Wave-Packet Effects:

a solution for isospin anomalies in vector-meson decay

Kenji Nishiwaki

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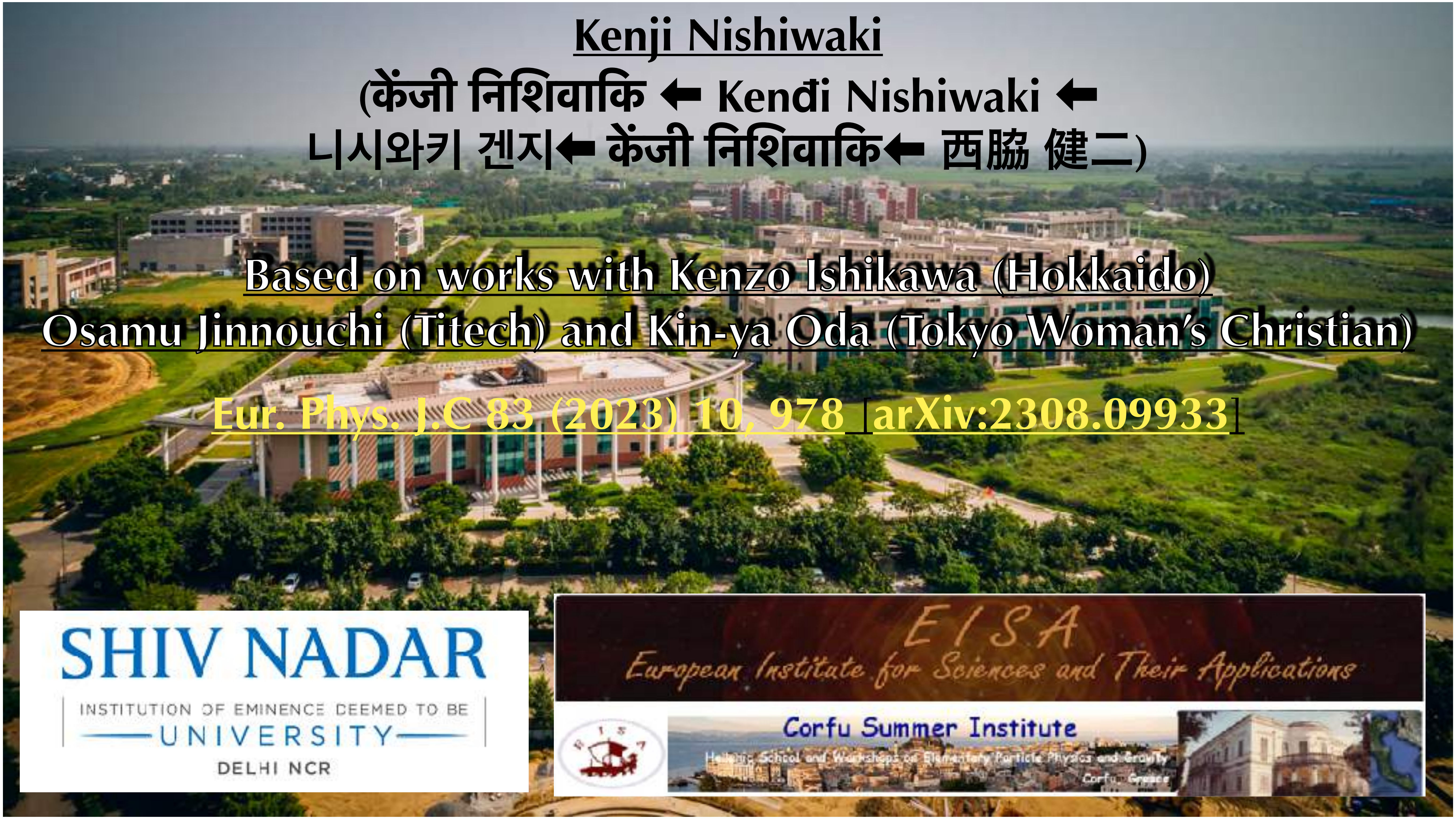
니시와키 겐지← केंजी निशिवाकि← 西脇 健二)

Based on works with Kenzo Ishikawa (Hokkaido)
Osamu Jinnouchi (Titech) and Kin-ya Oda (Tokyo Woman's Christian)

Eur. Phys. J.C 83 (2023) 10, 978 [arXiv:2308.09933]



Talk @ Workshop on the Standard Model and Beyond, 1st Aug. 2024 [Sun]

Intro: BSM

- Physics **B**eyond the **S**tandard **Model** is necessary due to
  - discussing issues that the SM cannot mention
    (e.g., dark matter, dark energy, inflation)
  - addressing (experimental) anomalies that the SM cannot explain
    (e.g., muon g-2, R_D)

Intro: BSM

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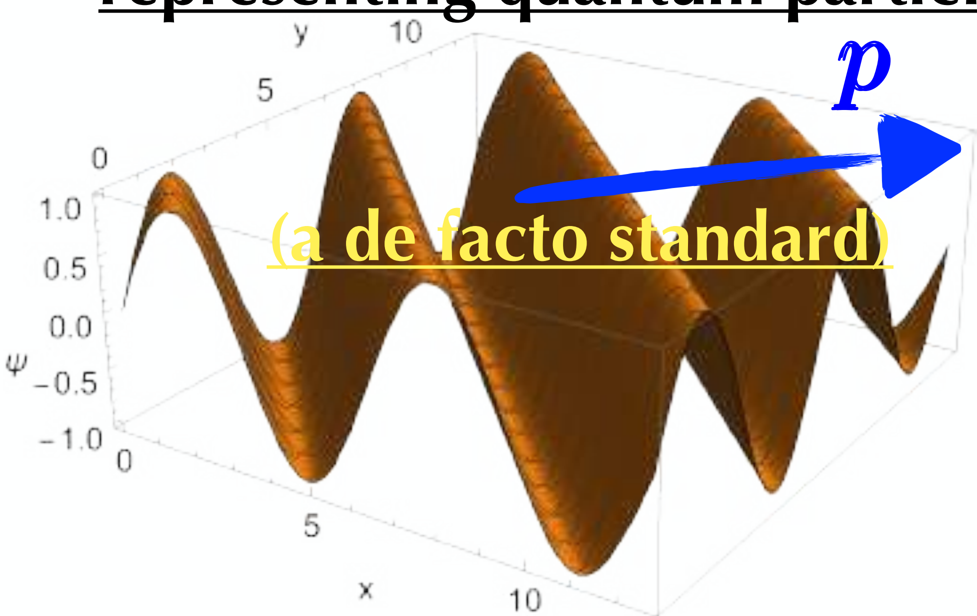
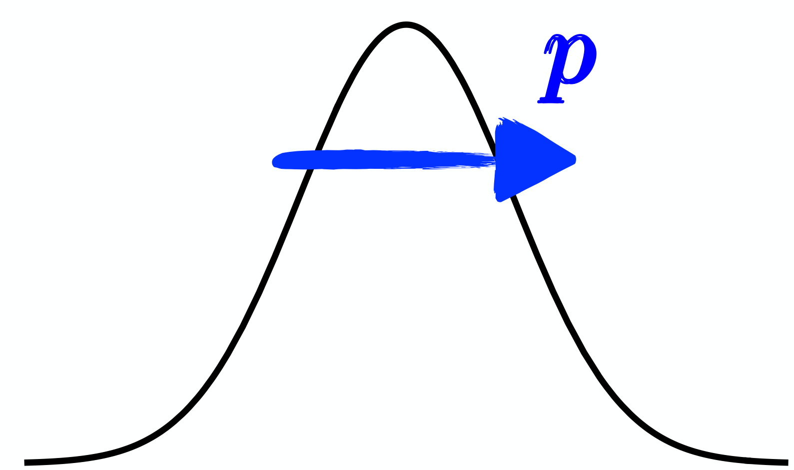
  - discussing issues that the SM cannot mention (e.g., dark matter, dark energy, inflation)
  - addressing (experimental) anomalies that the SM cannot explain (e.g., muon g-2, R_D)
- We can focus on <u>another BSM</u>: **B**eyond the **S**tandard **Method:**

The plane-wave form is widely used for representing quantum particles.



"beyond"

Wave-packet form for quantum particles

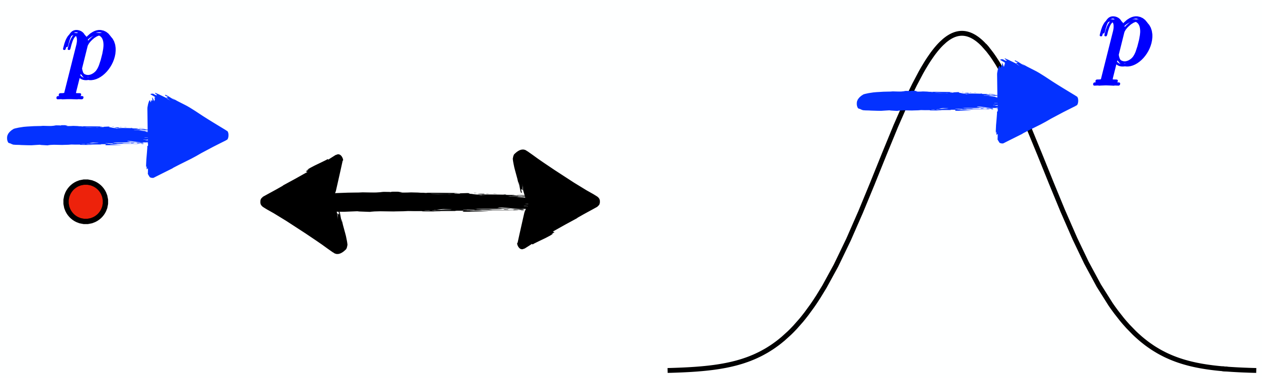


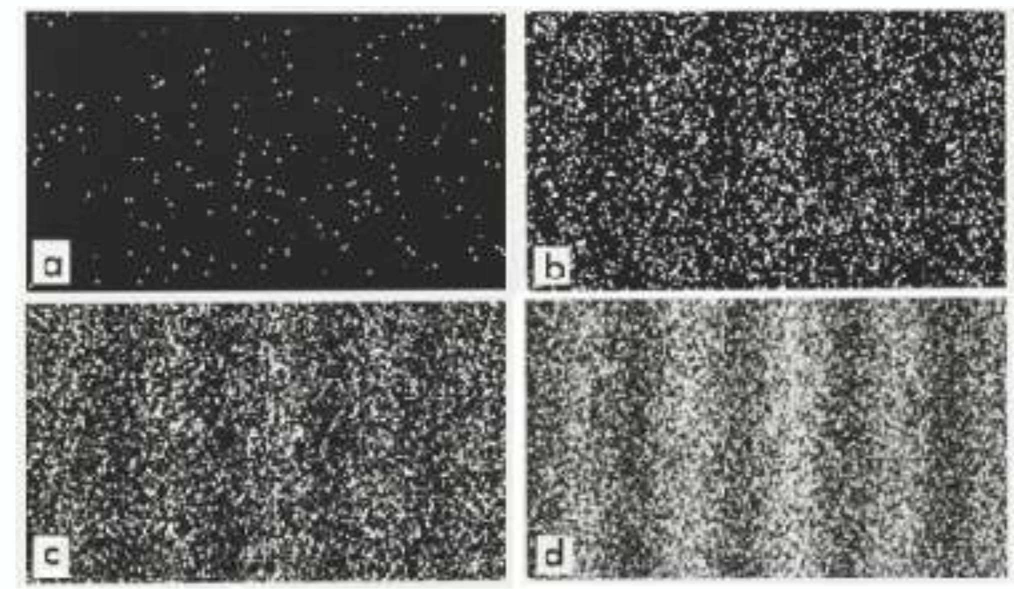
$$\psi(t,\boldsymbol{x}) \sim e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}} \quad \left(= e^{+ip_\mu x^\mu}\right)$$

note: (metric) = diag(-1,1,1,1);
taking afterward: $\hbar = c = 1$

Intro: how about locality?

- We remember that
  wave profiles need to be localised.





[A. Tonomura, Proceedings of the National Academy of Sciences, USA, 102, 14952 (2005)]

- In conclusion,
  - The plane-wave description of quantum particles well describes part of necessary properties of particles.
  - On the other hand, however, the plane wave **lacks some nature of quantum particles, at least the locality**.

By use of a **localised wave (wave packet)**,
we can **overcome** this difficulty and obtain
the **full information of quantum transitions**!

