Propagation of spinors on the angular deformed NC BH background

Nikola Konjik (University of Belgrade)

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Done in colaboration with:

Andjelo Samsarov, Institute Rudjer Boskovic, Zagreb,

Based on:

Propagation of spinors on a noncommutative spacetime: equivalence of the formal and the effective approach, Published in: Eur.Phys.J.C 83 (2023) 5, 387

Noncommutative scalar quasinormal modes of the Reissner–Nordström black hole, Published in:

Class.Quant.Grav. 35 (2018) 17, 175005

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Content

Noncommutative geometry

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Approaches to NC geometry $\star$-product, NC spectral triple, NC vierbein formalism, matrix models,...

NC space-time from the ρ-Minkowski (angular) twist

Twist is used to deform a symmetry Hopf algebra
Twist $\mathcal{F}$ is invertible bidifferential operator from the universal enveloping algebra of the symmetry algebra

We work in 4D and deform the space-time by the following twist

$$\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta_{ab}X^a \bigotimes X^b}$$
$$[X^a, X^b] = 0, \quad \text{a,b=1,2} \qquad X_1 = \partial_0 \text{ and } X_2 = x\partial_y - y\partial_x$$
$$\mathcal{F} = \mathrm{e}^{\frac{-ia}{2}(\partial_0 \otimes (x\partial_y - y\partial_x) - (x\partial_y - y\partial_x)\otimes \partial_0)}$$

Bilinear maps are deformed by twist!
Bilinear map μ
$\mu : X \times Y \to Z$
$\mu_\star = \mu \mathcal{F}^{-1}$

Commutation relations between coordinates are:

$[\hat{x}^0, \hat{x}] = ia\hat{y}$, All other commutation relations are zero

$[\hat{x}^0, \hat{y}] = -ia\hat{x}$

Our approach: deform space-time by an Abelian twist to obtain commutation relations

Angular twist in curved coordinates $X_1 = \partial_0$ and $X_2 = \partial_\varphi$

-supose that metric tensor $g_{\mu\nu}$ does not depend on t and φ coordinates

-Hodge dual becomes same as in commutative case

Scalar $U(1)_\star$ gauge theory

If a one-form gauge field $\hat{A} = \hat{A}_\mu \star dx^\mu$ is introduced to the model through a minimal coupling, the relevant action becomes

$$\begin{aligned} S[\hat{\phi}, \hat{A}] &= \int \Big(d\hat{\phi} - i\hat{A} \star \hat{\phi}\Big)^+ \wedge_\star *_H \Big(d\hat{\phi} - i\hat{A} \star \hat{\phi}\Big) \\ &\quad - \int \frac{\mu^2}{4!} \hat{\phi}^+ \star \hat{\phi} \epsilon_{abcd}\ e^a \wedge_\star e^b \wedge_\star e^c \wedge_\star e^d \\ &= \int d^4x \sqrt{-g} \star \Big(g^{\mu\nu} \star D_\mu \hat{\phi}^+ \star D_\nu \hat{\phi} - \mu^2 \hat{\phi}^+ \star \hat{\phi}\Big) \end{aligned}$$

After expanding action and varying with respect to Φ^+ EOM is

$$g^{\mu\nu}\left(D_\mu D_\nu\phi - \Gamma^\lambda_{\mu\nu}D_\lambda\phi\right) - \frac{1}{4}\theta^{\alpha\beta}g^{\mu\nu}\left(D_\mu(F_{\alpha\beta}D_\nu\phi) - \Gamma^\lambda_{\mu\nu}F_{\alpha\beta}D_\lambda\phi\right.$$

$$\left.-2D_\mu(F_{\alpha\nu}D_\beta\phi) + 2\Gamma^\lambda_{\mu\nu}F_{\alpha\lambda}D_\beta\phi - 2D_\beta(F_{\alpha\mu}D_\nu\phi)\right) = 0$$

Scalar field in the Reissner–Nordström background

RN metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix}$$

with $f = 1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2}$ which gives two horizons (r_+ and r_-)
Q-charge of RN BH
M-mass of RN BH
Non-zero components of gauge fields are $A_0 = -\frac{qQ}{r}$ i.e. $F_{r0} = \frac{qQ}{r^2}$
q-charge of scalar field

EOM for scalar field in RN space-time

$$\Big(\frac{1}{f}\partial_t^2 - \Delta + (1-f)\partial_r^2 + \frac{2MG}{r^2}\partial_r + 2iqQ\frac{1}{rf}\partial_t - \frac{q^2Q^2}{r^2 f}\Big)\phi$$
$$+\frac{aqQ}{r^3}\Big((\frac{MG}{r} - \frac{GQ^2}{r^2})\partial_\varphi + rf\partial_r\partial_\varphi\Big)\phi = 0$$

where a is $\theta^{t\varphi}$.

