Generating quantum geometry from gauged quantum mechanics

@ ``Workshop on Noncommutative and Generalized Geometry in String Theory, Gauge Theory and Related Physical Models‘’
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Quantum mechanics and M(atrix) theory

``On the relation between the quantum mechanics of Heisenberg, Born, and Jordan, and that of Schroedinger'', *Schroedinger* (Annalen der Physik *1926*)

Matrix mechanics (1925) ⟷ Wave mechanics (1926)

(from wiki)

$$F^{kl} = \int \rho(x) u_k(x) [F, u_l(x)] dx,$$

$$F_{ij} = \langle \psi_i | F(x, p = -i\partial_x) | \psi_j \rangle$$

(from wiki)

M(atrix) theory in string theory

BFSS ('97) IKKT('97) . . .

$$S = \frac{1}{4}\mathrm{tr}([X_\mu, X_\nu]^2) + \cdots$$

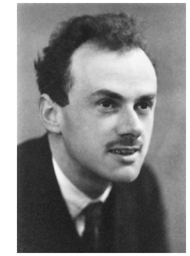
Conventional non-commutative scheme

Canonical quantization

Born, Jordan (1925)

(from wiki)

Dirac (1925)



(from wiki)

Poisson bracket $\{x, p\}$ → Commutator $[\hat{x}, \hat{p}] = i\hbar\{x, p\}$

Conventional non-commutative scheme

$$\{x, y\} \rightarrow [\hat{x}, \hat{y}]$$

Symplectic manifold — Quantization of symplectic manifold

This is the basic idea behind the conventional quantization methods:

Deformation quantization, geometric quantization, Berezin-Toeplitz quantization …

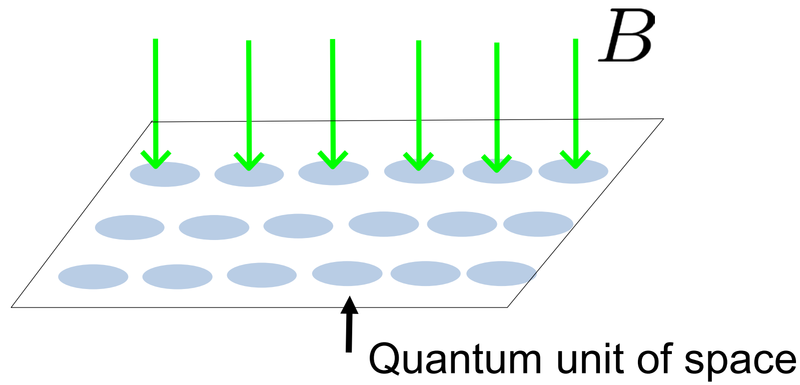
Restricted to symplectic manifolds → General manifold ?
Restricted to the commutator formalism → General NCG such as Nambu bracket ?

NCG only in the lowest Landau level ?

QM on magnetic plane

$$X = x + i\frac{1}{B}D_y \qquad Y = y - i\frac{1}{B}D_x$$

$$\Rightarrow \quad [X, Y] = i\frac{1}{B}$$



``Noncommutative field theory'' *Douglas, Nekrasov* Rev.Mod.Phys. 73 *(2001)* 977

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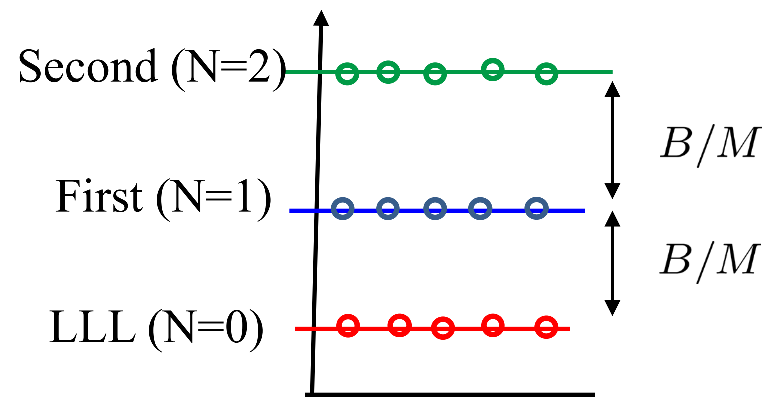
G. Other results
V. Applications to the Quantum Hall Effect
A. The lowest Landau level
B. The fractional quantum Hall effect
VI. Mathematical Aspects



NCG geometry appears in the LLL, but why lowest Landau level ???