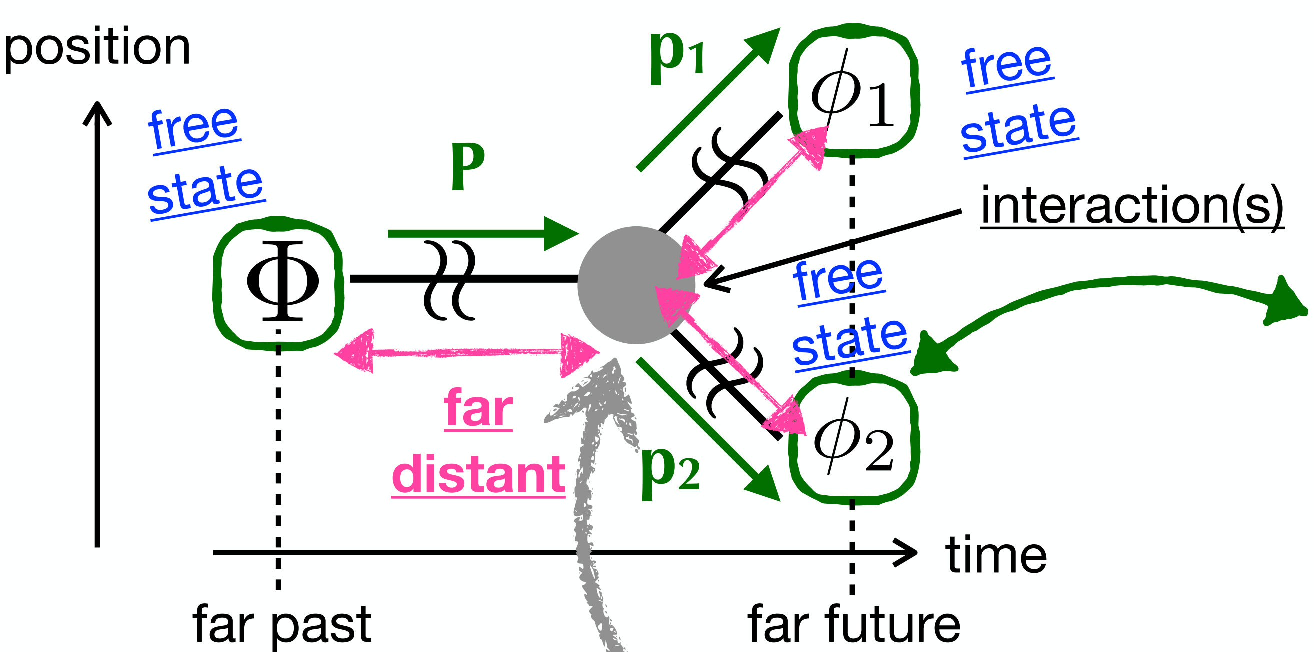
Problem in plane-wave S-matrix

[QFT textbooks]

☑ Plane-wave S-matrix (1→2) def.:

$$S_{\rm PW} = \overset{\text{out free state}}{\langle \boldsymbol{p}_1, \boldsymbol{p}_2|} \mathrm{T} e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \overset{\text{in free state}}{|\boldsymbol{P}_0\rangle}$$

$$= \underbrace{(2\pi)^4\delta^4(P_{\rm out} - P_{\rm in})}_{\text{manifest energy-momentum conservation}} \times \underbrace{(iM_{\rm PW})}_{\text{(factorised) amplitude}}$$



momentum eigenstates
(external free: also mass eigenstates;
$E_i^2 = \mathbf{p}_i^2 + m_i^2$)

taking all possible configurations
(up to an order)

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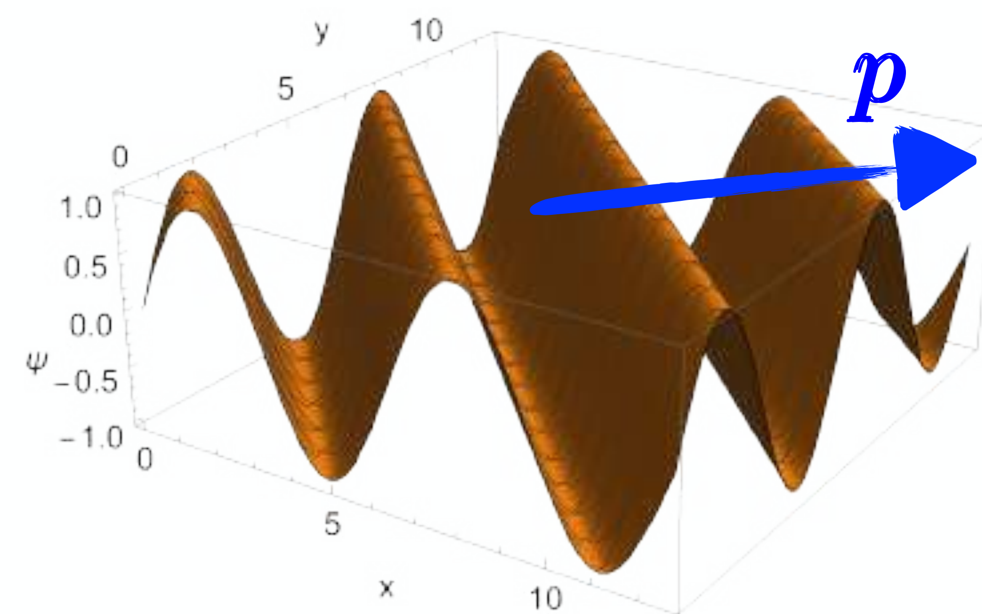
- Corresponding probability is given as $|S_{\mathrm{PW}}|^2$. $(2\pi)^{-4}$[(Volume)(Time) → ∞]
- $|S_{\mathrm{PW}}|^2$ is ill-defined due to $|\delta^4(P_{\mathrm{out}}-P_{\mathrm{in}})|^2 = \delta^4(P_{\mathrm{out}}-P_{\mathrm{in}}) \times \delta^4(0)$.

⇒ **Only** the averaged (per V and T) frequencies of events **is calculable.**

(T_{in} (= T_{initial}) = $-\infty$, T_{out} (= T_{final}) = $+\infty$) We will see soon later.

Why the problem happens?

Plane Wave is non-normalisable!



What is calculable?

- So, what can we do in the plane-wave formalism?
  - $\psi(t, \boldsymbol{x}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}V}} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}}$ "literal normalisation"
  - $[\text{(PW) phase space}] = \frac{(V)d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{(V)d^3\boldsymbol{p}_2}{2E_2(2\pi)^3}$

$|S_{\rm PW}|^2 \times [\text{phase space}] \times [\text{flux}]$

$$= \underbrace{(2\pi)^4\delta^4(P_{\rm out} - P_{\rm in})\frac{1}{2E_{\rm in}}|M_{\rm PW}|^2 \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3}}_{\text{well defined (The volume is cancelled out.)}} \times \underbrace{TV}_{\text{ill-defined! (since T, V}\to\infty)}$$

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$$|S_{\text{PW}}|^2 \times [\text{phase space}] \times [\text{flux}]$$
$$= (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) \frac{1}{2E_{\text{in}}} |M_{\text{PW}}|^2 \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3} \times TV$$

$$\frac{|S_{\text{PW}}|^2 \times [\text{phase space}] \times [\text{flux}]}{TV}$$
$$= (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) \frac{1}{2E_{\text{in}}} |M_{\text{PW}}|^2 \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3}$$

well defined!

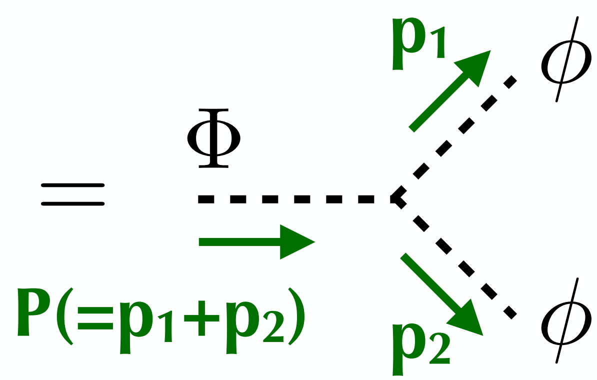
The frequency per time (= Γ: decay rate) is <u>well defined and calculable</u>.

(SKIPPABLE) DETAILS

As we know very well,

- In the case of $\hat{H}_{\rm int}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$,

the plane-wave amplitude; taking a **simple** form, easily derived via **Feynman rules**

$\circ\ iM_{\rm PW}(\Phi \to \phi\phi) =$ 

$= -i\kappa,$

(for $\mathbf{P}_{\rm in} = \mathbf{0}$)

$$\Gamma(\Phi \to \phi\phi) = \frac{\kappa^2}{32\pi m_\Phi}\sqrt{1 - \frac{4m_\phi^2}{m_\Phi^2}}$$

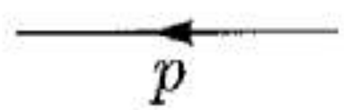
(SKIPPABLE) DETAILS

[Peskin, Schroeder]

A.1 Feynman Rules

In all theories it is understood that momentum is conserved at each vertex, and that undetermined loop momenta are integrated over: $\int d^4p/(2\pi)^4$. Fermion (including ghost) loops receive an additional factor of (-1), as explained on page 120. Finally, each diagram can potentially have a symmetry factor, as explained on page 93.

ϕ^4 theory: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

Scalar propagator: 

$$= \frac{i}{p^2 - m^2 + i\epsilon} \tag{A.1}$$

ϕ^4 vertex: 
$$= -i\lambda \tag{A.2}$$

External scalar: 
$$= 1 \tag{A.3}$$

plitude;
orm,
man rules

[Peskin, Schroeder]

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A.1 Feynman Rules

In all theories it is understood that momentum is conserved at each vertex, and that undetermined loop momenta are integrated over: $\int d^4p/(2\pi)^4$. Fermion (including ghost) loops receive an additional factor of (-1), as explained on page 120. Finally, each diagram can potentially have a symmetry factor, as explained on page 93.

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Scalar propagator: $$= \frac{i}{p^2 - m^2 + i\epsilon} \qquad \text{(A.1)}$$

The differential decay rate of an unstable particle to a given final state is

$$d\Gamma = \frac{1}{2m_{\mathcal{A}}}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}(m_{\mathcal{A}} \to \{p_f\})\right|^2 (2\pi)^4\delta^{(4)}(p_{\mathcal{A}} - \textstyle\sum p_f). \quad \text{(A.57)}$$

For the special case of a two-particle final state, the Lorentz-invariant phase space takes the simple form

$$\left(\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}(\textstyle\sum p_i - \sum p_f) = \int \frac{d\Omega_{\text{cm}}}{4\pi}\frac{1}{8\pi}\left(\frac{2|\mathbf{p}|}{E_{\text{cm}}}\right), \qquad \text{(A.58)}$$

S-matrix in Gaussian basis

☑ S-matrix (1→2) def.:

[Note: as in the plane-wave basis,
but by the creation/annihilation
operators for wave packets]

$$\mathcal{S} := \langle \overbrace{\mathcal{P}_1, \mathcal{P}_2}^{\substack{\text{out} \\ \text{free state}}} | \mathrm{T} e^{-i \int_{T_\text{in}}^{T_\text{out}} dt\, \hat{H}_\text{int}^{(I)}(t)} \overbrace{|\mathcal{P}_0\rangle}^{\substack{\text{in} \\ \text{free state}}}$$

$$\left[\mathcal{P}_i = \{\sigma_i, \underbrace{X_i^0 (= T_i), \boldsymbol{X}_i}_{=: X_i} \boldsymbol{P}_i\}\right]$$

This describes the amplitude for the **finite probability/frequency** of the **event** with **fully-described initial & final particle states!**

“additional” information

Normalisability of Gaussian can makes *S itself finite*!

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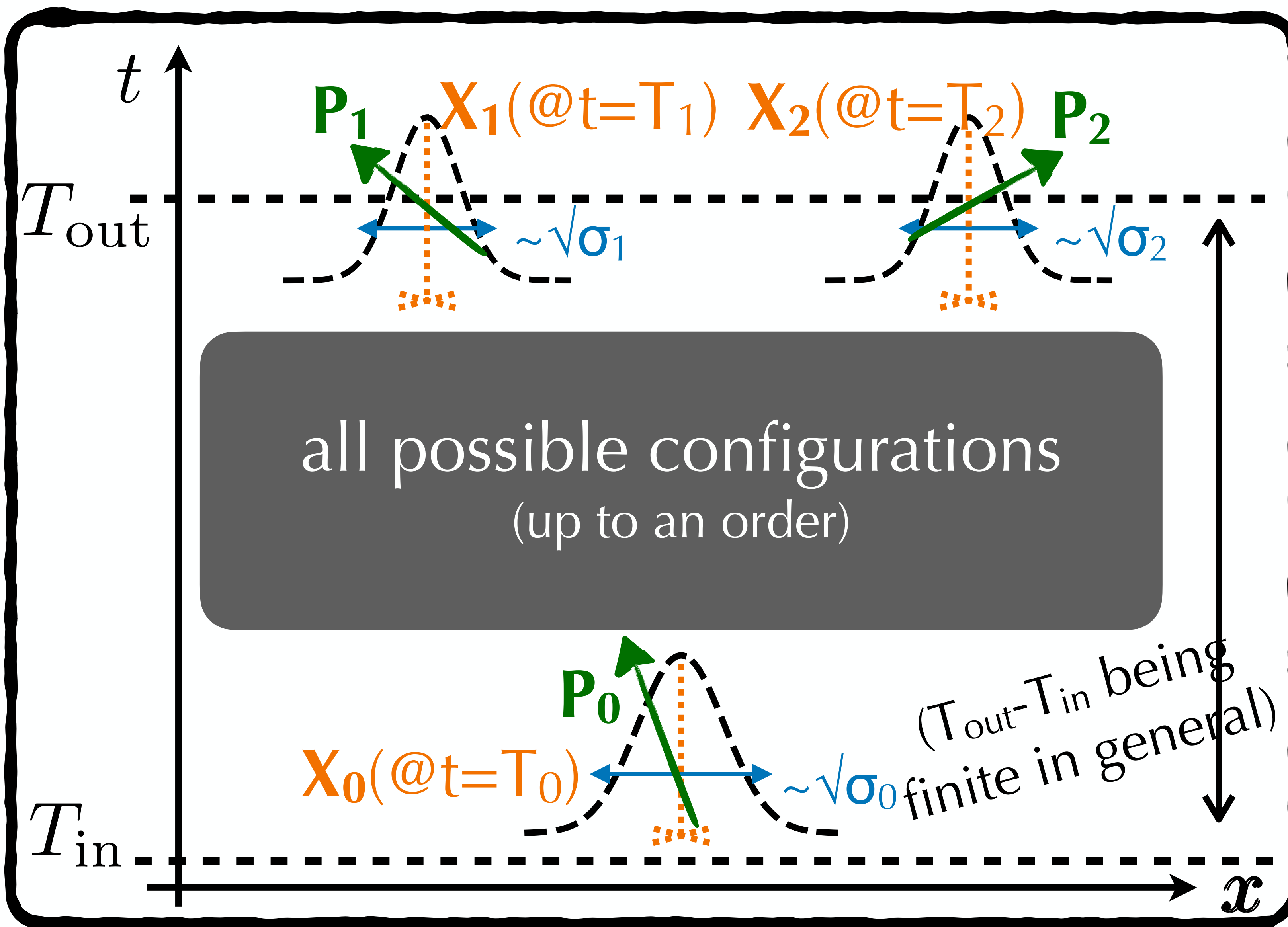


Normalisability of Gaussian can makes *S itself finite*!

