Metric Geometry and Differential Forms

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Main message

There is an exotic, unfamiliar to most people construction that encodes a metric on a space into a collection of differential forms on the same space

More precisely

(Metric, unit spinor) $\Leftrightarrow$ (Collection of differential forms)

PDE's on the metric $\Leftrightarrow$ PDE's on differential forms

The purpose of the talk is to describe a collection of examples, as well as the principle which explains why these examples exist

At the end of the talk I will explain why a physicist would (should) care

4D geometry via triples of 2-forms

This is the canonical example that exhibits all non-triviality, as well as provides a pattern for higher D

Let $\Sigma^1, \Sigma^2, \Sigma^3 \in \Lambda^2(M)$ be a triple of 2-forms such that

$$\Sigma^1 \wedge \Sigma^1 = \Sigma^2 \wedge \Sigma^2 = \Sigma^3 \wedge \Sigma^3$$

$$\Sigma^i \wedge \Sigma^j = 0, \quad i \neq j$$

Can be encoded more compactly as $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

We will refer to such a triple of 2-forms satisfying the algebraic conditions as SU(2) structure

Definition: a G-structure is a reduction of the principal GL(n,R) bundle of frames over M to a G-subbundle

Proposition: the GL(4,R) stabiliser of a triple of 2-forms (satisfying the algebraic conditions) is SU(2)

Follows from the following two propositions

Proposition: the symmetric pairing defined via $g_\Sigma(\xi, \eta) v_g = \frac{1}{6} \epsilon^{ijk} i_\xi \Sigma^i \wedge i_\eta \Sigma^j \wedge \Sigma^k$ $\qquad \xi, \eta \in TM$

is a Riemannian metric on M $\qquad$ $\Sigma^i \wedge \Sigma^i$ is a natural orientation

It follows that the GL(4,R) stabiliser of Σ^i is inside SO(4,R)

Proposition: Σ^i are self-dual 2-forms (in the orientation defined by Σ^i) in the metric g_Σ

It follows that the stabiliser of Σ^i is SU(2) that does not act on Λ^+

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Remark: we note, for later purposes that

$$\dim(\mathrm{GL}(4,\mathbb{R})/\mathrm{SU}(2)) = 16 - 3 = 13 = \underset{\text{metric}}{\underset{\nearrow}{10}} + \underset{\text{extra}}{\underset{\nwarrow}{3}}$$

It is thus clear that Σ^i encode more than a metric

$$10 = \dim(\mathrm{GL}(4,\mathbb{R})/\mathrm{SO}(4))$$

We will later see that this is (metric, unit spinor)

We also note that $\dim(\{\Sigma^i\}/\mathrm{constraints}) = 18 - 5 = 13$

All this seems exotic, it is not clear why this works, and how to generalise it. This will be explained later.

PDE's for an SU(2) structure

Our task is now to see how natural PDE's on the metric (e.g. Einstein equations) can be encoded as PDE's on 2-forms

Proposition: let ∇ be the Levi-Civita connection for the metric defined by Σ^i

There exists a triple of 1-forms A^i such that $\nabla\Sigma^i + \epsilon^{ijk}A^j\Sigma^k = 0$

Remark: A^i is called the "intrinsic torsion" of the SU(2) structure

Remark: can project the definition relation for A^i on the space of 3-forms $d\Sigma^i + \epsilon^{ijk}A^j \wedge \Sigma^k = 0$

This determines A^i completely

In particular, this means that A^i is completely determined by the exterior derivatives $d\Sigma^i$

Proposition: let $F^i = dA^i + \frac{1}{2}\epsilon^{ijk}A^j \wedge A^k$

Then $F^i_{\mu\nu} = \frac{1}{2}R_{\mu\nu}{}^{\alpha\beta}\Sigma^i_{\alpha\beta}$

Riemann curvature

Proposition: In 4D, the SD/ASD decomposition of Riemann is

$$\text{Riemann} = \begin{pmatrix} W^+ + \text{scalar} & Rc_0 \\ Rc_0 & W^- + \text{scalar} \end{pmatrix}$$

Ricci tracefree

The Einstein equations (in the absence of matter)

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Equivalent to $Rc_0 = 0$

In view of the previous remarks, equivalent to $\boxed{F^i = M^{ij}\Sigma^j}$

Where M^{ij} is an arbitrary 3x3 matrix (automatically symmetric by a version of the Bianchi identity)

