{p^2=-\mu^2} &= 0 \\ \frac{\mathrm{d}\Pi(p^2)}{\mathrm{d}p^2}\Bigg|_{p^2=-\mu^2} &= 0 \end{aligned} \tag{10}$$

# (Off-Shell)Renormalization conditions - Coupling Constant

- The coupling constant $\lambda$ is defined as the magnitude $\mathcal{M}$ of the scattering $\phi\phi \to \phi\phi$

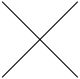
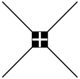
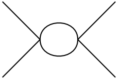
$$\langle\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle = \mathcal{M}\frac{i}{p_1^2}\frac{i}{p_2^2}\frac{i}{p_3^2}\frac{i}{p_4^2}(2\pi)^d\delta\left(\sum p_{in} - \sum p_{out}\right) \quad (11)$$

- For the renormalization of the coupling constant $\lambda$ we define the following renormalization condition:

$$\mathcal{M} = -i\lambda\ , \text{ at } S.P.(\mu) \text{ with } s^2 = t^2 = u^2 = -\mu^2 \quad (12)$$

where $S.P.(\mu)$ is the Symmetric Point.

The one loop contributions of the $\phi\phi \to \phi\phi$ scattering are given by:.

$\mathcal{M} =$  $+$  $+$  $+ (t, u)$-channels

$$= -i\lambda - i\lambda\delta_\lambda^{(1)} - \frac{(-i\lambda)^2}{2} \sum_{p^2=s,t,u} L_1(p^2) \tag{13}$$

$$L_1(p^2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} = i \frac{\Gamma\left(2 - \frac{d}{2}\right) \left[\Gamma\left(\frac{d}{2} - 1\right)\right]^2}{(4\pi)^{d/2} \Gamma(d-2)} \left(-p^2\right)^{\frac{d}{2}-2} \tag{14}$$

Using the renormalization condition we evaluate the counterterm $\delta_\lambda^{(1)}$

$$\delta_\lambda^{(1)} = -i\frac{3\lambda}{2} L_1(-\mu^2) \tag{15}$$

Which implies that for $d = 4 - \epsilon$ with $\epsilon \to 0$

$$\mathcal{M} = -i\lambda - i\frac{\lambda^2}{2(4\pi)^2} \sum_{p^2=s,t,u} \ln\left(\frac{-p^2}{\mu^2}\right) \tag{16}$$

# One-loop Renormalization and $\beta$-function

We will calculate the $\beta$-function by applying the Callan-Symanzik equation on $\mathcal{M}$:

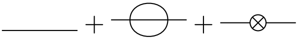
$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\partial_\lambda + 4\gamma_\phi\right]\langle\phi\phi\phi\phi\rangle = 0 \tag{17}$$

Since $\gamma_\phi \sim \mathcal{O}(\lambda^2)$, solving for $\beta_\lambda$ gives:

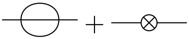
$$\beta_\lambda = -3\frac{\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3) \tag{18}$$

# Two-loop Renormalization and $\gamma$-function

The massless interacting propagator is given by the following diagrams:

$$\langle\phi(p)\phi(-p)\rangle =$$  (19)

In this case:

$$-i\Pi(p^2) =$$  $$= -\frac{i}{p^4}\frac{\lambda^2}{6}S_1(p^2) - \frac{i}{p^2}\delta_\phi \tag{20}$$

$$S_1(p^2) = \int \frac{\mathrm{d}^d k_{1,2}}{(2\pi)^{2d}} \frac{1}{k_1^2 k_2^2 \left(k_1 + k_2 - p\right)^2} = \frac{(-1)^{d-4}}{(4\pi)^d} \frac{\Gamma(3-d)\left[\Gamma\left(\frac{d}{2}-1\right)\right]^3}{\Gamma\left(\frac{3d}{2}-3\right)} (-p^2)^{d-3} \tag{21}$$

Using the renormalization condition we get $\delta_\phi$ and as a consequence the renormalized propagator for $d \to 4$:

$$\delta_\phi = -\frac{1}{-\mu^2}\frac{\lambda^2}{6}S_1(-\mu^2) \Rightarrow \langle\phi(p)\phi(-p)\rangle = \frac{i}{p^2}\left[1 + \frac{\lambda^2}{12(4\pi)^2}\ln\left(\frac{-p^2}{\mu^2}\right)\right] \tag{22}$$

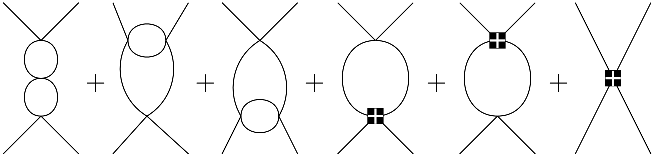
Solving the CS- equation for $\gamma_\phi$ we get:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\partial_\lambda + 2\gamma_\phi\right]\langle\phi(p)\phi(-p)\rangle = 0 \Rightarrow \gamma_\phi = \boxed{\frac{\lambda^2}{12(4\pi)^4} + \mathcal{O}(\lambda^3)} \tag{23}$$