- First proposal by coherent state:
  [Ishikawa, Shimomura (0508303)]
- Claims on various phenomena by Ishikawa-san et. al.
  e.g. [Ishikawa, Jinnouchi, Kubota, Sloan, Tatsuishi (1901.03019)]

Experiment by the same group → (1907.01264)
$({}^{22}\mathrm{Na} \to {}^{22}\mathrm{Ne}^* e^+ \nu, e^+(e^-) \to 2\gamma)$

Short Summary

For the same focused physical 1→2 process,

(note: we can similarly construct those for m→n processes.)

✱ plane-wave S-matrix:

- with partial information
- not suitably normalised

$$S_{\mathrm{PW}} = \overset{\text{out}}{\underset{\text{free state}}{\langle \boldsymbol{p}_1, \boldsymbol{p}_2|}} \mathrm{T}e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\, \hat{H}^{(I)}_{\mathrm{int}}(t)} \overset{\text{in}}{\underset{\text{free state}}{|\boldsymbol{P}_0\rangle}}$$

$$= (2\pi)^4 \delta^4(P_{\mathrm{out}} - P_{\mathrm{in}}) \times (iM_{\mathrm{PW}})$$

not equal

more informative

(suitable limits/ marginalisations)

✱ **Gaussian S-matrix:**

- with **full** information
- **normalised** appropriately

$$\mathcal{S} := \overset{\text{out}}{\underset{\text{free state}}{\langle \mathcal{P}_1, \mathcal{P}_2|}} \mathrm{T}e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\, \hat{H}^{(I)}_{\mathrm{int}}(t)} \overset{\text{in}}{\underset{\text{free state}}{|\mathcal{P}_0\rangle}}$$

$$\left[\mathcal{P}_i = \{\boxed{\sigma_i, \underbrace{X_i^0(=T_i), \boldsymbol{X}_i}_{=:X_i}} \boldsymbol{P}_i\}\right]$$

"additional" information

Short Summary

For the same focused physical 1→2 process,

(note: we can similarly construct those for m→n processes.)

✱ |plane-wave S-matrix|^2:

- with partial information
- not suitably normalised

$|S_{\mathrm{PW}}|^2$ itself is **ill** defined. (dimensionful, relative frequency)

$$d\Gamma = \frac{1}{2E_{\mathrm{in}}} \frac{|S_{\mathrm{PW}}|^2}{TV} \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3}$$

External states are characterised by momenta.

✱ **| Gaussian S-matrix |^2:**

- with **full** information
- **normalised** appropriately

$|\mathcal{S}|^2$ **itself is well defined.** (dimensionless, absolute frequency)

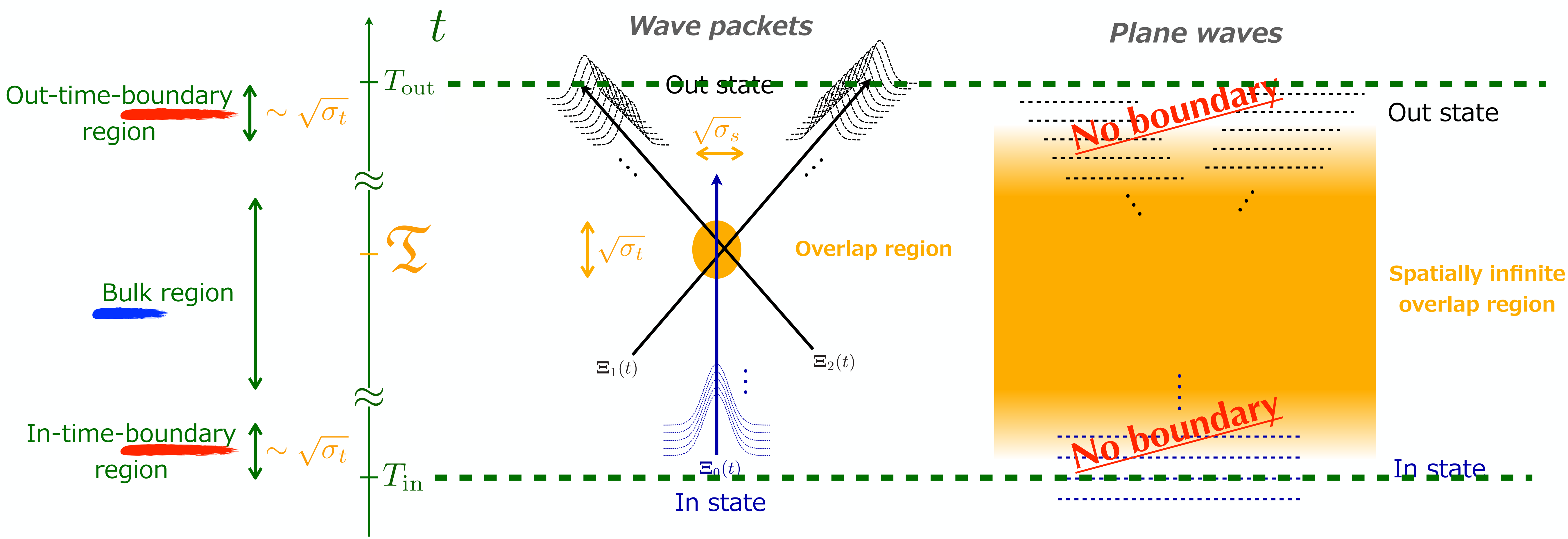
$$dP = |\mathcal{S}|^2 \frac{d^3\boldsymbol{X}_1 d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{d^3\boldsymbol{X}_2 d^3\boldsymbol{p}_2}{(2\pi)^3}$$

External states are characterised by momenta and positions of centres.

Table of Contents

Structure of Transitions

- The plane-wave S-matrix has no **time boundaries** (only in **time bulk**).
- The **wave-packet** S-matrix has **time boundaries** (also in **time bulk**).



What's next?

- We examined the simplest $1 \to 2$ case in wave-packet formalism.
  Now, we will be interested in

1. How about the $2 \to 2$ full scattering, including the production part?

→ No detailed discussion today

2. When does the wave-packet effect become significant?

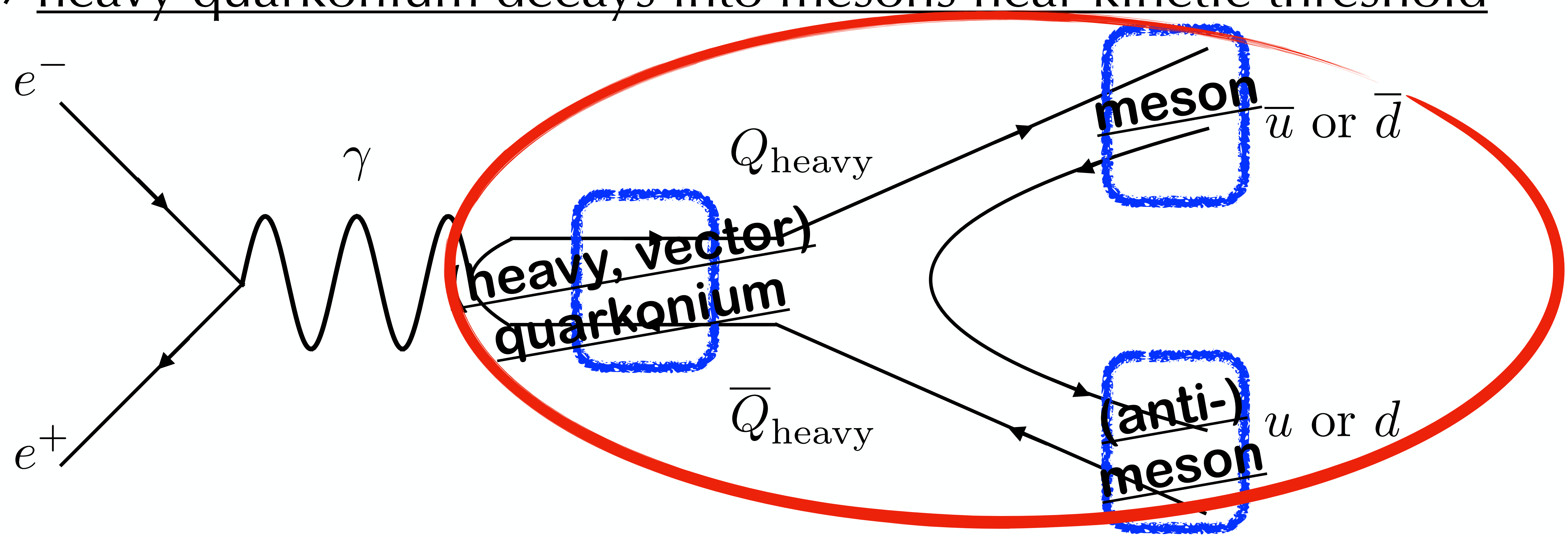
→ **Today's main topic**

- So, the `best' process to see a wave-packet intrinsic nature requires
  - ○ domination of the boundary contribution, e.g., via a narrow phase space
  - ○ resonant production & decay

    +

  - ○ experimental anomalies being reported

We found such a process!
⇒ heavy quarkonium decays into mesons near kinetic threshold



Anomaly in heavy vector quarkonium decays

- For each heavy vector quarkonium (V), two dominant decay branches are "$V \to P^+P^-$" and "$V \to P^0\overline{P^0}$".
  - P^+ is the EM-charged one; (anti-particle of P^+) = P^-
  - P^- is the EM-neutral one; (anti-particle of P^0) = P^0

$$\phi(s\bar{s}) \to K^+(u\bar{s})\, K^-(s\bar{u})\,, \quad \psi(c\bar{c}) \to D^+\big(c\bar{d}\big)\, D^-(d\bar{c})\,, \quad \Upsilon\big(b\bar{b}\big) \to B^+\big(u\bar{b}\big)\, B^-(b\bar{u})\,,$$

$$\phi(s\bar{s}) \to K^0(d\bar{s})\, \overline{K^0}\big(s\bar{d}\big) \to K^0_{\mathrm{L}} K^0_{\mathrm{S}}, \quad \psi(c\bar{c}) \to D^0(c\bar{u})\, \overline{D^0}(u\bar{c})\,, \quad \Upsilon\big(b\bar{b}\big) \to B^0\big(d\bar{b}\big)\, \overline{B^0}\big(b\bar{d}\big)\,,$$

"Kaons" "D-mesons" "B-mesons"

Anomaly in heavy vector quarkonium decays

- For each heavy vector quarkonium (V), two dominant decay branches are "V→P+P-" and "V→$P^0\overline{P^0}$".
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$$\phi(s\bar{s}) \to K^0(d\bar{s})\,\overline{K^0}\left(s\bar{d}\right) \to K^0_{\rm L}K^0_{\rm S}, \quad \psi(c\bar{c}) \to D^0(c\bar{u})\,\overline{D^0}(u\bar{c})\,, \quad \Upsilon\left(b\bar{b}\right) \to B^0\left(d\bar{b}\right)\overline{B^0}\left(b\bar{d}\right),$$

"Kaons" "D-mesons" "B-mesons"

Measuring Isospin breaking

$$R_\phi := \frac{\Gamma(\phi \to K^+K^-)}{\Gamma\left(\phi \to K^0_{\rm L}K^0_{\rm S}\right)}, \qquad R_\psi := \frac{\Gamma(\psi \to D^+D^-)}{\Gamma\left(\psi \to D^0\overline{D^0}\right)}, \qquad R_\Upsilon := \frac{\Gamma(\Upsilon \to B^+B^-)}{\Gamma\left(\Upsilon \to B^0\overline{B^0}\right)}.$$

- **PDG values:**

$$R_\phi^{\rm PDG} = 1.45 \pm 0.03, \qquad R_\psi^{\rm PDG} = 0.798 \pm 0.010, \qquad R_\Upsilon^{\rm PDG} = 1.058 \pm 0.024.$$

- **plane-wave results:**

$$R_\phi^{\rm plane} = \frac{g^2_{\phi+}}{g^2_{\phi 0}}(1.5156 \pm 0.0033)\,, \quad R_\psi^{\rm plane} = \frac{g^2_{\psi+}}{g^2_{\psi 0}}(0.6915 \pm 0.0046)\,, \quad R_\Upsilon^{\rm plane} = \frac{g^2_{\Upsilon+}}{g^2_{\Upsilon 0}}(1.047 \pm 0.026)\,.$$

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2.1σ 9.5σ!! 0.32σ

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[for isospin-symmetric case ($g_{P+} = g_{P0}$)]

- The mass difference in the final states deviates the ratio from unity.
- In the **plane-wave** calculation, R(Φ) and R(ψ) depend on only the masses in the isospin-symmetric limit ($g_+ = g_0$).

$$\widehat{\mathcal{H}}^{(\mathrm{I})}_{\mathrm{int,eff}} = i g_{V+} \mathcal{V}^\mu \left[\mathcal{P}^+ \partial_\mu \mathcal{P}^- - \mathcal{P}^- \partial_\mu \mathcal{P}^+\right] + i g_{V0} \mathcal{V}^\mu \left[\mathcal{P}^0 \partial_\mu \overline{\mathcal{P}^0} - \overline{\mathcal{P}^0} \partial_\mu \mathcal{P}^0\right]$$

It should be good since $m_u \sim m_d \sim O(1)$ MeV, while $m_s \sim O(10^2)$ MeV and $m_c \sim O(1)$ GeV.