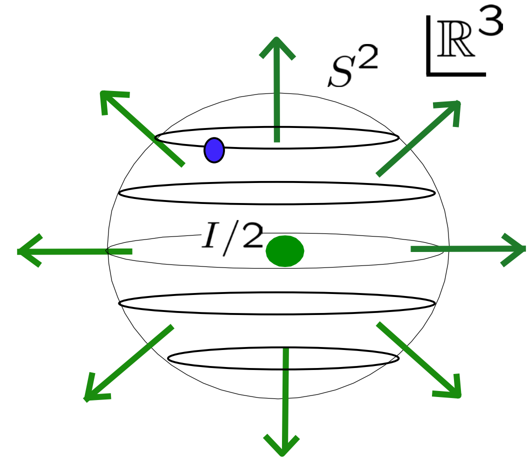
Why magnetic field ???

The Landau model on a two-sphere

Let's consider the simplest case. *Wu, Yang ('76)* *Haldane ('83)*

$$H = -\frac{1}{2M}\sum_{i=1}^{3} {D_i}^2\Big|_{r=1} = \frac{1}{2M}\sum_{i=1}^{3} {\Lambda_i}^2$$

$$(\Lambda_i = -i\epsilon_{ijk}x_j D_k)$$



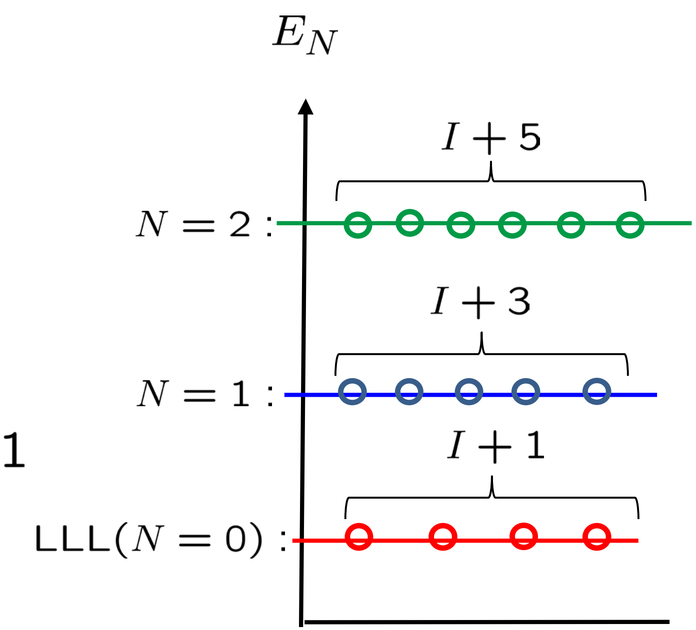
⇓

SU(2) Casimir index

$$\ell = N + \frac{I}{2} \qquad (N, I = 0, 1, 2, \cdots)$$

Landau levels

$$E_N = \frac{1}{2M}(I(N+\frac{1}{2}) + N(N+1))$$



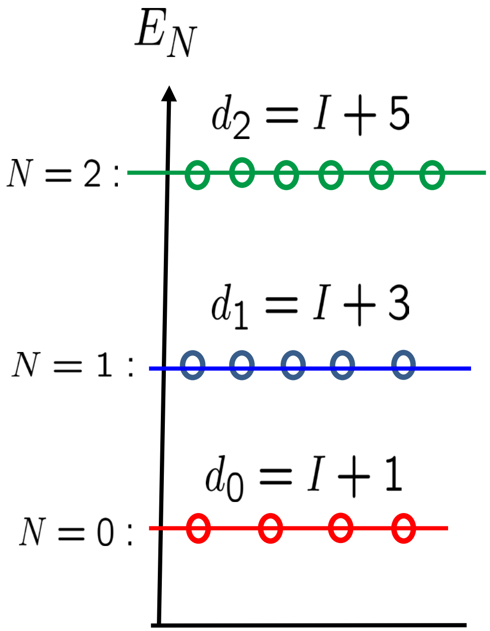
Eigenstates= SU(2) irreps. : monopole harmonics