# Two-loop Renormalization and $\beta$-function

The 2-loop contributions of the $s$-channel of the $\phi\phi \to \phi\phi$ scattering are given by the following diagrams:

$\mathcal{M}^{(2)}(s) =$



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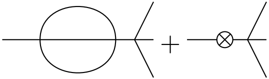
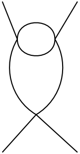
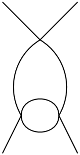


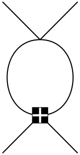
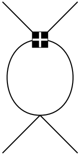
$$= \frac{i\lambda^3}{4}\left[L_1(s)\right]^2 - i\lambda^3 D(s) + 2\frac{\delta_\lambda^{(1)}\lambda^2}{2}L_1(s) - i\delta_{\lambda,s}^{(2)}\lambda - i\frac{\lambda^3}{6}\frac{S_1(q^2)}{q^2} - i\lambda\delta_\phi \quad (24)$$

**The first 5 diagrams contain overlapping divergences!**

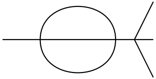
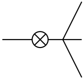
# Two-loop Renormalization and $\beta$-function

 + 

$$= -i\lambda^3 D(s) = \frac{i\lambda^3}{2}\left[\int_k \frac{L_1(k+p_2)+L_1(k+p_3)}{k^2\left(k+p_s\right)^2}\right] \ni \frac{2\ln(-s)}{\epsilon} \tag{25}$$

 = 

$$= \frac{\lambda}{2}\delta_\lambda L_1(s) = \frac{-i}{2}\frac{3\lambda^3}{2}L_1(s)L_1(-\mu^2) \ni \frac{-2\ln(-s)}{\epsilon} \tag{26}$$

 + 

$$= -i\frac{\lambda^3}{12(4\pi)^4}\ln\left(\frac{-4q^2}{3\mu^2}\right) \tag{27}$$

# Two-loop Renormalization and $\beta$-function

$$\begin{aligned}\mathcal{M}^{(2)}(s) =& \frac{i\lambda^3}{4}\left\{\left[L_1(s) - L_1(-\mu^2)\right]^2 - \left[L_1(-\mu^2)\right]^2\right\} \\ &- i\lambda^3\left[L_1(-\mu^2)L_1(s) + D(s)\right] \\ &- i\delta^{(2)}_{\lambda,s}\lambda - i\frac{\lambda^3}{12(4\pi)^4}\ln\left(\frac{-4q^2}{3\mu^2}\right)\end{aligned} \tag{28}$$

The overlapping divergences get cancelled out since for $d = 4 - \epsilon$ with $\epsilon \to 0$

$$\begin{aligned}\left[L_1(-\mu^2)L_1(s) + D(s)\right](4\pi)^4 =& -\frac{2}{\epsilon^2} + \frac{2\ln\left(\frac{\mu^2 e^\gamma}{4\pi}\right) - 3}{\epsilon} + \frac{1}{2}\ln^2\left(\frac{-s}{\mu^2}\right) - \ln\left(\frac{-s}{\mu^2}\right) \\ &- \frac{1}{2}\left[G(p_s, p_2) + G(p_s, p_3)\right] + (\text{mom. ind. terms})\end{aligned} \tag{29}$$

# Two-loop Renormalization and $\beta$-function

The s-channel counterterm is given by:

$$\delta^{(2)}_{\lambda,s} = -\frac{\lambda^2}{4}\left[L_1(-\mu^2)\right]^2 - \lambda^2\left[L_1(-\mu^2)L_1(-\mu^2) + D(-\mu^2)\right] \tag{30}$$

Since the other two channels $(t, u)$ will give exactly the same contribution, the total renormalized magnitude of the 4-point function up to $\mathcal{O}(\lambda^3)$ is:

$$\begin{aligned}\mathcal{M} = & -i\lambda - i\frac{\lambda^2}{2(4\pi)^2}\sum_{p^2=s,t,u}\ln\left(\frac{-p^2}{\mu^2}\right) - i\frac{3\lambda^3}{4(4\pi)^4}\sum_{p^2=s,t,u}\ln^2\left(\frac{-p^2}{\mu^2}\right)\\ & + \frac{i\lambda^3}{(4\pi)^4}\sum_{p^2=s,t,u}\ln\left(\frac{-p^2}{\mu^2}\right) - i\frac{\lambda^3}{12(4\pi)^4}\sum_{i=1,2,3,4}\ln\left(\frac{-4q_i^2}{3\mu^2}\right)\\ & + \frac{i\lambda^3}{2(4\pi)^4}\sum_{k=s,t,u}\hat{G}(p_k, p_2, p_3)\end{aligned} \tag{31}$$