- $\Gamma\left(\phi \to K^+K^-\right) = \frac{2}{3}\left(\frac{g_+^2}{4\pi}\right)\frac{|\boldsymbol{k}|^3}{m_\phi^2}$,

$$|\boldsymbol{k}| = \frac{1}{2}\left(m_\phi^2 - 4m_{K^+}^2\right)^{1/2}$$

- $R_{\mathrm{th}} := \left.\frac{\Gamma\left(\phi \to K^+K^-\right)}{\Gamma\left(\phi \to K^0\overline{K^0}\right)}\right|_{\mathrm{th}} =: \left(\frac{g_+^2}{g_0^2}\right) R_{\mathrm{FGR2}}$

$$= \cancel{\left(\frac{g_+^2}{g_0^2}\right)}\left(\frac{m_\phi^2 - 4m_{K^+}^2}{m_\phi^2 - 4m_{K^0}^2}\right)^{3/2}.$$

[Branon,Escribano,Lucio,Pancheri, hep-ph/0003273]

- Isospin breaking and QED corrections do not resolve the discrepancy of R(Φ).
- Here, the form factor [ψ(r=0)] of the vector meson is cancelled out in R.

Note:

$$m_\phi = (1019.461 \pm 0.016)\ \mathrm{MeV}, \quad 2m_{K^+} = (987.354 \pm 0.032)\ \mathrm{MeV}, \quad 2m_{K^0} = (995.222 \pm 0.026)\ \mathrm{MeV}, \quad \Gamma_\phi = (4.249 \pm 0.013)\ \mathrm{MeV},$$

$$m_\psi = (3773.7 \pm 0.4)\ \mathrm{MeV}, \quad 2m_{D^+} = (3739.32 \pm 0.10)\ \mathrm{MeV}, \quad 2m_{D^0} = (3729.68 \pm 0.10)\ \mathrm{MeV}, \quad \Gamma_\psi = (27.2 \pm 1.0)\ \mathrm{MeV},$$

$$m_\Upsilon = (10579.4 \pm 1.2)\ \mathrm{MeV}, \quad 2m_{B^+} = (10558.7 \pm 0.24)\ \mathrm{MeV}, \quad 2m_{B^0} = (10559.3 \pm 0.24)\ \mathrm{MeV}, \quad \Gamma_\Upsilon = (20.5 \pm 2.5)\ \mathrm{MeV}.$$

Form of Gaussian wave-packet S-matrix

☑ $$S_{V\to P\overline{P}} = i g_{\text{eff}} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A} \right)^{3/4} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} \left(2\pi\sigma_s\right)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$
$$\times\, e^{-\frac{\Gamma_V}{2}(\mathfrak{T} - T_0 + i\sigma_t\delta\omega) + \frac{\Gamma_V^2\sigma_t}{8}}\, \widetilde{F}(|\boldsymbol{V}_1 - \boldsymbol{V}_2|)$$

a finite form

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a finite form

form factor for the vector quarkonium

○ $F(\boldsymbol{r}) = \dfrac{N}{\sqrt{2\pi R_0}} \dfrac{e^{-\frac{r}{R_0}}}{r}$

Fourier transform & normalising

Non-rel Approx.

$$\widetilde{F}(|\boldsymbol{V}_1 - \boldsymbol{V}_2|) := \frac{1}{\left(\frac{R_0 m_P (\boldsymbol{V}_1 - \boldsymbol{V}_2)}{2} \right)^2 + 1}$$

[approximate form of (s-wave) ground state under a Coulomb potential **(beyond "r=0" approximation)**]

[Fischbach, Overhauser, Woodahl, hep-ph/0112170]

(c.f., a similar introduction for Φ→2K)

In this order,
the form factor depends on
final-state configurations

→ NOT factored out in R_V.

Form of Gaussian wave-packet S-matrix

☑ $$S_{V\to P\overline{P}} = i g_{\rm eff} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A}\right)^{3/4} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$
$$\times\, e^{-\frac{\Gamma_V}{2}(\mathfrak{T} - T_0 + i\sigma_t \delta\omega) + \frac{\Gamma_V^2 \sigma_t}{8}} \widetilde{F}(|\boldsymbol{V}_1 - \boldsymbol{V}_2|)$$

a finite form

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○ $$F(\boldsymbol{r}) = \frac{N}{\sqrt{2\pi R_0}} \frac{e^{-\frac{r}{R_0}}}{r}$$

$$\xrightarrow[\text{Non-rel Approx.}]{\text{Fourier transform \& normalising}}$$

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[approximate form of

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○ $$dP_{V\to P\overline{P}} = \frac{d^3\boldsymbol{X}_1 d^3\boldsymbol{P}_1}{(2\pi)^3} \frac{d^3\boldsymbol{X}_2 d^3\boldsymbol{P}_2}{(2\pi)^3} \left|S_{V\to P\overline{P}}\right|^2$$

Non-relativistic approximations work fine.

○ $$R_V^{\rm WP} := \frac{P_{V\to P^+P^-}}{P_{V\to P^0\overline{P^0}}}$$ **(the Ratio in terms of transition probability)**

(SKIPPABLE) DETAILS

Form of Gaussian wave-packet S-matrix

$$S_{V\to P\overline{P}} = i g_{\text{eff}} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A} \right)^{3/4} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} \left(2\pi\sigma_s\right)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

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$$\widehat{\mathcal{H}}^{(\mathrm{I})}_{\text{int,eff}} = i g_{V+} \mathcal{V}^\mu \left[\mathcal{P}^+ \partial_\mu \mathcal{P}^- - \mathcal{P}^- \partial_\mu \mathcal{P}^+ \right]$$

$$+\, i g_{V0} \mathcal{V}^\mu \left[\mathcal{P}^0 \partial_\mu \overline{\mathcal{P}^0} - \overline{\mathcal{P}^0} \partial_\mu \mathcal{P}^0 \right]$$

$$\overline{|g_{\text{eff}}|^2} := \frac{g_V^2}{3} \sum_{\lambda_0} \left| \varepsilon_\mu(P_0, \lambda_0)\left(P_1^\mu - P_2^\mu\right) \right|^2 = \frac{g_V^2}{3} (\boldsymbol{P}_1 - \boldsymbol{P}_2)^2.$$

$g_+ = g_0$ ($\to g$) is suggested via the isospin symmetry (u⇔d)

Form of Gaussian wave-packet S-matrix

(SKIPPABLE) DETAILS

$$S_{V\to P\overline{P}} = i g_{\text{eff}} N_V \boxed{\left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A}\right)^{3/4}\right)} \boxed{e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}} G(\mathfrak{T})$$
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normalisation factors of free Gaussians

overlaps of the wave packets (including approximated Energy-Momentum conservation)

Form of Gaussian wave-packet S-matrix

(SKIPPABLE) DETAILS

☑ $$S_{V\to P\overline{P}} = i g_{\mathrm{eff}} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A}\right)^{3/4}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$
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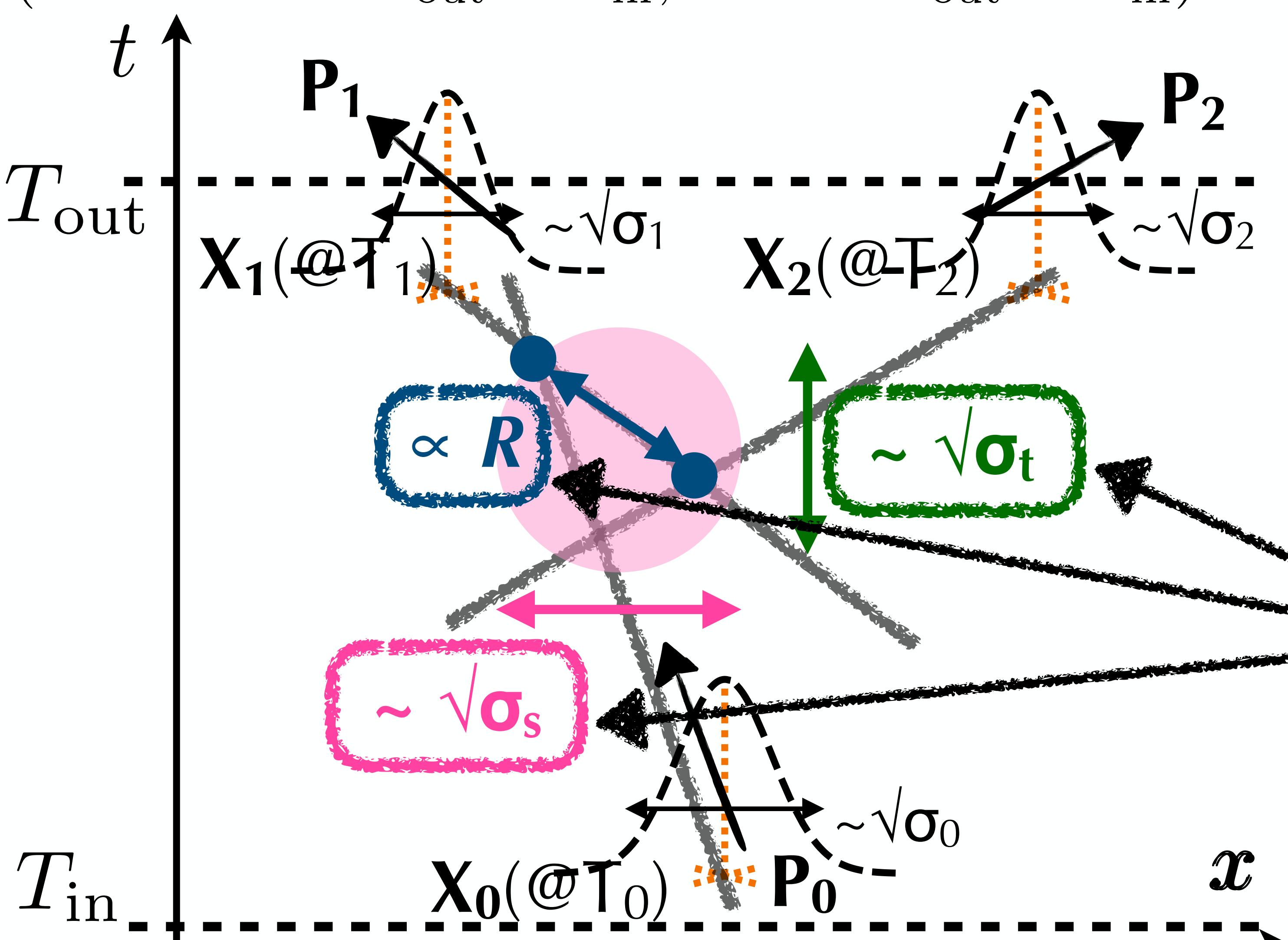
• Geometrical variables characterise *S*.

$(\delta\omega \sim \delta E := E_{\mathrm{out}} - E_{\mathrm{in}},\ \delta \boldsymbol{P} := \boldsymbol{P}_{\mathrm{out}} - \boldsymbol{P}_{\mathrm{in}})$



They are functions of
X_i, P_i, σ_i (i=0,1,2)

(SKIPPABLE) DETAILS

Form of Gaussian wave-packet S-matrix

☑ $$S_{V\to P\overline{P}} = ig_{\rm eff}N_V\left(\prod_{A=0}^{2}\frac{1}{\sqrt{2E_A}}\left(\frac{1}{\pi\sigma_A}\right)^{3/4}\right)e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$
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• Geometrical variables characterise *S*.