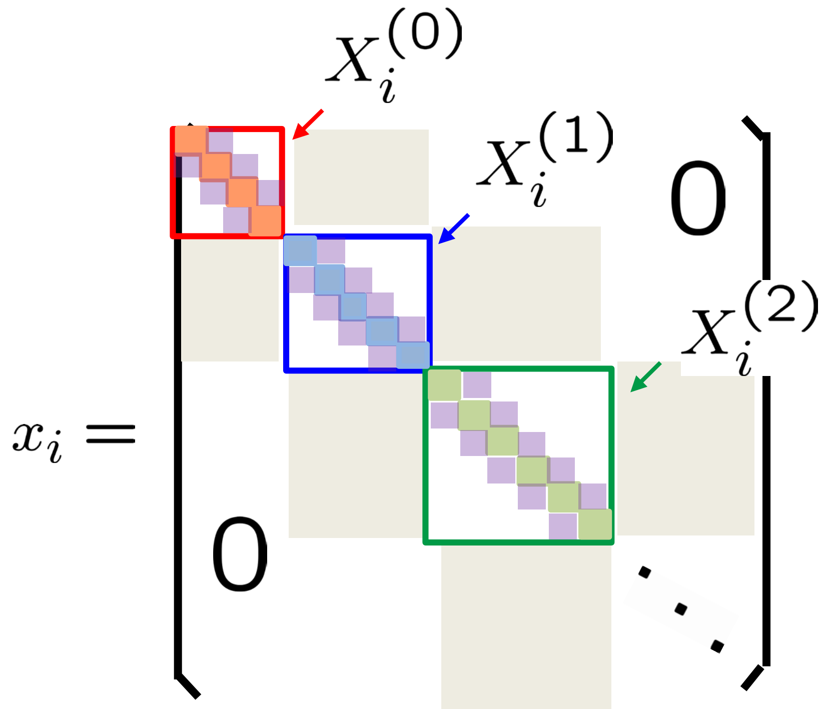
$$d_N = 2\ell + 1 = I + 2N + 1$$

$$Y_m^{(N)}(\theta,\phi) \qquad (m = \ell, \ell-1, \cdots, -\ell)$$

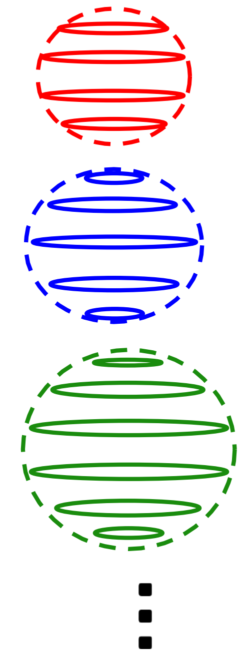
magnetic quantum #

Fuzzy geometry in arbitrary Landau levels





KH [’*16*]



Fuzzy two-spheres

Matrix coordinates

$$(X_i^{(N)})_{mn} = \langle Y_m^{(N)} | x_i | Y_n^{(N)} \rangle = 2\alpha_N (S_i^{(l=N+\frac{I}{2})})_{mn} \qquad (\sum_{i=1}^{3} x_i x_i = 1)$$

$$\sum_{i=1}^{3} X_i^{(N)} X_i^{(N)} = 4\alpha_N{}^2 l(l+1) \cdot 1 \qquad [X_i^{(N)}, X_j^{(N)}] = 2i\alpha_N \epsilon_{ijk} X_k^{(N)}$$

Hoppe (‘*82*)

Madore (‘92)

The non-commutatve geometry appears also in the higher Landau levels!

Behind the scene

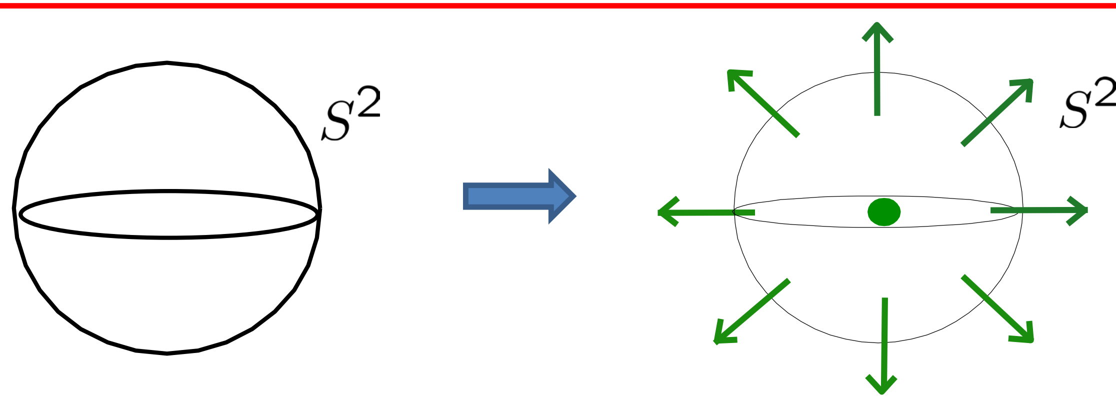
$S^2 \Rightarrow S^2_{\text{fuzzy}}$

- Global sym. of manifold, SO(3)
- Points on the manifold (= infinite dof)
  : a closed set under the SO(3) group action
- Stabilizer group SO(2)
  = sym. that does not change a point

- Global sym. group of the system, SU(2)
- Irreps. (= finite dof) of SU(2)
  : a closed set under the SU(2) group action
- Gauge group U(1)
  = sym. that does not change physical states

(External symmetry → Internal symmetry)

$S^2 \simeq SO(3)/SO(2)$ ⇒ QM with SU(2) global sym. and U(1) gauge sym.



(Magnetic field is a consequence of the gauge symmetry.)