From the CS equation we get:

$$\boxed{\beta_\lambda = \frac{3\lambda^2}{(4\pi)^2} - \frac{17\lambda^3}{3(4\pi)^4} + \mathcal{O}(\lambda^4)} \tag{32}$$

# Outline

# Definitions

## Definition of Bare Operator

$$\mathcal{O}_{n,0}(x) = \lim_{\{x_i\}\to x} \phi_0(x)\phi_0(x_1)\cdots\phi_0(x_{n-1}) \tag{33}$$

## Definition of Bare Correlation functions

$$\langle \mathcal{O}_{n,0}(x)\phi_0(y)\phi_0(z)\rangle = \frac{\langle 0|\, T\left\{\mathcal{O}_{n,0}(x)\phi_0(y)\phi_0(z)e^{iS^{(0)}_{int}[\phi_0;\lambda_0]}\right\}|0\rangle}{\langle 0|\, T\left\{e^{iS^{(0)}_{int}[\phi_0;\lambda_0]}\right\}|0\rangle} \tag{34}$$

## Definition of Renormalized Operator

$$\mathcal{O}_n(x) = Z_{\mathcal{O}}^{-1}\mathcal{O}_{n,0}(x) \tag{35}$$

## Definition of Renormalized Correlation functions

$$\langle \mathcal{O}_n(x)\phi(y)\phi(z)\rangle = Z_{\mathcal{O}_n}^{-1} Z_{\phi}^{-1}\langle \mathcal{O}_{n,0}\phi_0\phi_0\rangle \tag{36}$$

# Renormalization (of Composite operator) Algorithm

**Steps:**

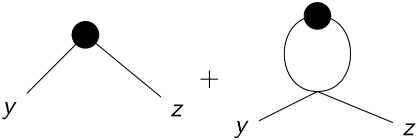
1. We apply Wick's theorem to the bare Lagrangian in order to get the bare $(n+2)$-point $\langle \phi_0(x) \cdots \phi_0(x_{n-1}) \phi_0(y) \phi_0(z) \rangle$
2. We consider the limit $\{x_i\} \to x$ and evaluate the corresponding loop diagrams.
3. We impose the appropriate renormalization conditions and solve for $Z_{\mathcal{O}_n}$
4. We input the value of $Z_{\mathcal{O}_n}$ back in (36) and obtain the expression for the renormalized 3pt function
5. With the use of Callan Symanzik equation we extract the Conformal Data of Composite operators.

# The example of $\mathcal{O}_2$ operator

The bare 3pt function of $\mathcal{O}_{2,0}$ is given by the following limit:

$$\left\langle \mathcal{O}_{2(0)}(x)\phi_0(y)\phi_0(z)\right\rangle = \lim_{y_1 \to y} \langle \phi_0(x_1)\phi_0(x_2)\phi_0(y_1)\phi_0(y)\rangle \tag{37}$$

which corresponds to the following diagrams up to $\mathcal{O}(\lambda_0)$




(38)

In momentum space the 3-point function is given by:

$$\langle\langle \mathcal{O}_{2,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle = \frac{i}{p_2^2}\frac{i}{p_3^2}\left[2 + i\lambda_0 L_1(p_1^2)\right] \tag{39}$$

where

$$\langle\langle \cdots \rangle\rangle = \langle \cdots \rangle \, (2\pi)^d \delta(p_1 + p_2 + p_3) \tag{40}$$

# Renormalization condition of $\mathcal{O}_2$

The general form of the Poincaré invariant 3-point function is the following:

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle = \frac{c_{abc}}{|x_{1,2}|^{\alpha}\,|x_{1,3}|^{\beta}\,|x_{2,3}|^{\gamma}}\ ,\ |x_{i,j}| = |x_i - x_j| \tag{41}$$

using dimensional analysis:

$$\alpha + \beta + \gamma = \sum_{i=a,b,c} [\mathcal{O}_i] \tag{42}$$

we can choose

$$\begin{aligned} \alpha &= [\mathcal{O}_a] + [\mathcal{O}_b] - [\mathcal{O}_c] \\ \beta &= [\mathcal{O}_a] + [\mathcal{O}_c] - [\mathcal{O}_b] \\ \gamma &= [\mathcal{O}_c] + [\mathcal{O}_b] - [\mathcal{O}_a] \end{aligned} \tag{43}$$

# Renormalization condition of $\mathcal{O}_2$

In our case $\mathcal{O}_a = \mathcal{O}_2$ and $\mathcal{O}_b = \mathcal{O}_c = \phi$