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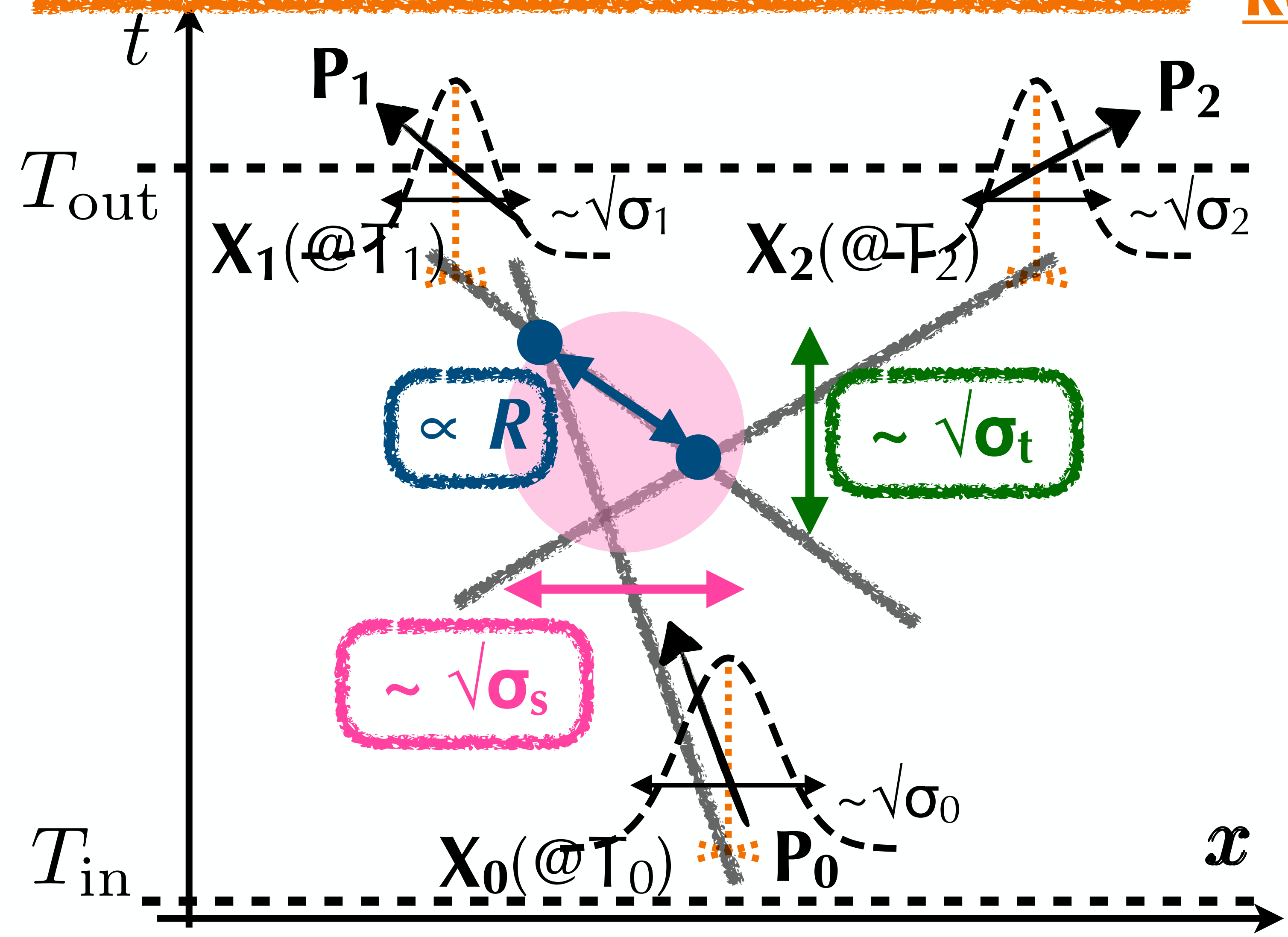

The limit ($\sigma_s\to\infty$ and $\sigma_t\to\infty$) ⇒

Recovery of the energy-momentum conservation

Note:

$$\left(\sqrt{\frac{\sigma}{2\pi}}e^{-\frac{\sigma}{2}(p-p_0)^2}\underbrace{\longrightarrow}_{\sigma\to\infty}\delta(p-p_0)\right)$$

(SKIPPABLE) DETAILS

Form of Gaussian wave-packet S-matrix

$$S_{V\to P\overline{P}} = i g_{\text{eff}} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A} \right)^{3/4} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

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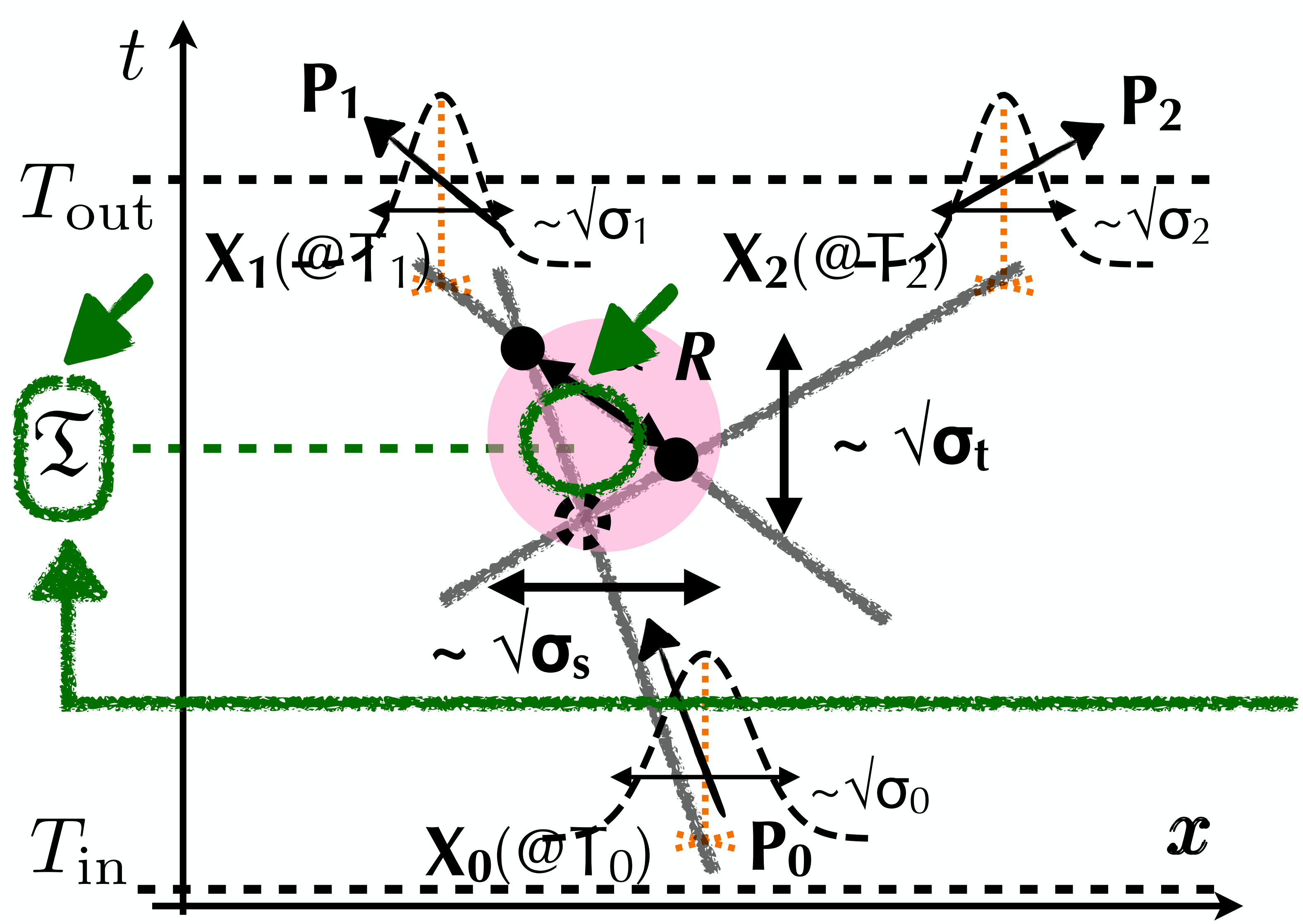
“window function”

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).



$$\mathfrak{T} := \sigma_t \frac{\overline{\boldsymbol{V}} \cdot \overline{\boldsymbol{\mathfrak{x}}} - \overline{\boldsymbol{V} \cdot \boldsymbol{\mathfrak{x}}}}{\sigma_s}$$

$$\boldsymbol{\mathfrak{x}}_A := \boldsymbol{\Xi}_A(0) \; (= \boldsymbol{X}_A - \boldsymbol{V}_A X_A^0)$$

determined by the trajectories
(configurations of
external particles)

Form of Gaussian wave-packet S-matrix

(SKIPPABLE) DETAILS

$$S_{V\to P\overline{P}} = i g_{\text{eff}} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A}\right)^{3/4}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$
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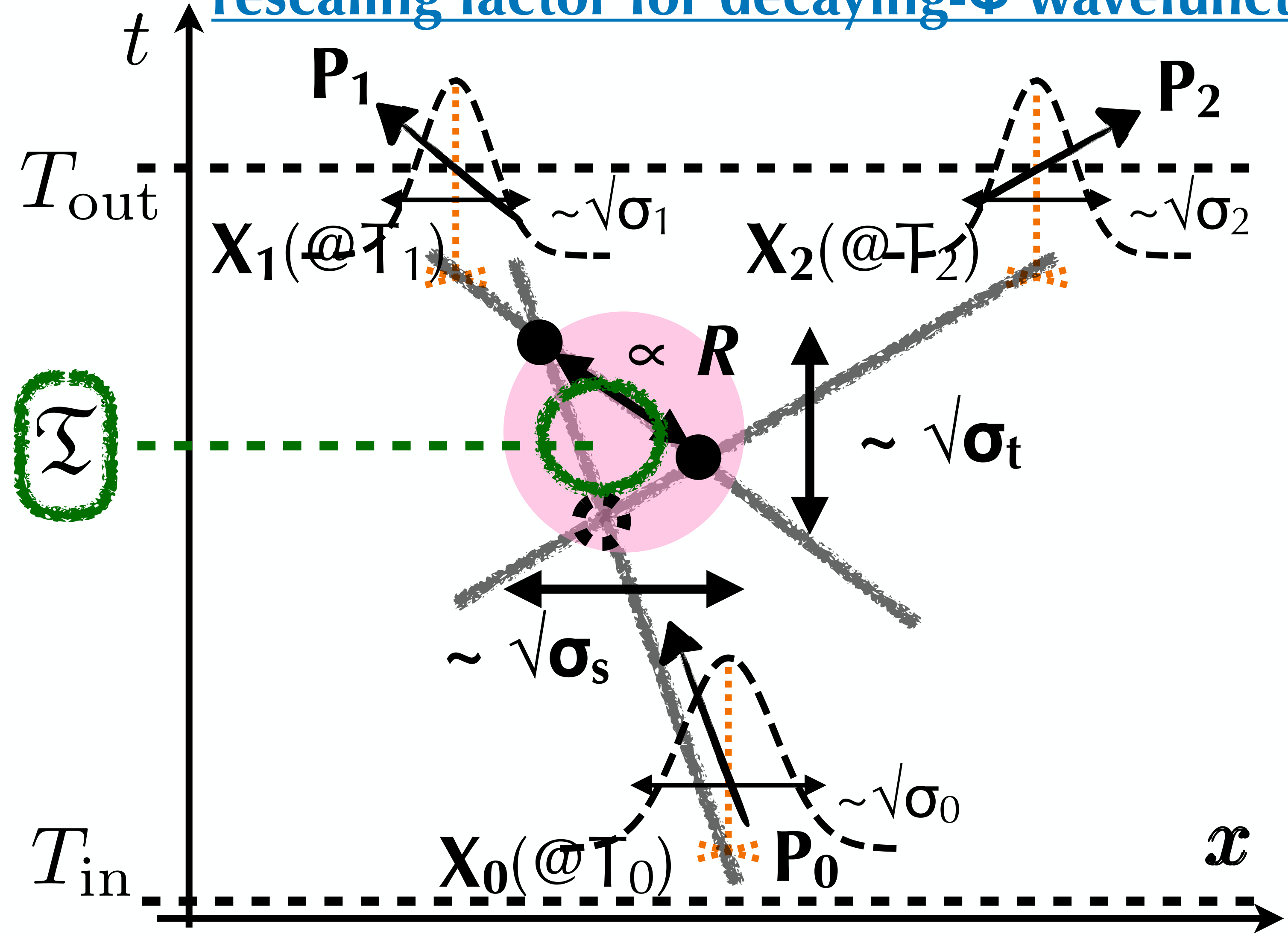
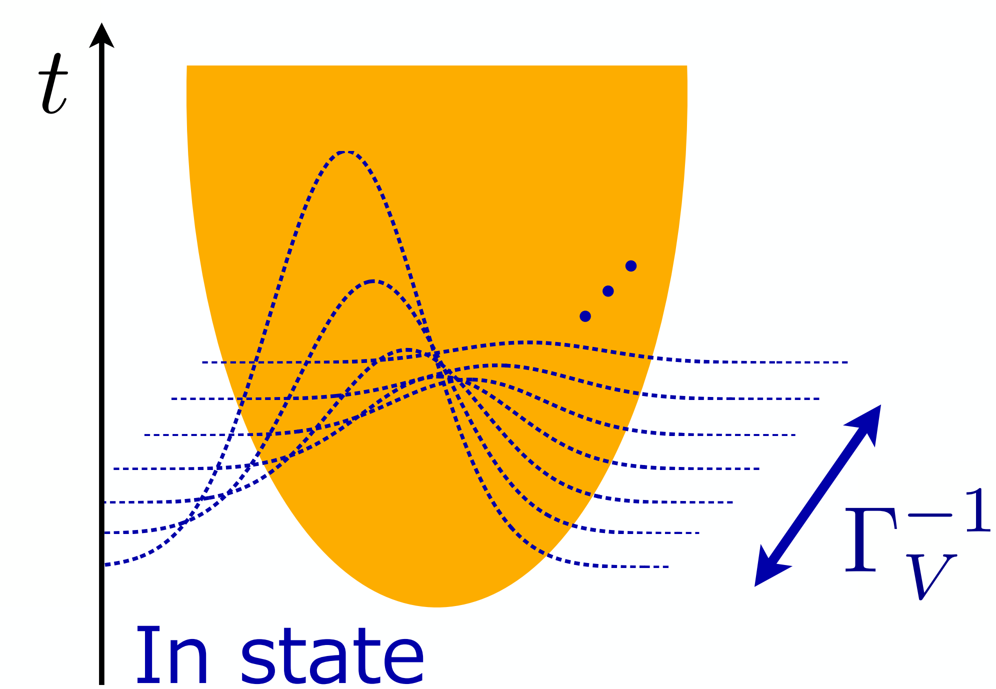
$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

"window function"

rescaling factor for decaying-Φ wavefunction



The off-shell-ness (decaying nature) of the initial state is taken into account (**Weisskopf-Wigner Approx.**).