General prescription

$$\mathcal{M} \simeq G/H \Rightarrow \mathcal{M}_{\text{fuzzy}}$$

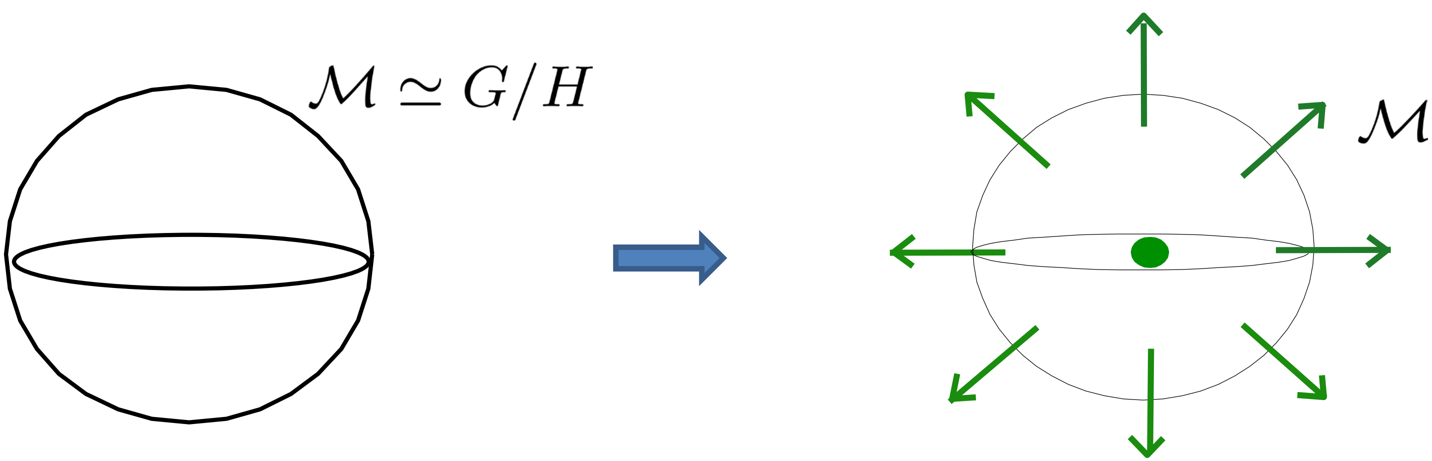
Just replace SU(2) → G, and U(1) → H.

$\mathcal{M} \simeq G/H$ ⇒ QM with gauge symmetry H on the manifold $\mathcal{M}$

$$H = -\frac{1}{2M}\sum_i {D_i}^2\Big|_{\mathcal{M}}$$

⇒ Solve the eigenvalue problem ⇒ Matrix coordinates $\mathcal{M}_{\text{fuzzy}}$

$$|\psi_\alpha^{(N)}\rangle \qquad (X_a^{(N)})_{\alpha\beta} = \langle\psi_\alpha^{(N)}|x_a|\psi_\beta^{(N)}\rangle$$



Noticeable points

- The quantum geometry is not postulated a priori, but naturally emerges in the context of physics.
- The original system is totally physical, and its background framework is a consistent Hilbert space of QM.

→ Maybe, we do not need to worry about mathematical inconsistency!

- Following the prescription, we can automatically construct the fuzzy manifold corresponding to G/H

→ Not restricted to symplectic manifolds. Odd D is also OK!

``Relativistic Landau models and generation of fuzzy spheres” *K.H.* (‘16)

as a consequence of level projection. In this work, we proactively utilize the level projection as an effective tool to generate fuzzy geometry. The level projection is specifically

Find a new quantum geometry from quantum mechanics !

Relativistic model [*‘16*], Susy model [*‘16*], three-sphere [*‘17*], four-sphere [*’20, ’21. ‘23*]…