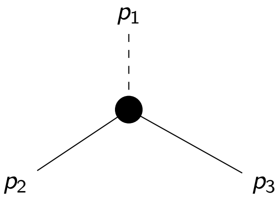
$$\langle \mathcal{O}_2(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{c_{\mathcal{O}_2\phi\phi}}{\left|x_{1,2}\right|^{d-2}\left|x_{1,3}\right|^{d-2}\underbrace{\left|x_{2,3}\right|^{0}}_{1}} \tag{44}$$

We apply a Fourier transformation and set $c_{\mathcal{O}_2\phi\phi} = 2\left[\frac{4\pi^{d/2}}{\Gamma(d-2)}\right]^{-2} i^2$

$$\langle \mathcal{O}_2(p_1)\phi(p_2)\phi(p_3)\rangle = 2\frac{i}{p_2^2}\frac{i}{p_3^2}(2\pi)^d\,\delta^{(d)}(p_1+p_2+p_3) \tag{45}$$

which corresponds to the following Feynman diagram :

$$\langle \mathcal{O}_2(p_1)\phi(p_2)\phi(p_3)\rangle = \tag{46}$$



# $\mathcal{O}(\lambda)$ renormalization of $\mathcal{O}_2$ operator

The renormalization condition for the 3-point function of $\mathcal{O}_2$ operator is the following one:

$$\langle\langle \mathcal{O}_2(p_1)\phi(p_2)\phi(p_3)\rangle\rangle = 2\frac{i}{p_2^2}\frac{i}{p_3^2}\ , \text{ at S.P. } p_1^2 = p_2^2 = p_3^2 = -\mu^2 \tag{47}$$

By imposing this renormalization condition we obtain:

$$Z_{\mathcal{O}_2} = 1 + i\frac{\lambda}{2}L_1(-\mu^2) + \mathcal{O}(\lambda^2) \tag{48}$$

and as a result for $d = 4 - \epsilon$ with $\epsilon \to 4$:

$$\langle\langle \mathcal{O}_2(p_1)\phi(p_2)\phi(p_3)\rangle\rangle = 2\frac{i}{p_2^2}\frac{i}{p_3^2}\left[1 + \frac{\lambda}{2(4\pi)^2}\ln\left(\frac{-p_1^2}{\mu^2}\right) + \mathcal{O}(\lambda^2)\right] \tag{49}$$

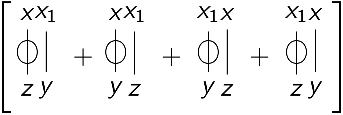
Using the CS equation we obtain the anomalous dimension of $\mathcal{O}_2$:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + 2\gamma_\phi + \Gamma_{\mathcal{O}_2}\right]\langle\langle \mathcal{O}_2(p_1)\phi(p_2)\phi(p_3)\rangle\rangle = 0 \Rightarrow \boxed{\Gamma_{\mathcal{O}_2} = \frac{\lambda}{(4\pi)^2}} \tag{50}$$

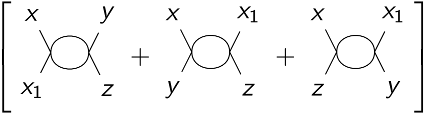
# $\mathcal{O}(\lambda^2)$ renormalization of $\mathcal{O}_2$

There are two kinds of $\mathcal{O}(\lambda^2)$ contributions

$$\langle \mathcal{O}_{2,0}(x)\phi_0(y)\phi_0(z)\rangle_{(\text{sunset})} = \lim_{x_1 \to x}$$




$$\langle \mathcal{O}_{2,0}(x)\phi_0(y)\phi_0(z)\rangle_{(\text{candy})} = \lim_{x_1 \to x}$$




(51)

Moving to momentum space the total $\mathcal{O}(\lambda_0^2)$ contribution of the bare 3-point function is given below:

$$\langle \mathcal{O}_{2,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle_{\mathcal{O}(\lambda^2)} = \frac{i}{p_2^2}\frac{i}{p_3^2}\left\{-\frac{\lambda_0^2}{2}\left(L_1(p_1)\right)^2 - \frac{\lambda_0^2}{2}\int \frac{L_1(k-p_2)+L_1(k-p_3)}{k^2\,(k+p_1)^2} - 2\frac{\lambda_0^2}{6}\left[\frac{S_1(p_2)}{p_2^2}+\frac{S_1(p_3)}{p_3^2}\right]\right\} \tag{52}$$

# $\mathcal{O}(\lambda^2)$ renormalization of $\mathcal{O}_2$

We have to take into account the quantum correction of the coupling constant $\lambda$

$$\lambda_0 = Z_\lambda Z_\phi^{-2} \lambda \tag{53}$$

and the renormalized primary field $\phi$.