Form of Gaussian wave-packet S-matrix

(SKIPPABLE) DETAILS

$$S_{V\to P\overline{P}} = i g_{\text{eff}} N_V \left(\prod_{A=0}^{2} \frac{1}{\sqrt{2E_A}} \left(\frac{1}{\pi\sigma_A}\right)^{3/4} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T}) \times e^{-\frac{\Gamma_V}{2}(\mathfrak{T} - T_0 + i\sigma_t\delta\omega) + \frac{\Gamma_V^2\sigma_t}{8}} \widetilde{F}(|\boldsymbol{V}_1 - \boldsymbol{V}_2|)$$

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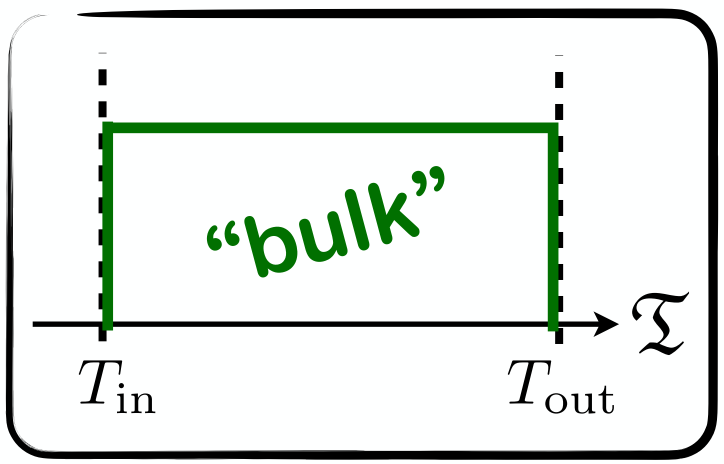
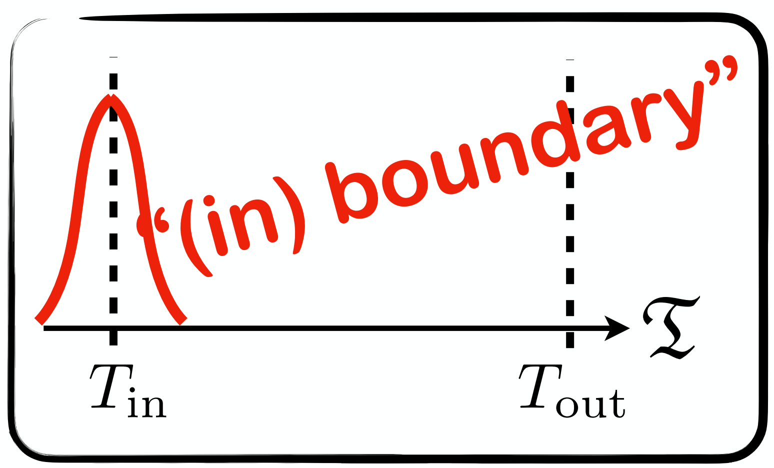
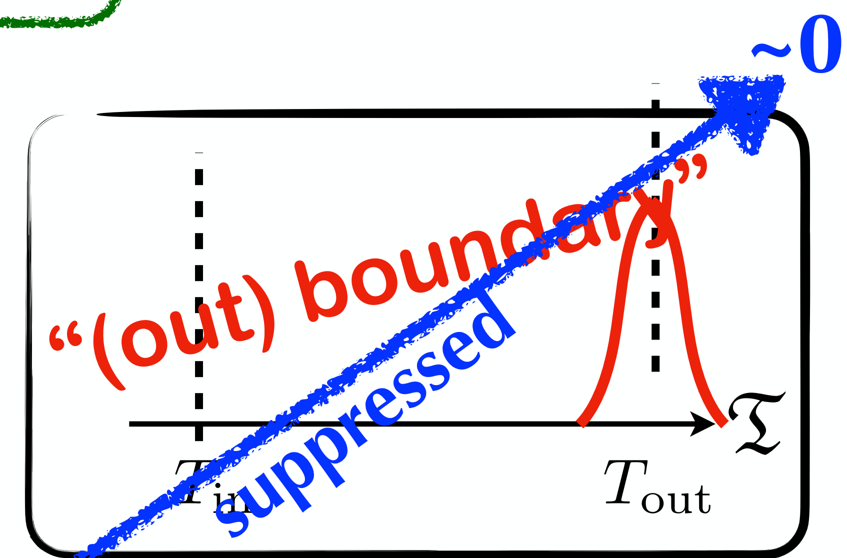
"window function"







(by taking $T_{\text{out}} \to \infty$)

$$G(\mathfrak{T}) \simeq W(\mathfrak{T}) - \frac{1}{2} e^{-\frac{\left(\mathfrak{T} - T_{\text{in}} - \frac{\Gamma_V\sigma_t}{2}\right)^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega\left(\mathfrak{T} - T_{\text{in}} - \frac{\Gamma_V\sigma_t}{2}\right)} \sqrt{\frac{2\sigma_t}{\pi}} \frac{1}{\mathfrak{T} - T_{\text{in}} - \frac{\Gamma_V\sigma_t}{2} + i\sigma_t\delta\omega}$$
$$+ \frac{1}{2} e^{-\frac{\left(\mathfrak{T} - \frac{\Gamma_V\sigma_t}{2} - T_{\text{out}}\right)^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega\left(\mathfrak{T} - \frac{\Gamma_V\sigma_t}{2} - T_{\text{out}}\right)} \sqrt{\frac{2\sigma_t}{\pi}} \frac{1}{\mathfrak{T} - T_{\text{out}} - \frac{\Gamma_V\sigma_t}{2} + i\sigma_t\delta\omega}$$

Table of Contents

Predictions for the Ratio

- For $\psi \to D^+D^-$ and $\psi \to D^0\overline{D^0}$

Overall normalisation does not contribute to the ratio R(ψ).

Parameters: $m_\psi,\ m_{D^+}\,(=m_{D^-}),\ m_{D^0}\,(=m_{\overline{D^0}}), \tau_\psi\,(=\Gamma_\psi^{-1});\ R_0\ \sigma_D\ \cancel{\sigma_\psi};\ \cancel{N_d}$

within saddle-point approx.

Experimental data are available.

wave-packet profile

Parameter of Form factor [Length]

Size of Wave Packet (final D & Dbar) [Length²]

Size of Wave Packet (initial ψ) [Length²]

Renormalisation of initial-ψ's wavefunction (due to decaying nature)

Note 1:

$$\left[\lambda_{\text{de-Brogile}} = \mathcal{O}\left(10^{-2}\,\text{MeV}^{-1}\right)\right]$$

$$\lambda_{\text{de-Brogile}} \lesssim \sqrt{\sigma_D}$$

Note 2: T_{in} provides overall effects (NOT to the ratio).

Predictions for the Ratio

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Overall normalisation
does not contribute to the ratio R(ψ).

Parameters: $m_\psi,\ m_{D^+}\,(=m_{D^-}),\ m_{D^0}\,(=m_{\overline{D^0}}), \tau_\psi\,(=\Gamma_\psi^{-1});\ R_0,\ \sigma_D,\ \cancel{\sigma_\psi};\ \cancel{N_d}$

Experimental data are available.

wave-packet profile

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Parameter of Form factor [Length]

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Note:

[Reminder]

- $$\mathrm{d}P_{V\to P\overline{P}} = \frac{\mathrm{d}^3\boldsymbol{X}_1\mathrm{d}^3\boldsymbol{P}_1}{(2\pi)^3}\frac{\mathrm{d}^3\boldsymbol{X}_2\mathrm{d}^3\boldsymbol{P}_2}{(2\pi)^3}\left|S_{V\to P\overline{P}}\right|^2$$

Non-relativistic approximations work fine.

- $$R_V^{\mathrm{WP}} := \frac{P_{V\to P^+P^-}}{P_{V\to P^0\overline{P^0}}}$$ **(the Ratio in terms of transition probability)**

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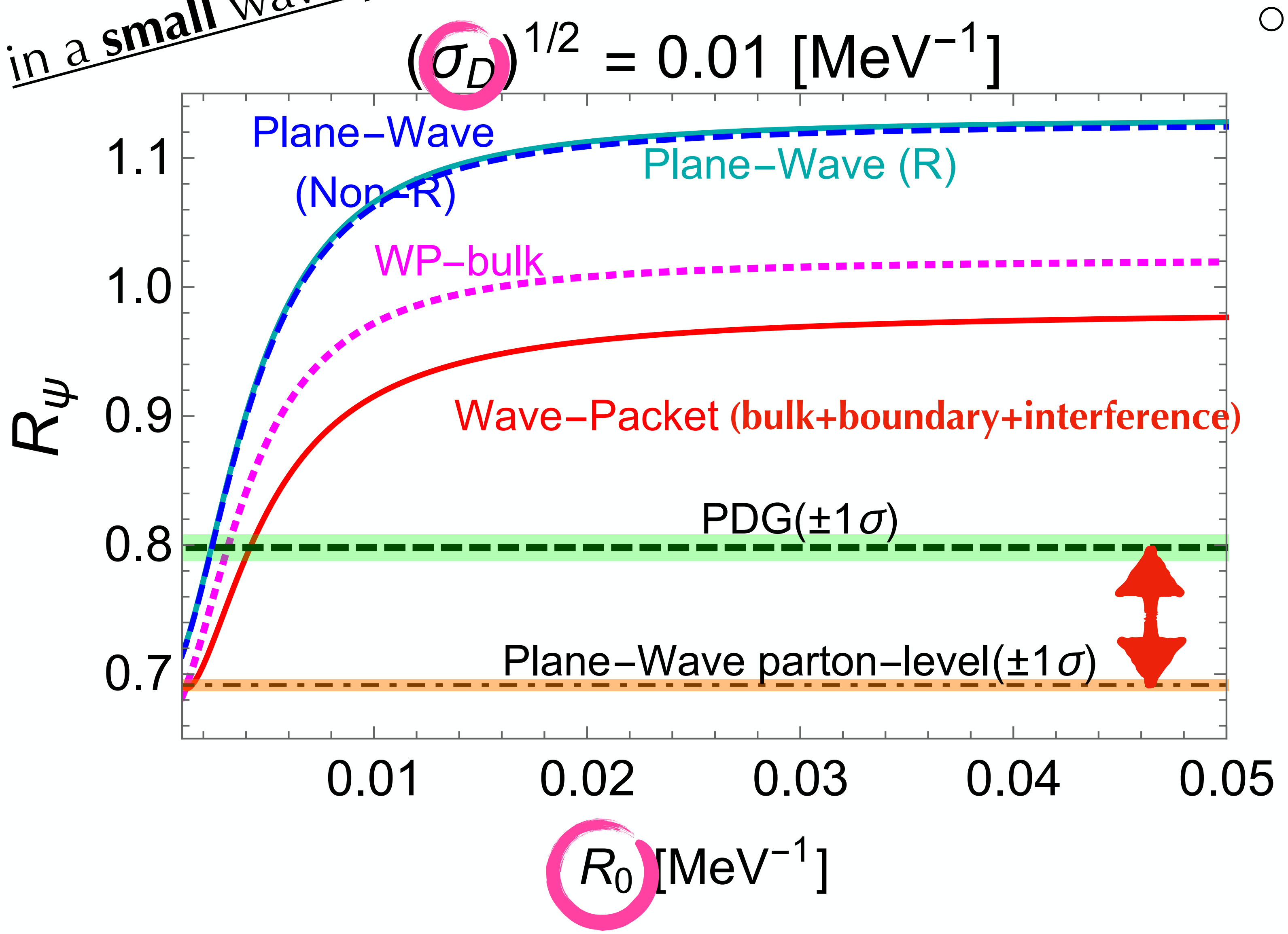
Experimental data are available. wave-packet profile