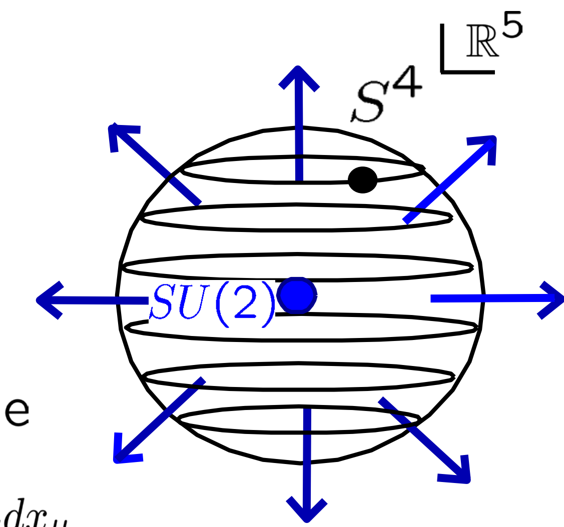
From idea to a concrete model

Classical geometry $S^4 \simeq SO(5)/SO(4)$

Stabilizer group $SO(4) \sim SU(2) \otimes SU(2)$

⇩

Gauge symmetry $SU(2)$ ➡ Yang's $SU(2)$ monopole



$$A = -\frac{1}{r(r+x_5)}\eta^i_{\mu\nu} S_i^{(I/2)} x_\nu dx_\mu$$

Quantum mechanics

$$H = -\frac{1}{2M}\sum_{a=1}^{5} {D_a}^2\Big|_{r=1}$$

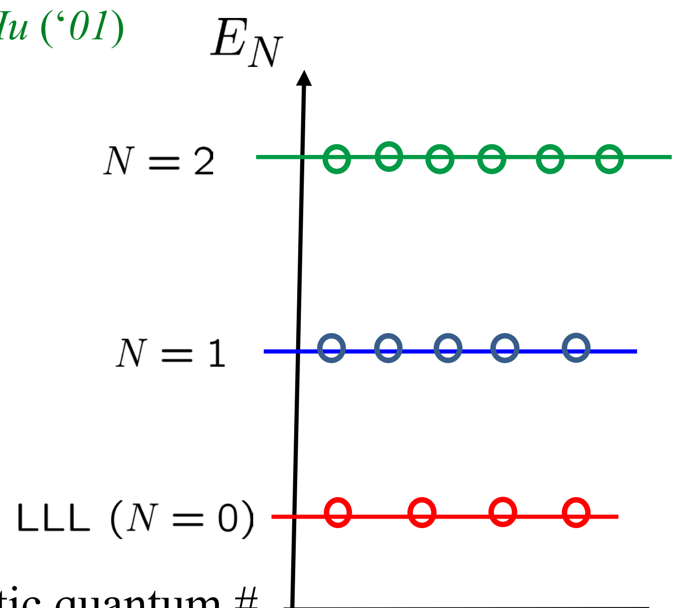
Yang ('78) *Zhang, Hu ('01)*

SO(5) Casimir index (p,q) $q = N, \quad p = N + I$

$N = 0, 1, 2, \cdots$

Landau levels:

$$E_N = \frac{1}{2M}(N(N+3) + I(N+1))$$



Eigenstates = SO(5) irreps. : SU(2) monopole harmonics

$Y^{(N)}_{j,m_j;\ k,m_k}(\xi,\chi,\theta,\phi)$ $SU(2) \bigotimes SU(2) = SO(4)$ magnetic quantum #

Matrix coordinates

KH ['23, '20]

$$(X_a^{(N)})_{\alpha\beta} = \langle Y_\alpha^{(N)} | x_a | Y_\beta^{(N)} \rangle \qquad (\sum_{a=1}^{5} x_a x_a = 1)$$

⇨ In LLL

$$X_a^{(N=0)} = \frac{1}{I+4}(\overbrace{\gamma_a \otimes 1 \otimes \cdots \otimes 1}^{I} + \overbrace{1 \otimes \gamma_a \otimes \cdots \otimes 1}^{I} + \cdots + \overbrace{1 \otimes 1 \otimes \cdots \otimes \gamma_a}^{I})_{\text{sym}}$$

reproduces the Berezin-Toeplitz result derived by *Ishiki, Matsumoto, Muraki['18]*

Fuzzy S4

Grosse, Klimcik, Presnajdar ['96] Castelino, Lee, Taylor ['98] Ramgoolam ['01] Ho, Ramgoolam ['02]

Kimura ['02] Sheikh-Jabbari ['04] DeBellis, Saemann, Szabo ['10] Steinacker ['15] …

⇨ In higher LLs

$$X_{\mu=1,2,3,4}^{(N)} = \sum_{\sigma,\tau=+,-} \langle \sin\xi \rangle \ \langle y_\mu \rangle \ \delta_{j',j+\frac{\sigma}{2}} \ \delta_{k',k+\frac{\tau}{2}} \qquad X_5^{(N)} = -\frac{2n+I+2}{(2N+I+2)(2N+I+4)} \cdot 2s \cdot \delta_{j,j'}\delta_{k,k'}\delta_{m_j,m_j'}\delta_{m_k,m_k'}$$

$$\langle \sin\xi \rangle = -\frac{4s}{(2N+I+2)(2N+I+4)}\sqrt{(N-n+\frac{1-\sigma}{2})(N+n+I+2+\frac{1+\sigma}{2})} + \cdots$$

$$\langle y_1 \rangle = \frac{\sqrt{(2j+1)(2k+1)}}{2}(-1)^{n+I+\frac{\tau}{2}} \begin{Bmatrix} j+\frac{\sigma}{2} & k+\frac{\tau}{2} & \frac{I}{2} \\ k & j & \frac{1}{2} \end{Bmatrix} \sum_{\kappa=+,-} (-1)^{\frac{\kappa-1}{2}} C_{\frac{1}{2},\frac{\kappa}{2};j,m_j}^{j+\frac{\sigma}{2},m_j'} C_{\frac{1}{2},\frac{\kappa}{2};k,m_k}^{k+\frac{\tau}{2},m_k'}$$