$$\phi_0 = Z_\phi^{1/2} \phi \tag{54}$$

with $Z_\phi = 1 + \delta_\phi$ and $Z_\lambda = 1 + \delta_\lambda$

$$\begin{aligned} \delta_\lambda^{(1)} &= -i \frac{3\lambda}{2} L_1(-\mu^2) \\ \delta_\phi &= -\frac{1}{-\mu^2} \frac{\lambda^2}{6} S_1(-\mu^2) \end{aligned} \tag{55}$$

The bare 3-point function can be written as:

$$\langle\langle \mathcal{O}_{2,0}(p_1) \phi_0(p_2) \phi_0(p_3) \rangle\rangle = \frac{i}{p_2^2} \frac{i}{p_3^2} C^{\mathbf{bare}}_{\mathcal{O}_2 \phi\phi} \tag{56}$$

# $\mathcal{O}(\lambda^2)$ renormalization of $\mathcal{O}_2$

This $C^{\textbf{bare}}_{\mathcal{O}_2\phi\phi}$ can be expressed in terms of renormalized $\lambda$:

$$C^{\textbf{bare}}_{\mathcal{O}_2\phi\phi} = 2 + i\lambda L_1(p_1) + i\delta_\lambda \lambda L_1(p_1) - \frac{\lambda^2}{2}\left[(L_1(p_1))^2 + 2D(p_1^2)\right] - 2\frac{\lambda^2}{6}\left[\frac{S_1(p_2)}{p_2^2} + \frac{S_1(p_3)}{p_3^2}\right] \tag{57}$$

**This expression is free from overlapping divergences!**

Using the $\mathcal{O}(\lambda)$ results from the renormalization of $\mathcal{O}_2$ we obtain:

$$C^{\textbf{R}}_{\mathcal{O}_2\phi\phi} = 2\left\{1 + \frac{\lambda}{2(4\pi)^2}\ln\left(\frac{-p_1^2}{\mu^2}\right) + \frac{\lambda^2}{2(4\pi)^4}\ln^2\left(\frac{-p_1^2}{\mu^2}\right) - \frac{\lambda^2}{2(4\pi)^4}\ln\left(\frac{-p_1^2}{\mu^2}\right) + \frac{\lambda^2}{12(4\pi)^4}\left[\ln\left(\frac{-p_2^2}{\mu^2}\right) + \ln\left(\frac{-p_3^2}{\mu^2}\right)\right] - \frac{\lambda^2}{2(4\pi)^4}\hat{G}\right\} \tag{58}$$

$$\hat{G} = \int_0^1 \mathrm{d}y\mathrm{d}z \frac{z}{1-z}\left[\ln\left(\frac{-2yz(1-z)p_1\cdot p_2 + yz(1-yz)p_1^2 + z(1-z)p_2^2}{p_1^2\left[z^2y(1+y) - z(1+2y)\right]}\right) + (p_2 \leftrightarrow p_3)\right] \tag{59}$$

# $\mathcal{O}(\lambda^2)$ renormalization of $\mathcal{O}_2$

From the CS equatio we obtain:

$$\boxed{\Gamma_{\mathcal{O}_2} = \frac{\lambda}{(4\pi)^2} - \frac{5}{6}\frac{\lambda^2}{(4\pi)^4}} \tag{60}$$

In agreement with the general result presented in[4]

## A closer look at $\hat{G}$ term

This term does not contribute in the Callan-Symanzik equation. It is important to note that $\hat{G}(p_1, p_2, p_3)$ is a scale invariant term since:

$$\hat{G}(p_1, p_2, p_3) = \hat{G}(\alpha p_1, \alpha p_2, \alpha p_3) \tag{61}$$

Also obeys the Dilatation Ward Identity:

$$\left[-p_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial p_2} - p_3\frac{\partial}{\partial p_3} + 2\Delta_\phi + \Delta_{\mathcal{O}_2} - 2d\right]\lambda^2\frac{i}{p_2^2}\frac{i}{p_3^2}\hat{G}(p_1, p_2, p_3) = \mathcal{O}(\lambda^3) \tag{62}$$

[4] Johan Henriksson. "The critical O(N) CFT: Methods and conformal data". In: *Physics Reports* 1002 (Feb. 2023), pp. 1–72. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2022.12.002. URL: http://dx.doi.org/10.1016/j.physrep.2022.12.002.