Einstein equations in the language of 2-forms

Curvature of the SO(3) connection (intrinsic torsion) A^i is self-dual as a 2-form

Worth emphasising that all the equations are written in terms of the exterior derivative on forms

Corollary: Assume that $d\Sigma^i = 0$ Triple of closed 2-forms satisfying $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

Then $A^i = 0, F^i = 0$ and thus by previous discussion $Rc = 0, W^+ = 0$

Ricci-flat, half-flat 4D spaces are known to be hyper-Kahler

They have a triple of integrable complex structures satisfying $IJ = K$

Action principles: Second-order action

$$S[\Sigma] = \int_M \Sigma^i \epsilon^{ijk} A^j(\Sigma) A^k(\Sigma)$$

Critical points - g_Σ Ricci-flat

First-order action

In both of these need to remember that $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

$$S[\Sigma, A] = \int_M \Sigma^i F^i$$

Plebanski action

$$S[\Sigma, A, \Psi] = \int_M \Sigma^i F^i - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right)\Sigma^i \Sigma^j$$

Many other things one can do with this formalism, but need to move on to understand why it is possible

Spinors and the geometric (squaring) map

Let M be spin, and let S be the bundle of spinors

The basic fact about spinors $S \otimes S = \oplus_{k=0}^{n} \Lambda^k$

Will refer to this as the geometric (squaring) map, because its result is a collection of geometric objects - differential forms

Spinors in 4D:

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \gamma_i \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$$

Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Dirac spinors are 4-component

Weyl spinors are 2-component

$$S = S_+ \oplus S_-, \qquad S_\pm \sim \mathbb{C}^2$$

γ-matrices are off-diagonal

$$\gamma : S_+ \to S_-$$

and vice versa

Invariant inner product on $S_\pm$

$$S_\pm \ni \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\langle \psi_1, \psi_2 \rangle = \psi_1^T \epsilon \psi_2, \qquad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

where $\psi_{1,2}$ are both either in S_+ or in S_-

Invariant conjugation on $S_\pm$ $\hat{\psi} = \epsilon \psi^*$

Spin(4) stabiliser of a spinor in S_+ is SU(2)

Squaring map in 4D

Can define $\omega := \frac{i}{2}\langle \overline{\psi}, \gamma_{[\mu}\gamma_{\nu]}\psi\rangle dx^\mu \wedge dx^\nu$ real $\Omega := \frac{i}{2}\langle \psi, \gamma_{[\mu}\gamma_{\nu]}\psi\rangle dx^\mu \wedge dx^\nu$ complex

A simple computation gives $\omega = V^i_\psi \Sigma^i, \qquad \Omega = m^i_\psi \Sigma^i$

where $\Sigma^i = dx^4 \wedge dx^i - \frac{1}{2}\epsilon^{ijk} dx^j \wedge dx^k$ is the basis of self-dual 2-forms on $\mathbb{R}^4$

and $\vec{V}_\psi = (2\mathrm{Re}(\alpha^*\beta), 2\mathrm{Im}(\alpha^*\beta), |\alpha|^2 - |\beta|^2) \in \mathbb{R}^3$ $(\vec{V}_\psi, \vec{V}_\psi) = (|\alpha|^2 + |\beta|^2)^2 = \langle \overline{\psi}, \psi\rangle^2$

$\vec{m}_\psi = (-\alpha^2 + \beta^2, -i(\alpha^2 + \beta^2), 2\alpha\beta)$ $(\vec{m}_\psi, \vec{m}_\psi) = 0, \quad (\vec{m}_\psi, \vec{V}_\psi) = 0, \quad (\vec{m}_\psi, \vec{m}^*_\psi) = 2\langle \overline{\psi}, \psi\rangle^2$

The data (ω, Ω) is not arbitrary but satisfies $\boxed{\Omega \wedge \Omega = 0, \quad \Omega \wedge \omega = 0, \quad 2\Omega \wedge \overline{\Omega} = \omega^2}$

Alternatively, the triple $(\mathrm{Re}\Omega, \mathrm{Im}\Omega, \omega)$ is an SU(2) structure in the sense previously defined

Summary: (Metric, unit spinor) $\Rightarrow$ SU(2) structure, 2-forms arise as $\hat{\psi} \otimes \psi, \psi \otimes \psi, \hat{\psi} \otimes \hat{\psi}$

Dimensions of these spaces match and every SU(2) structure comes by this construction from some metric and unit spinor