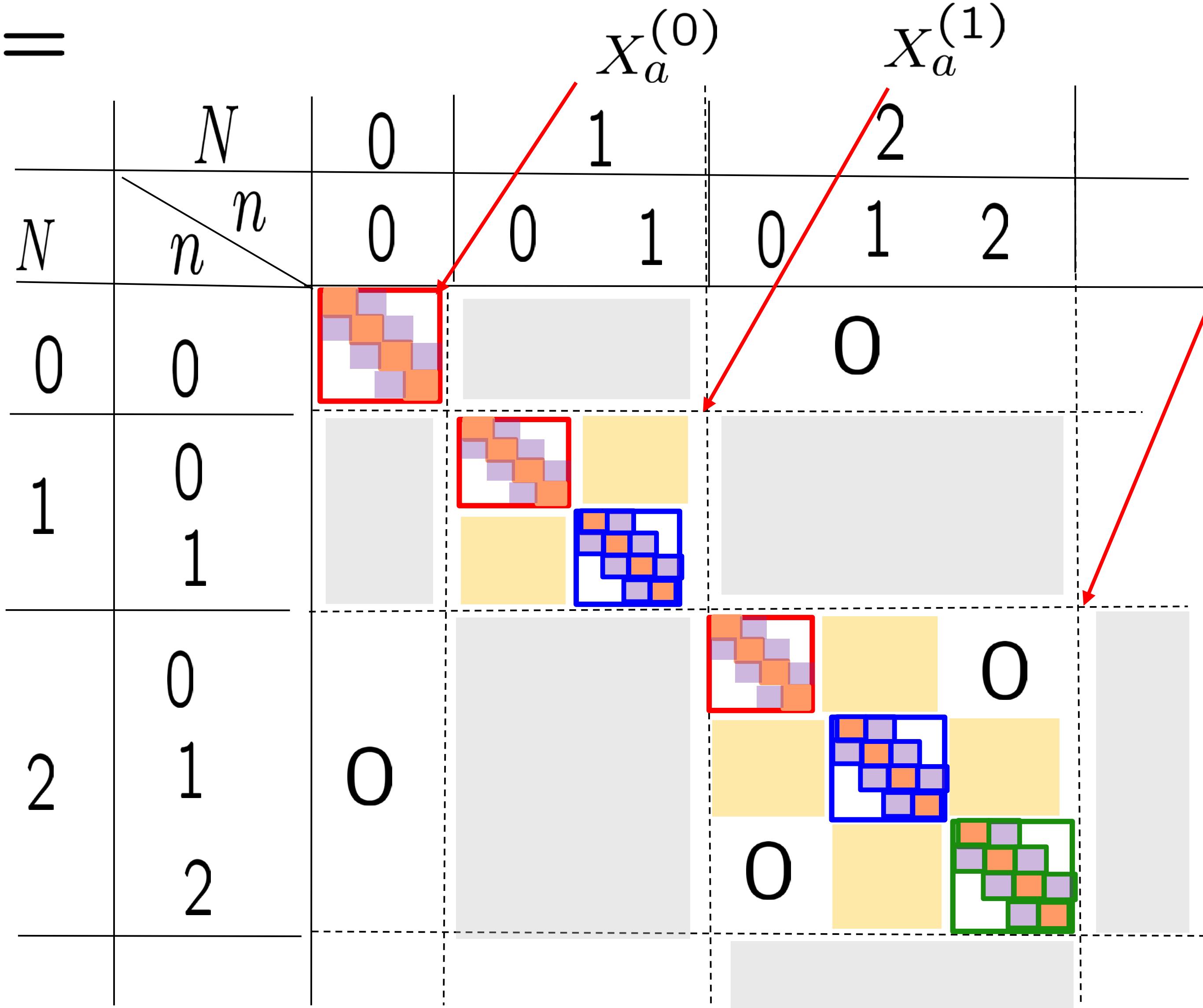
$$\sum_{a=1}^{5} X_a^{(N)} X_a^{(N)} = c_1(N,I)\ 1 \qquad [X_a^{(N)}, X_b^{(N)}, X_c^{(N)}, X_d^{(N)}] = c_3(N,I)\epsilon_{abcde} X_e$$