# Outline

# K-operators & Correlation Functions

## Definition of K-operators

$$\begin{aligned} K_{1,0}(x) &= \partial_\nu \phi_0(x) \partial^\nu \phi_0(x) \\ K_{2,0}(x) &= \Box \phi_0^2(x) \\ K_{3,0}(x) &= \phi_0(x) \Box \phi_0(x) \end{aligned} \tag{63}$$

## Definition of Correlation functions

$$\begin{aligned} \langle K_{1,0}(x)\phi_0(y)\phi_0(z)\rangle &= \lim_{x_1 \to x} \left[ \left( \partial^{(x_1)} \cdot \partial^{(x)} \right) \langle \phi_0(x_1)\phi_0(x)\phi_0(y)\phi_0(z)\rangle \right] - (\text{c.t.}) \\ \langle K_{2,0}(x)\phi_0(y)\phi_0(z)\rangle &= \Box_x \lim_{x_1 \to x} \langle \phi_0(x)\phi_0(x_1)\phi_0(y)\phi_0(z)\rangle - (\text{c.t.}) \\ \langle K_{3,0}(x)\phi_0(y)\phi_0(z)\rangle &= \lim_{x_1 \to x} \Box_x \langle \phi_0(x)\phi_0(x_1)\phi_0(y)\phi_0(z)\rangle - (\text{c.t.}) \end{aligned} \tag{64}$$

# Key-equations

## F-Identify

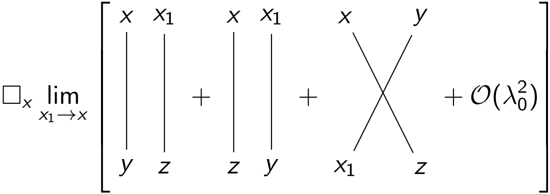
$$F_0(x) \equiv K_{2,0}(x) - 2K_{1,0}(x) - 2K_{3,0}(x) = 0 \tag{65}$$

## Equations of motion

$$K_{3,0} = -\frac{\lambda_0}{6}\mathcal{O}_{4,0} \tag{66}$$

# 3-point function of $K_2$ operator

This is the simplest case. We evaluate the following diagrams:




(67)

The bare 3-point function of $K_{3,0}$ operator takes the following form:

$$\langle\langle K_{3,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle = -p_1^2 \, \langle\langle \mathcal{O}_{2,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle \tag{68}$$

The renormalization procedure is exactly the same as in the case of $\mathcal{O}_2$ operator.

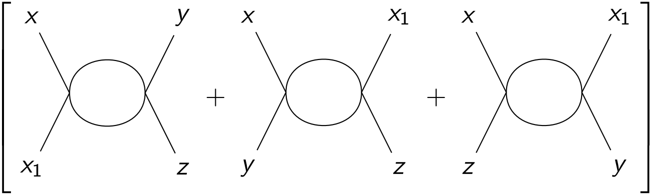
$$\langle\langle K_3(p_1)\phi(p_2)\phi(p_3)\rangle\rangle = -p_1^2 \, \langle\langle \mathcal{O}_2(p_1)\phi(p_2)\phi(p_3)\rangle\rangle \tag{69}$$

So, it is a matter of fact that:

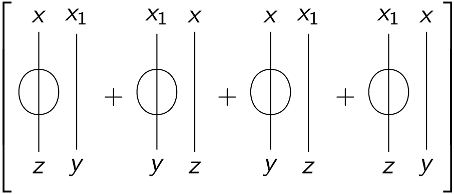
$$\Gamma_{K_2} = \Gamma_{\mathcal{O}_2} \tag{70}$$

# 3-point function of $K_3$ operator

The first non-vanishing contribution $\langle K_{3,0}\phi_0\phi_0\rangle$ is of order $\mathcal{O}(\lambda_0^2)$.

$\lim_{x_1\to x} \Box_x$ 

(71)

$\lim_{x_1\to x} \Box_x$ 

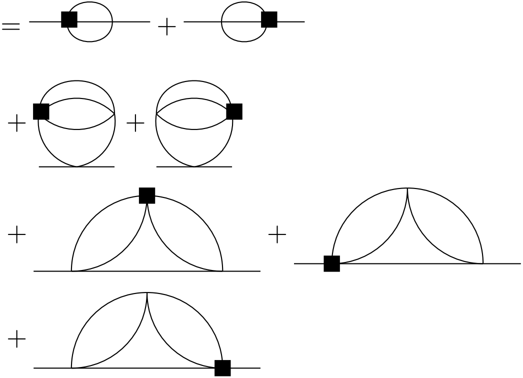
(72)

$$\langle\langle K_{3,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle = \frac{2\lambda_0^2}{3}\frac{i}{p_2^2}\frac{i}{p_3^2}\left[S_1(p_2^2)+S_1(p_3^2)\right] \tag{73}$$

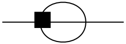
Using the equality following equality we can obtain the $\mathcal{O}(\lambda_0^3)$ expression of the 3-point function

$$\langle\langle K_{3,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle = -\frac{\lambda_0}{6}\langle\langle \mathcal{O}_{4,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle \quad (74)$$

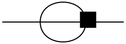
$$\langle\langle \mathcal{O}_{4,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle = \quad (75)$$



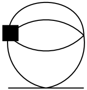
# Loop diagrams of the 3-point function of $K_3$