Fermions

Fermionic action coupled to EM field and curved space is

$$S_\star = \int \mathrm{d}^4x \; |e| \star \bar{\hat{\Psi}} \star \Big(i\gamma^\mu \big(\partial_\mu \hat{\Psi} - i\omega_\mu \star \hat{\Psi} - iq\hat{A}_\mu \star \hat{\Psi}\big) - m\hat{\Psi} \Big), \quad (1)$$

where

$$D_\mu \Psi = \partial_\mu \Psi - \frac{i}{2}\omega_\mu{}^{ab}\Sigma_{ab}\Psi - iqA_\mu \Psi. \quad (2)$$

After expanding the fields and $\star$-product with SW map we get

$$\begin{aligned} S_\star \;=\; & \int \mathrm{d}^4x \; |e| \bar{\Psi} \Big(i\gamma^\mu D_\mu \Psi - m\Psi \Big) \\ & + \frac{1}{2}\theta^{\alpha\beta} \Big(-iF_{\mu\alpha}\bar{\Psi}\gamma^\mu D_\beta^{\mathrm{U}(1)}\Psi - \frac{i}{2}\bar{\Psi}\gamma^\mu \omega_\mu F_{\alpha\beta}\Psi - \frac{1}{2}F_{\alpha\beta}\bar{\Psi}\big(i\gamma^\mu D_\mu^{\mathrm{U}(1} \end{aligned}$$

To get proper EOM, we have done the same procedure for fermions in RN metric (coupled to external EM field).
The result is

$$i\gamma^\mu\Big(\partial_\mu\Psi - i\omega_\mu\Psi - iA_\mu\Psi\Big) - m\Psi - \frac{ia}{2}\frac{qQ}{r^2}\sqrt{f}\gamma^1\partial_\phi\Psi = 0.$$

Duality picture

We have another way to get the equation of motion for scalar and fermionic field: Using the effective metric in commutative space

$$g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & -\frac{aqQ}{2}\sin^2\theta \\ 0 & 0 & -r^2 & 0 \\ 0 & -\frac{aqQ}{2}\sin^2\theta & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{3}$$

Two ways

- Noncommutative space with pure RN metric
- Commutative space with modified RN metric

NC QNM solutions

QNM

-We can better understand structure of the space-time by detecting of a gravitational waves

-Dominant phase of perturbed BH are dumped oscilations of geometry or matter called (Quasinormal modes) -special solutions of EoM

Set of boundary conditions which lead to QNM solutions are: on the horizon we have purely incoming waves, while in the infinity we have purely outgoing waves

Continued fractions method

If we want to get EoM in the following way

$$\frac{d^2\psi}{dy^2} + V\psi = 0$$

y must be

$$y = r + \frac{r_+}{r_+ - r_-}\left(r_+ - iamqQ\right)\ln(r - r_+) - \frac{r_-}{r_+ - r_-}\left(r_- - iamqQ\right)\ln(r - r_-)$$

y is modified tortoise coordinate for the RN metric

Asimptotic form of the EoM is

$$R(r) \to \begin{cases} Z^{out} e^{i\Omega y} y^{-1-i\frac{\omega qQ - \mu^2 M}{\Omega} - amqQ\Omega} & \text{za } y \to \infty \\ Z^{in} e^{-i\left(\omega - \frac{qQ}{r_+}\right)\left(1 + iam\frac{qQ}{r_+}\right)y} & \text{za } y \to -\infty \end{cases}$$

Combining asimptotic forms, general solution has the following form

$$R(r) = e^{i\Omega r}(r - r_-)^{\epsilon} \sum_{n=0}^{\infty} a_n \left(\frac{r - r_+}{r - r_-} \right)^{n+\delta} \tag{4}$$

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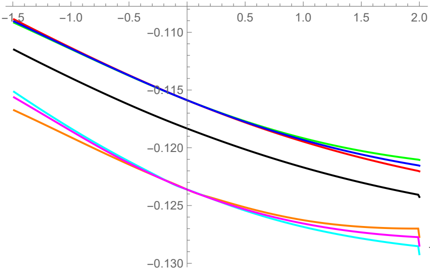
$$\delta = -s - i\frac{r_+^2}{r_+ - r_-}\left(\omega - \frac{qQ}{r_+}\right), \qquad \epsilon = -1 - 2s - iqQ + i\frac{r_+ + r_-}{2\Omega}\left(2\omega^2\right),$$

Inserting the eq (4) in the EoM we get 6-term recurence relations for a_n:

$$\begin{aligned} A_n a_{n+1} + B_n a_n + C_n a_{n-1} + D_n a_{n-2} + E_n a_{n-3} + F_n a_{n-4} = 0, \\ A_3 a_4 + B_3 a_3 + C_3 a_2 + D_3 a_1 + E_3 a_0 = 0, \\ A_2 a_3 + B_2 a_2 + C_2 a_1 + D_2 a_0 = 0, \\ A_1 a_2 + B_1 a_1 + C_1 a_0 = 0, \\ A_0 a_1 + B_0 a_0 = 0, \end{aligned}$$



−1.5
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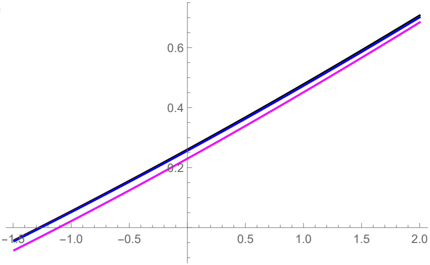
0.5
1.0
1.5
2.0
−0.110
−0.115
−0.120
−0.125
−0.130



0.6
0.4
0.2
−1.5
−1.0
−0.5
0.5
1.0
1.5
2.0

Outlook

- We constructed Angular twist which induces angular noncommutativity
- Angular NC scalar and vector gauge theory is constructed
- EOM is solved with QNM boundary conditions for scalar field coupled to RN geometry
- But this is toy model!
- Plan for future is to calculate electromagnetic and gravitational QNMs and to compare it with results from LIGO, VIRGO, LISA...
- We want to understand physics of the effective metric