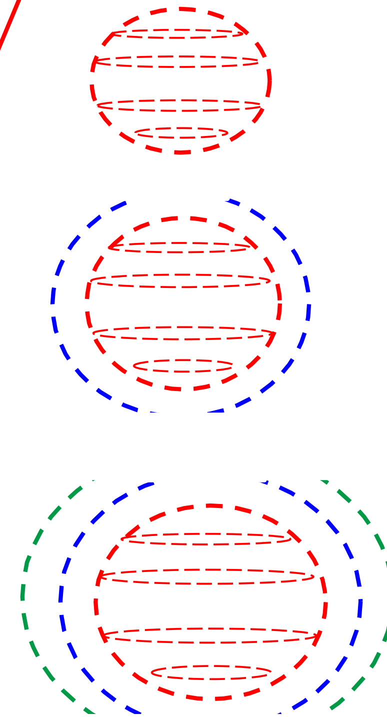
$$([X_1, X_2, X_3, X_4] \equiv \epsilon_{abcd5} X_a X_b X_c X_d)$$

Realization of the fuzzy four-sphere geometry !

Matrix geometry

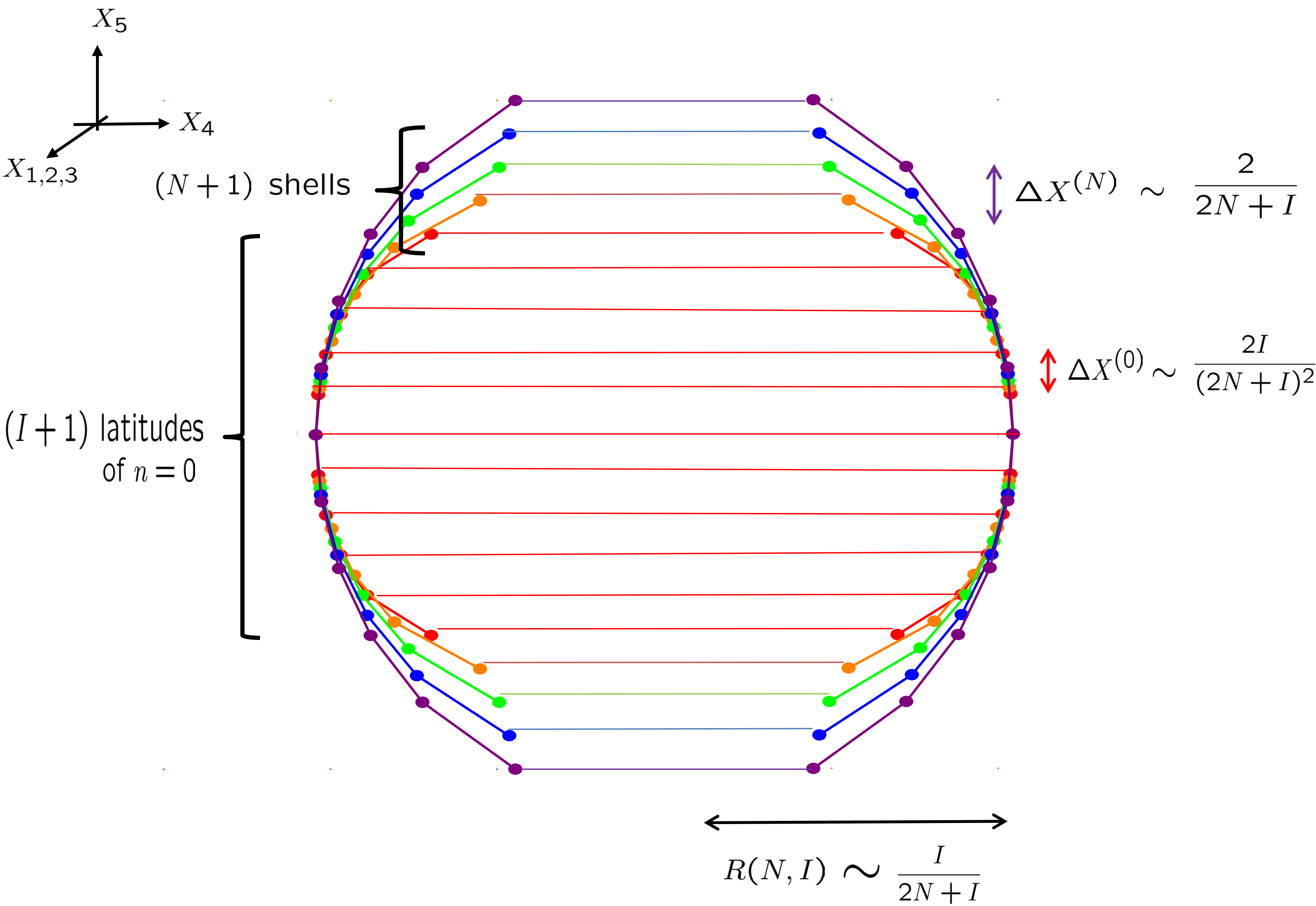
$x_a =$





The Nth LL geometry becomes a nested fuzzy four-sphere with N+1 shells.