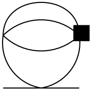
$$= -4\lambda_0 \frac{i}{p_2^2}\frac{i}{p_3^2} S_1(p_2^2) \quad (76)$$



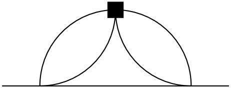
$$= -4\lambda_0 \frac{i}{p_2^2}\frac{i}{p_3^2} S_1(p_3^2) \quad (77)$$



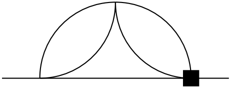
$$=$$



$$= -i3\lambda_0^2 \frac{i}{p_2^2}\frac{i}{p_3^2} ST({p_1}^2) \quad (78)$$



$$= -i6\lambda_0^2 \frac{i}{p_2^2}\frac{i}{p_3^2} \left[T(p_2,p_3) + (p_2 \leftrightarrow p_3)\right] \quad (79)$$



$$= -i\frac{3}{2}\lambda_0^2 TB(p_3^2) \quad (80)$$

# Values of the loop diagrams

$$ST(p^2) = \frac{i(-1)^{3d-12}}{(4\pi)^{3d/2}} \frac{\left[\Gamma\left(\frac{d}{2}-1\right)\right]^4 \Gamma(3-d)\Gamma\left(5-\frac{3d}{2}\right)\Gamma\left(\frac{3d}{2}-4\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma(4-d)\Gamma(2d-5)} \left(-p^2\right)^{\frac{3d}{2}-5} \tag{81}$$

$$TB(p^2) = i(-1)^{2d-10} \frac{\left[\Gamma\left(2-\frac{d}{2}\right)\right]^2 \left[\Gamma\left(\frac{d}{2}-1\right)\right]^5}{(4\pi)^{3d/2}\left[\Gamma\left(d-2\right)\right]^2} \frac{\Gamma\left(5-\frac{3d}{2}\right)}{\Gamma(4-d)} \frac{\Gamma\left(\frac{3d}{2}-4\right)}{\Gamma(2d-5)} \left(-p^2\right)^{\frac{3d}{2}-5} \tag{82}$$

$$T(p_2, p_3) = i\frac{(-1)^{2d-10}}{(4\pi)^{3d/2}} \frac{\left[\Gamma\left(\frac{d}{2}-1\right)\right]^4 \Gamma\left(5-\frac{3d}{2}\right)}{\left[\Gamma\left(d-2\right)\right]^2} \left(I_d(p_2, p_3) + I_f(p_2, p_3)\right) \tag{83}$$

Where $I_f$ is finite and $I_d$ is divergent and given by :

$$I_d(p_2, p_3) = 2\frac{\Gamma(d-3)\Gamma(d-2)}{(4-d)\Gamma(2d-5)}(-p_2^2)^{\frac{3d}{2}-5} + 2\frac{\Gamma(d-3)\Gamma\left(\frac{3d}{2}-4\right)}{(4-d)\Gamma\left(\frac{5d}{2}-7\right)} \left(-p_3^2\right)^{\frac{3d}{2}-5} \tag{84}$$

# $\mathcal{O}(\lambda^3)$ renormalization of $K_3$ operator and mixing

The renormalization condition is:

$$\langle\langle K_3(p_1)\phi(p_2)\phi(p_3)\rangle\rangle = \frac{2}{3}\lambda^2\frac{i}{p_2^2}\frac{i}{p_3^2}(p_2^2+p_3^2)\ ,\ \text{at } S.P \text{ with } p_1^2=p_2^2=p_3^2=-\mu^2 \tag{85}$$

The bare 3-point function is given by:

$$\begin{aligned}\langle\langle K_{3,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle\rangle &= p_1^2\frac{i}{p_2^2}\frac{i}{p_3^2}i\lambda^3\frac{ST(p_1^2)}{p_1^2} \\ &+ i\lambda^2\frac{i}{p_3^2}\left\{\frac{2}{3}\frac{S_1(p_2^2)}{p_2^2} - i\lambda L_1(-\mu^2)\frac{S_1(p_2^2)}{p_2^2} + i\frac{1}{4}\lambda\frac{TB(p_2^2)}{p_2^2} + i\lambda\frac{T(p_2,p_3)}{p_2^2}\right\} \\ &+ (p_2\leftrightarrow p_3)\end{aligned} \tag{86}$$

**There is mixing between $K_3$ and $K_2$ !**

The Callan Symanzik equation gets the following form:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + 2\gamma_\phi +\right]\langle K_i\phi\phi\rangle + \Gamma_{ij}\langle K_j\phi\phi\rangle = 0 \tag{87}$$