in a **small** wave packet



- **Parton-level (factored form factor) deviated from PDG (as reported).**

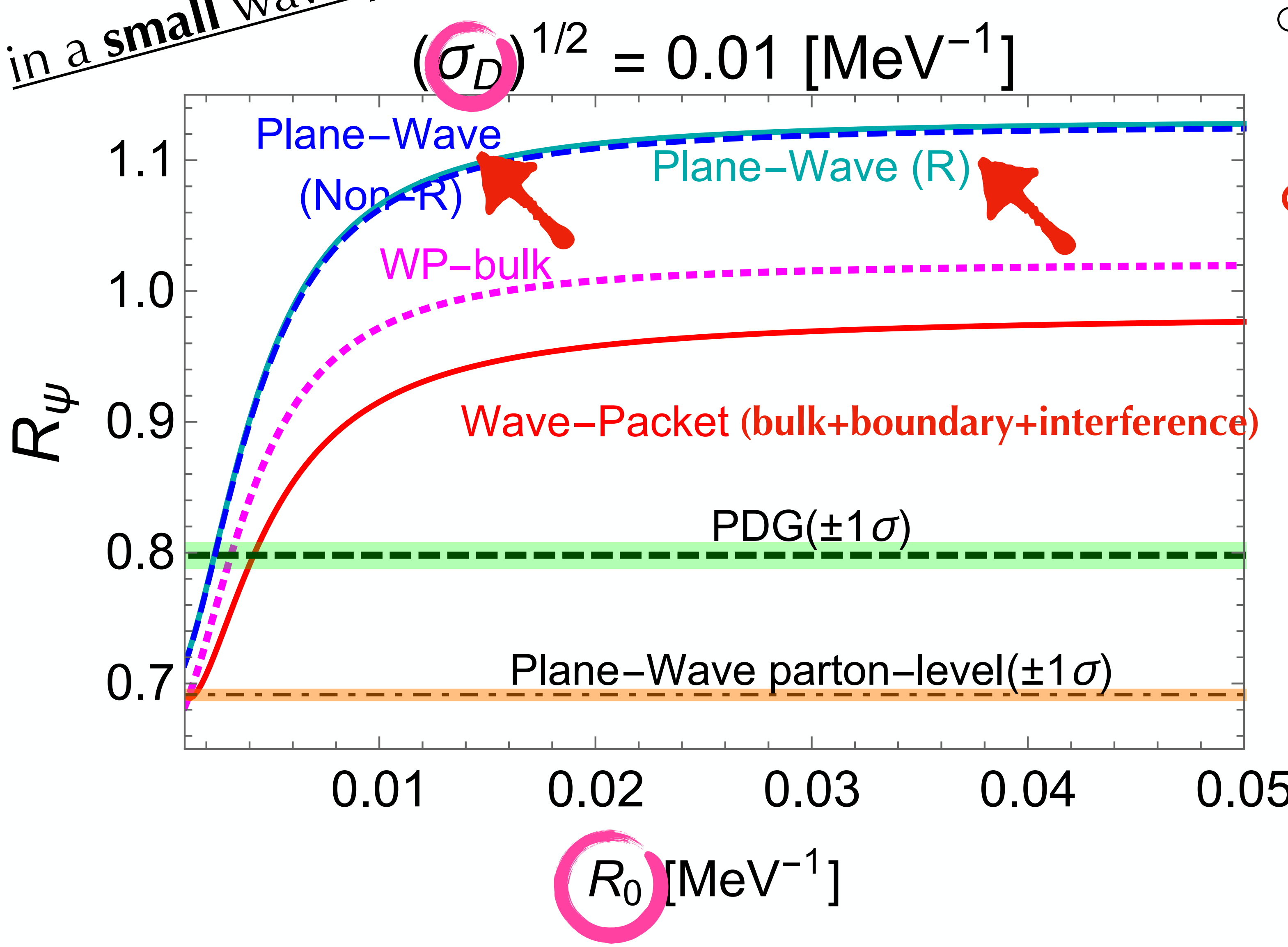
Predictions for the Ratio

- For $\psi \to D^+ D^-$ and $\psi \to D^0 \overline{D^0}$

Parameters: $m_\psi,\ m_{D^+}\ (= m_{D^-}),\ m_{D^0}\ (= m_{\overline{D^0}}), \tau_\psi\ (= \Gamma_\psi^{-1});\ R_0\ \sigma_D\ \cancel{\sigma_\psi};\ \cancel{N_d}$

Experimental data are available. wave-packet profile

in a **small** wave packet



- Parton-level (factored form factor) deviated from PDG (as reported).
- **The improved form factor provides the compositeness well. ⇒ good fits!**

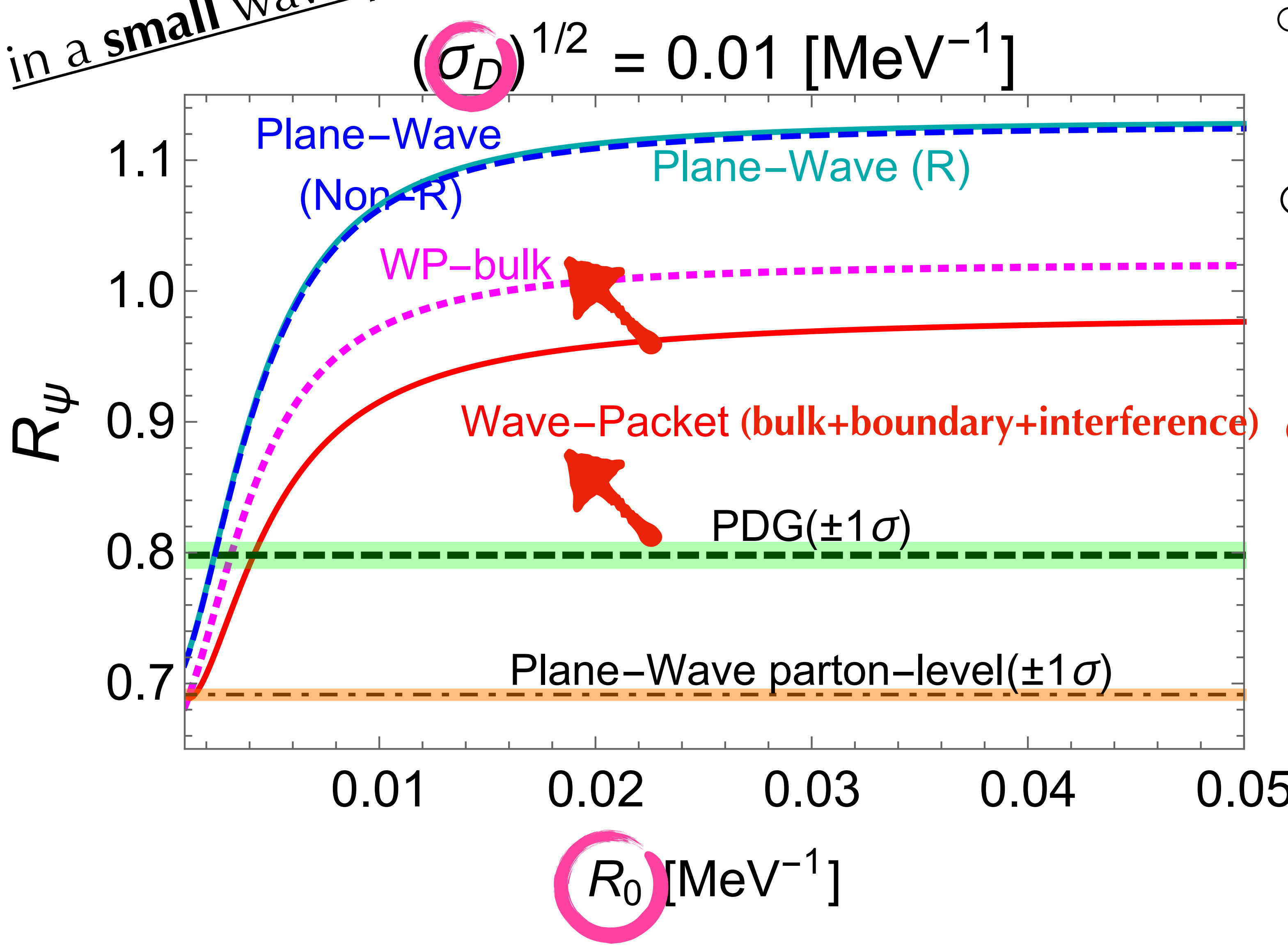
Predictions for the Ratio

- For $\psi \to D^+D^-$ and $\psi \to D^0\overline{D^0}$

Parameters: $m_\psi,\ m_{D^+}\ (= m_{D^-}),\ m_{D^0}\ (= m_{\overline{D^0}}), \tau_\psi\ (= \Gamma_\psi^{-1});\ R_0\ \sigma_D\ \cancel{\sigma_\psi}; \cancel{N_d}$

Experimental data are available. wave-packet profile

in a **small** wave packet



- Parton-level (factored form factor) deviated from PDG (as reported).
- The improved form factor provides the compositeness well. ⇒ good fits!
- **Wave-packet calculations explain the PDG result.**

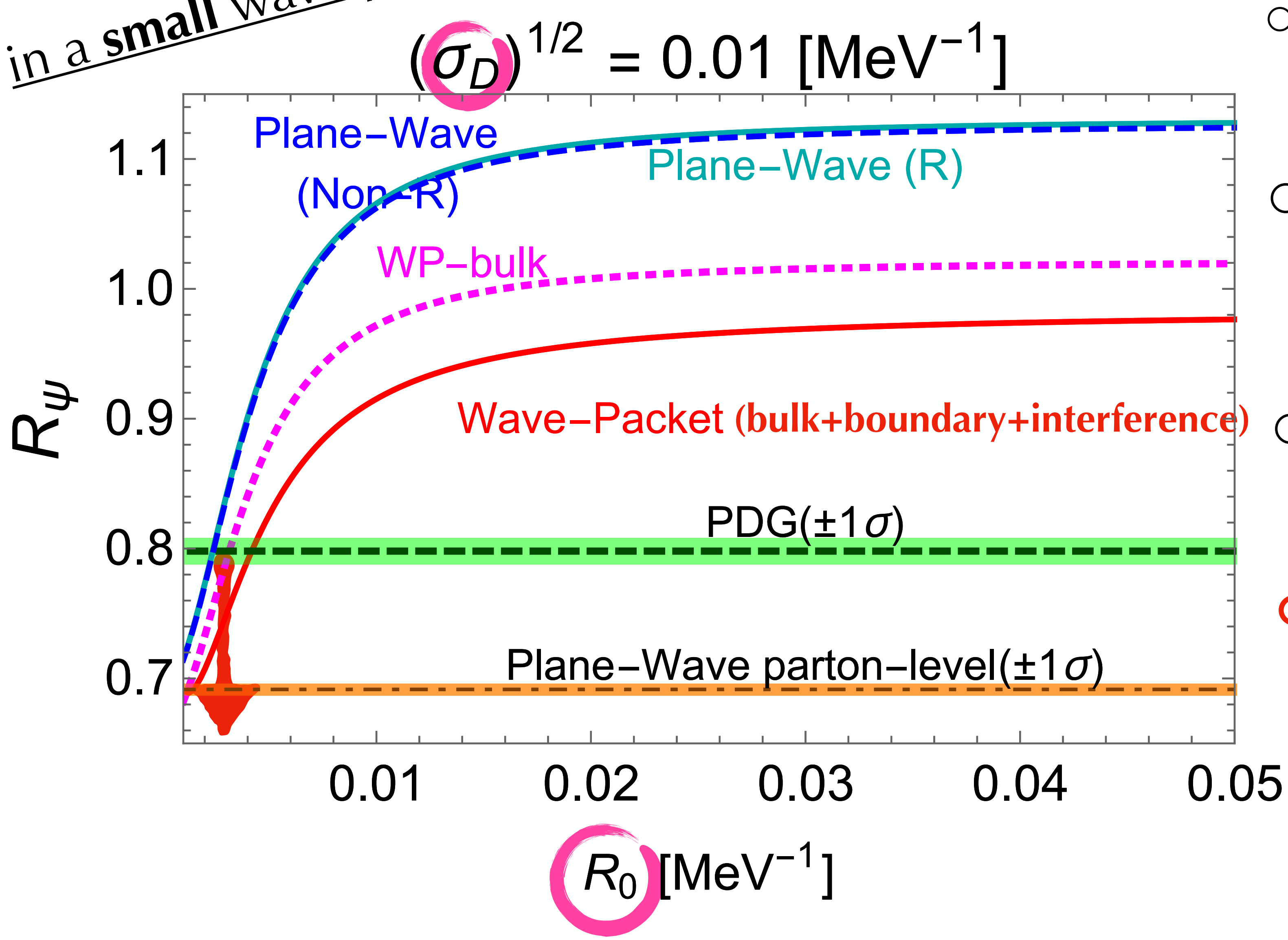
Predictions for the Ratio

- For $\psi \to D^+ D^-$ and $\psi \to D^0 \overline{D^0}$

Parameters: $m_\psi,\ m_{D^+}\,(= m_{D^-}),\ m_{D^0}\,(= m_{\overline{D^0}}), \tau_\psi\,(= \Gamma_\psi^{-1});\ R_0\ \sigma_D;\ \cancel{\sigma_\psi};\ \cancel{N_d}$

Experimental data are available. wave-packet profile

in a **small** wave packet



- Parton-level (factored form factor) deviated from PDG (as reported).
- The improved form factor provides the compositeness well. ⇒ good fits!
- Wave-packet calculations explain the PDG result.
- **$R_0 \sim O(10^{-2})$ MeV^{-1} ~ (QCD scale)$^{-1}$ is a reasonable choice.**