Towards higher D

Details of this construction are specific to 4D, but the general idea extends to any dimension

$$\text{(Metric, Unit spinor)} \xRightarrow{\text{Squaring Map}} \text{Collection of differential forms}$$

One can generally expect that there is a sufficient number of diff. forms

That determines both the metric and the spinor (up to sign)

Further examples: 6D

Geometric (squaring) map produces $\omega \in \Lambda^2, \Omega \in \Lambda^3_{\mathbb{C}}$

Sufficient to take $\omega \in \Lambda^2, C = \mathrm{Re}\Omega \in \Lambda^3$

These are subject to algebraic constraints

$$\omega \wedge C = 0, C \wedge \hat{C} = \frac{1}{6}\omega^3$$

The GL(6,R) stabiliser of these data is SU(3)

C determines an almost complex structure J

Here $\hat{C}(\cdot,\cdot,\cdot) = C(J\cdot, J\cdot, J\cdot)$

The metric is determined as $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$

Proposition: $d\omega = 0, \quad dC = 0 \quad \Rightarrow$

The metric determined by these data is special Kahler

Has parallel spinor, holonomy in SU(3)

In particular, Ricci flat

3-forms in 7D

Another example is obtained by taking a real unit spinor in 7D

Its GL(7,R) stabiliser is G_2

Geometric (squaring) map produces $C \in \Lambda^3, C^* \in \Lambda^4$

Sufficient to take $C \in \Lambda^3$ Not subject to any algebraic constraints

Determines the metric via $g_C(\xi,\eta)v_g = \frac{1}{6} i_\xi C \wedge i_\eta C \wedge C$

The extra information in C (on top of the metric) is either that of a unit spinor

Or that of a cross-product in TM $(\xi_1 \times \xi_2, \xi_3)_g = C(\xi_1, \xi_2, \xi_3)$

Proposition: $dC = 0, \quad dC^* = 0$ $\Longrightarrow$ The metric determined by C has holonomy in G_2

Has parallel spinor

In particular, Ricci flat

4-forms in 8D

Yet another example is obtained by taking a real Weyl unit spinor in 8D

Geometric (squaring) map produces $\Phi \in \Lambda^4$

GL(8,R) stabiliser Spin(7)

Subject to 27 independent algebraic constraints that are somewhat hard to characterise explicitly

The 4-form determines the metric, but the formula is more complicated than the previously encountered ones

The extra data in the 4-form (on top of the metric) is that it, together with a choice of a unit vector in TM, identifies TM with octonions

Proposition: $d\Phi = 0$ $\Rightarrow$ Parallel spinor, holonomy in Spin(7)

In particular Ricci-flat

PDE's

In all considered examples only the "natural" first-order PDE's on the differential forms are known:

Closure of the relevant differential forms

What is not know is what are the "best" second-order PDE's in each case

In 8D I have studied this question in 2403.16661 [math.DG]

The natural, written in the language of the exterior derivative PDE's describe gravity coupled to exotic matter

(coming from the spinor degree of freedom)

There is still a lot to be understood here

What is clear is that when we are to describe the dynamics of (metric, spinor) system there

are other natural PDE's that can be written, apart from Einstein equations

Phrased a a physics question, this is the question of the low energy dynamics of such a system

Physics motivations: Unification

All of the known to us physics requires the following types of fields:

Metric, gauge fields, scalar fields

Spinors

They all get unified by a metric in a space of sufficiently high dimension

Unified by a spinor in a sufficiently high dimension

With this in mind, the question of dynamics of (metric, spinor) in higher D is a very natural one

Physics motivations: Discrete gravity

Discretising gravity (putting it on a simplicial complex) is a natural approach to both numerical and quantum gravity

Works in 2D, 3D, but so far no real progress in higher dimensions

At the same time differential forms and the exterior derivative can naturally be discretised

This is what the spin foam approach to QG attempts, but so far there are serious issues with it

It is much easier to discretise differential forms rather than Lie algebra valued differential forms

All higher D examples I described work with ordinary differential forms

It is possible that some models of simplicial higher D quantum gravity (coupled to a spinor) can be produced along these lines

Summary

4D gravity can be described using triples of 2-forms rather than metrics. Extremely efficient formalism

The origin of this formalism lies in spinors. The relevant differential forms are produced by the squaring map

Many other examples described in dimension 6,7,8

In all known examples this gives the most efficient known way to describe geometry and impose PDE's on it

Do not use the metric to describe geometry. Use differential forms that originate in spinors

Thank you!