# Calculation of Anomalous Dimension Matrix

The relation between the bare and renormalized operators is given by introducing a mixing matrix $Z_{ij}$ such that:

$$K_{i,0} = Z_{ij} K_j \tag{88}$$

For the purpose of our analysis we will work with the inverse matrix

$$\left[Z^{-1}\right]_{ij} = \begin{bmatrix} \frac{1}{Z_{K_2}} & 0 \\ -Z_{32}\frac{1}{Z_{K_2} Z_{K_3}} & \frac{1}{Z_{K_3}} \end{bmatrix} \tag{89}$$

with :

$$K_i = \left[Z^{-1}\right]_{ij} K_{j,0} \tag{90}$$

We can write the renormalized expression as:

$$\langle K_3(p_1)\phi(p_2)\phi(p_3)\rangle = -\frac{Z_{32}}{Z_{K_3}}\langle K_2(p_1)\phi(p_2)\phi(p_3)\rangle + \frac{1}{Z_{K_3} Z_\phi}\langle K_{3,0}(p_1)\phi_0(p_2)\phi_0(p_3)\rangle \tag{91}$$

# Calculation of Anomalous Dimension Matrix

By imposing the renormalization condition we can obtain the expressions of $Z_{K_2}$ and $Z_{32}$

$$Z_{32} = i\frac{\lambda^3}{2}\frac{ST(-\mu^2)}{-\mu^2} + \mathcal{O}(\lambda^4) \tag{92}$$

$$Z_{K_3} = \left\{ \frac{S_1(p_2^2)}{p_2^2} - i\frac{3}{2}\lambda L_1(-\mu^2)\frac{S_1(p_2^2)}{p_2^2} + i\frac{3}{8}\lambda\frac{TB(p_2^2)}{p_2^2} + i\frac{3}{2}\lambda\frac{T(p_2,p_3)}{p_2^2} \right\}_{\text{S.P.}} + \mathcal{O}(\lambda^2) \tag{93}$$

Using this expressions we can get the the form of the renormalized 3-point function:

$$\begin{aligned}\langle\langle K_3(p_1)\phi(p_2)\phi(p_3)\rangle\rangle = \frac{2\lambda}{3}\frac{i}{p_2^2}\frac{i}{p_3^2}\left\{p_2^2\left[\lambda + \frac{9}{2}\frac{\lambda^2}{16\pi^2}\ln\left(\frac{-p_2^2}{\mu^2}\right)\right] + (p_2 \leftrightarrow p_3)\right\} \\ -p_1^2\frac{i}{p_2^2}\frac{i}{p_3^2}\frac{\lambda^3}{4(4\pi)^6}\ln\left(\frac{-p_1^2}{\mu^2}\right) + \mathcal{O}(\lambda^4)\end{aligned} \tag{94}$$

# Calculation of Anomalous Dimension Matrix

Acting with $\hat{R} \equiv \mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + 2\gamma_\phi$ on the renormalized 3-point function we get:

$$\begin{aligned}\hat{R}\left\langle K_3(p_1)\phi(p_2)\phi(p_3)\right\rangle = \left(-\frac{6\lambda}{(4\pi)^2} + \frac{\beta_\lambda}{\lambda}\right)\left\langle K_3(p_1)\phi(p_2)\phi(p_3)\right\rangle \\ - \frac{\lambda^3}{4(4\pi)^6}\left\langle K_2(p_1)\phi(p_2)\phi(p_3)\right\rangle\end{aligned} \tag{95}$$

From the above expression we can extract the form of the Anomalous Dimension Matrix

$$\Gamma_{ij} = \begin{bmatrix} \Gamma_{K_2} & 0 \\ \frac{\lambda^3}{4(4\pi)^6} & \frac{6\lambda}{(4\pi)^2} - \frac{\beta_\lambda}{\lambda} \end{bmatrix} \tag{96}$$

This result is in agreement with the one presented in[5]

[5] Lowell S Brown and John C Collins. “Dimensional renormalization of scalar field theory in curved space-time”. In: *Annals of Physics* 130.1 (1980), pp. 215–248. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(80)90232-8. URL: https://www.sciencedirect.com/science/article/pii/0003491680902328.

# Outline

# Constraints on the construction of Θ operator

1. $\Theta_0$ will be a a linear combination of $K_{2,0}$ and $K_{3,0}$ operators
2. The renormalized operator should have vanishing Anomalous Dimension

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + 2\gamma_\phi\right]\langle\Theta\phi\phi\rangle = 0 \tag{97}$$

3. The 3-point function should vanish when the system approaches the fixed point :

$$\langle\Theta\phi\phi\rangle \sim \beta_\lambda \tag{98}$$

## Next steps

1. Construction of $\Theta_0$ operator taking into account the above constraints.
2. Check if Θ is uniquely defined.
3. Proceed to the calculation of 2-point function and find the eigenvalue of $\mu\frac{\partial}{\partial\mu}\langle\Theta\Theta\rangle$

**Thank you!**