Nth Landau level matrix geometry



Algebraic property

1. The un-nested fuzzy S4 (the lowest Landau level matrix geometry)

- Lie algebraic structure *Ho, Ramgoolam ['02]*

$$[X_a, X_b] = 4i\Sigma_{ab} \qquad [X_a, \Sigma_{bc}] = i\delta_{ab}X_c - i\delta_{ac}X_b \qquad [\Sigma_{ab}, \Sigma_{cd}] = i\delta_{ac}\Sigma_{bd} - \cdots$$

→ SU(4) algebra

- Quantum Nambu structure $[X_a, X_b, X_c, X_d] = 4!\epsilon_{abcde}X_e$

2. The nested fuzzy S4 (the higher Landau level matrix geometry)

- No Lie algebraic structure

$$[X_a, X_b] \neq 4i\Sigma_{ab} \qquad [X_a, \Sigma_{bc}] = i\delta_{ab}X_c - i\delta_{ac}X_b \qquad [\Sigma_{ab}, \Sigma_{cd}] = i\delta_{ac}\Sigma_{bd} - \cdots$$

- Quantum Nambu structure

$$[X_a, X_b, X_c, X_d] = 4!\epsilon_{abcde}X_e$$

→ Pure quantum Nambu geometry!
(not captured by the commutator formalism)

Matrix model's new solutions

Yang-Mills matrix model : $S = \frac{1}{4}\mathrm{tr}([A_a, A_b]^2) + \frac{1}{5}\epsilon_{abcde}\mathrm{tr}(A_a A_d A_c A_d A_e)$

Ho, Ramgoolam ('02) *Kimura ('03)*

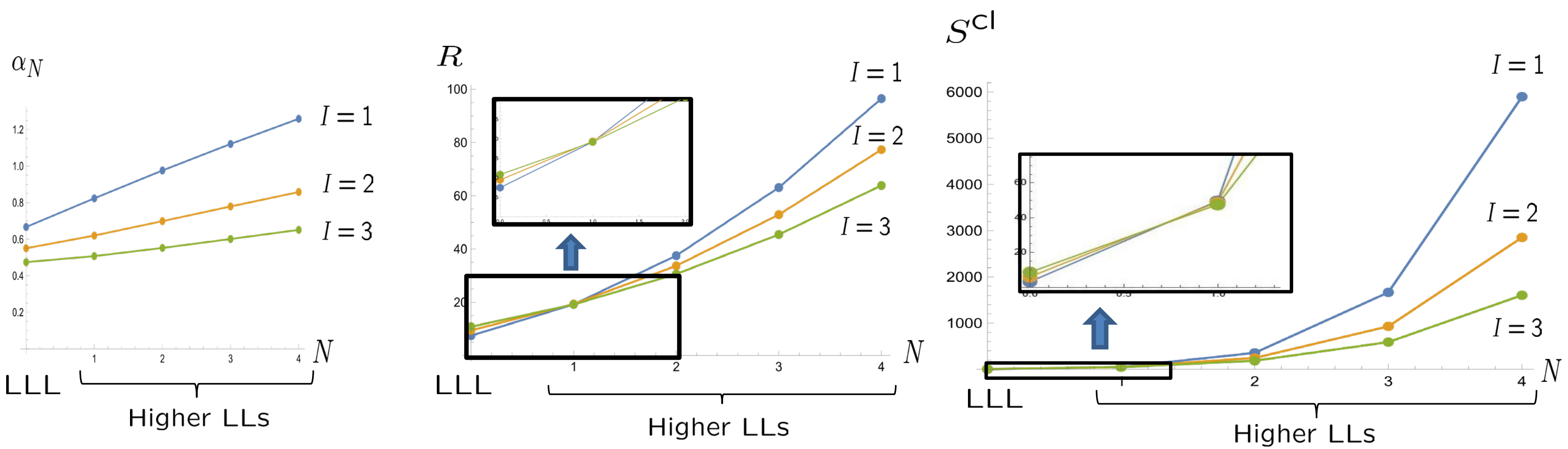
→ EOM : $[[A_a, A_b], A_b] = \epsilon_{abcde} A_b A_c A_d A_e$

Known solution : The un-nested fuzzy S4 (LLL matrix geometry)

$$A_a = \alpha_0 X_a^{(N=0)}$$

Castelino, Lee, Taylor ('98)

The nested fuzzy S4 (Higher LL matrix geometry) → New solutions $A_a = \alpha_N X_a^{(N=1,2,\cdots)}$



Summary

Conventional non-commutative scheme:

Quantization of classical manifolds

➡ Present non-commutative scheme:

Directly from quantum Hilbert space

``Quantum oriented''

- A concrete prescription for generating the matrix geometry of $\mathcal{M} = G/H$

- Obtained quantum space is interesting by itself: pure quantum Nambu geometry

- A practical method to generate new solutions of matrix model