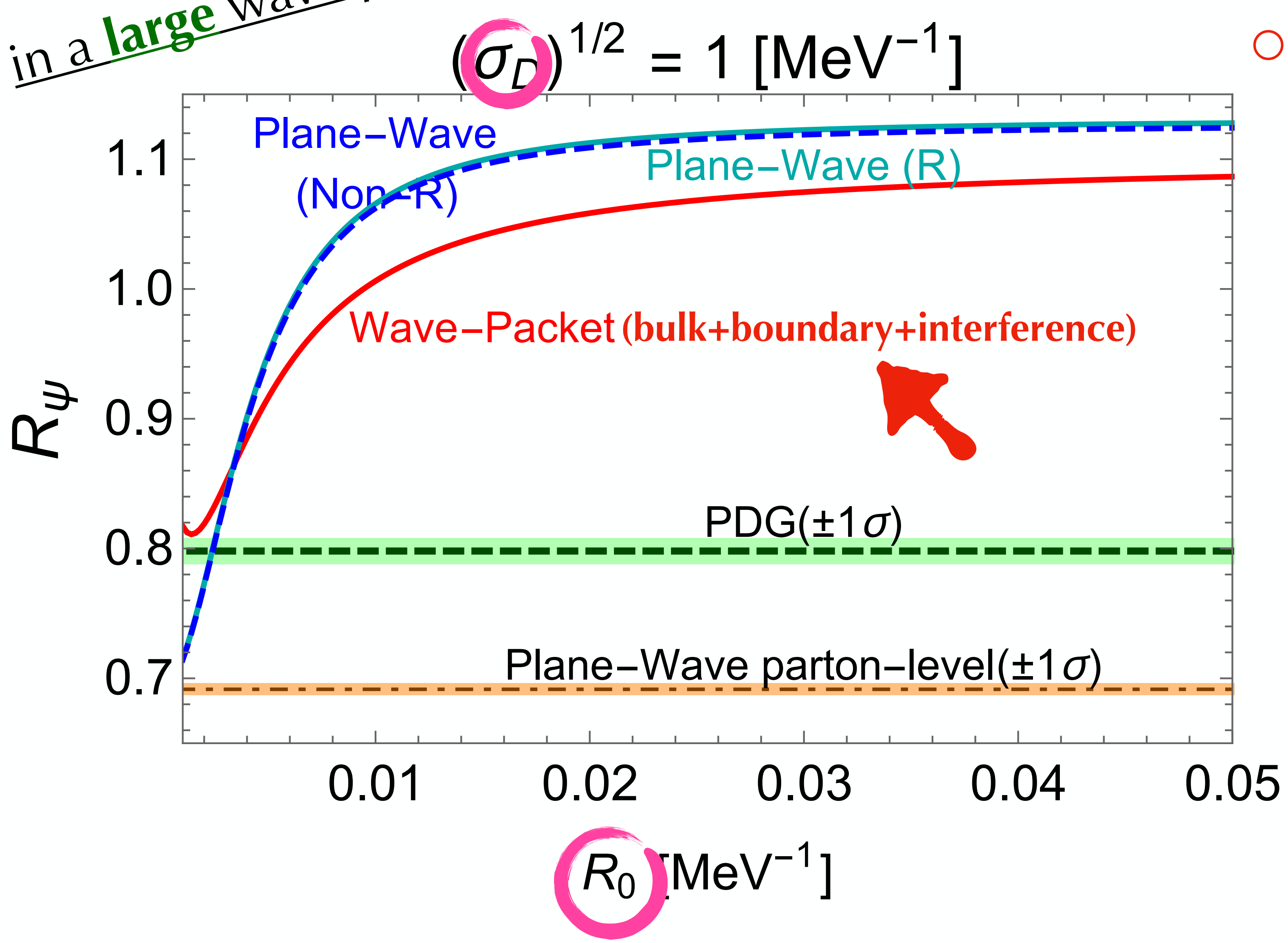
Predictions for the Ratio

- For $\psi \to D^+D^-$ and $\psi \to D^0\overline{D^0}$

Parameters: $m_\psi,\ m_{D^+}\,(=m_{D^-}),\ m_{D^0}\,(=m_{\overline{D^0}}), \tau_\psi\,(=\Gamma_\psi^{-1});\ R_0\ \sigma_D;\ \cancel{\sigma_\psi};\ \cancel{N_d}$

Experimental data are available. wave-packet profile

in a **large** wave packet



○ Bulk-only wave-packet result is far away from PDG (outside the shown region), also does not make sense.
⇒ **Boundary part should be taken.**

Predictions for the Ratio

- For $\psi \to D^+D^-$ and $\psi \to D^0\overline{D^0}$

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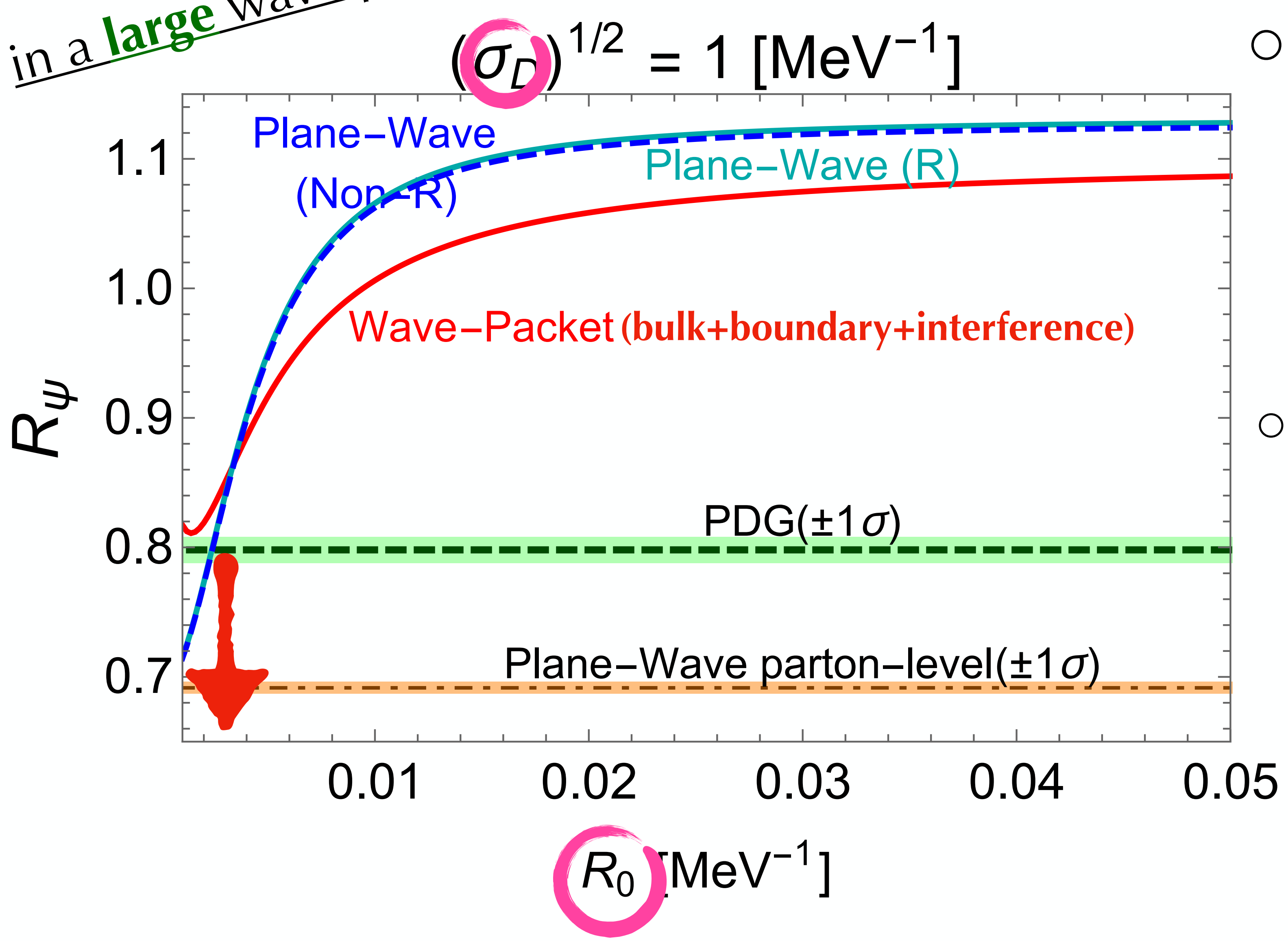
Experimental data are available. wave-packet profile

in a **large** wave packet

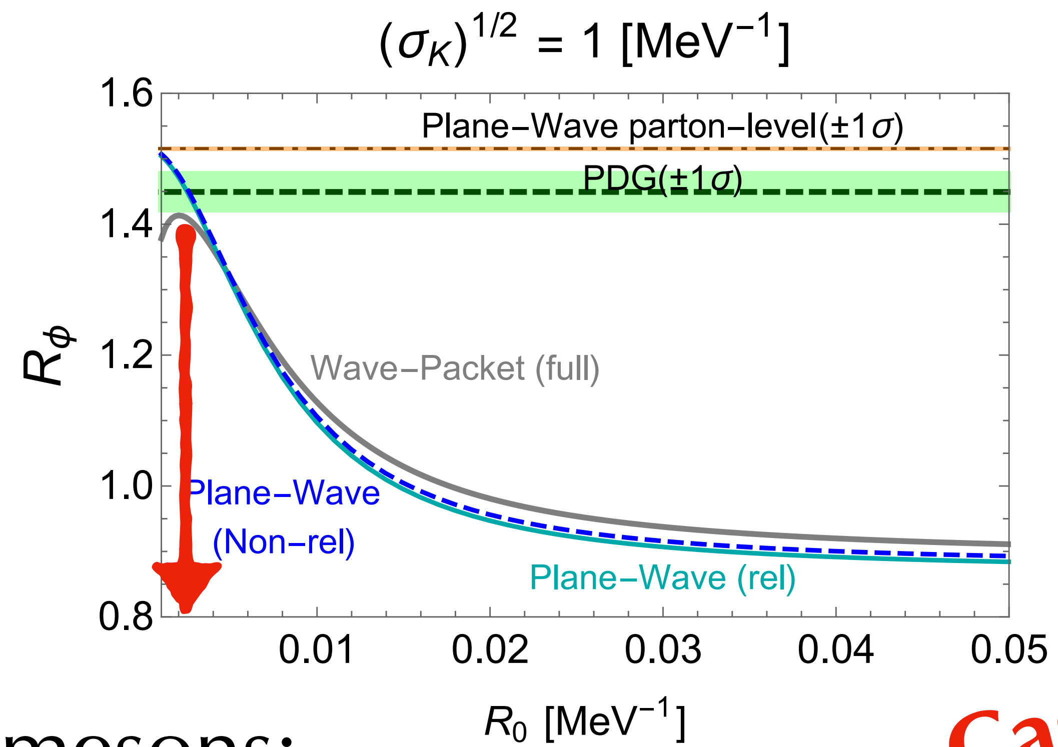


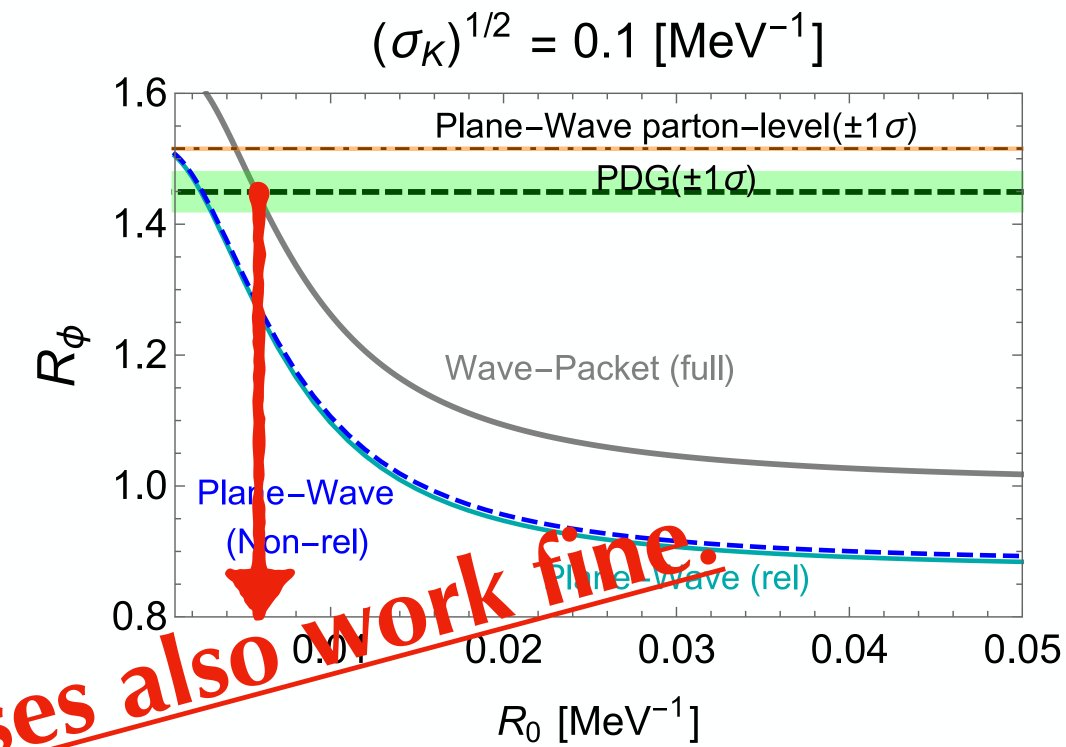
- Bulk-only wave-packet result is far away from PDG (outside the shown region), also does not make sense.
  ⇒ **Boundary part should be taken.**

- **R_0 for explaining PDG is similar for smaller and bigger wave packets.**

Predictions for the Ratio

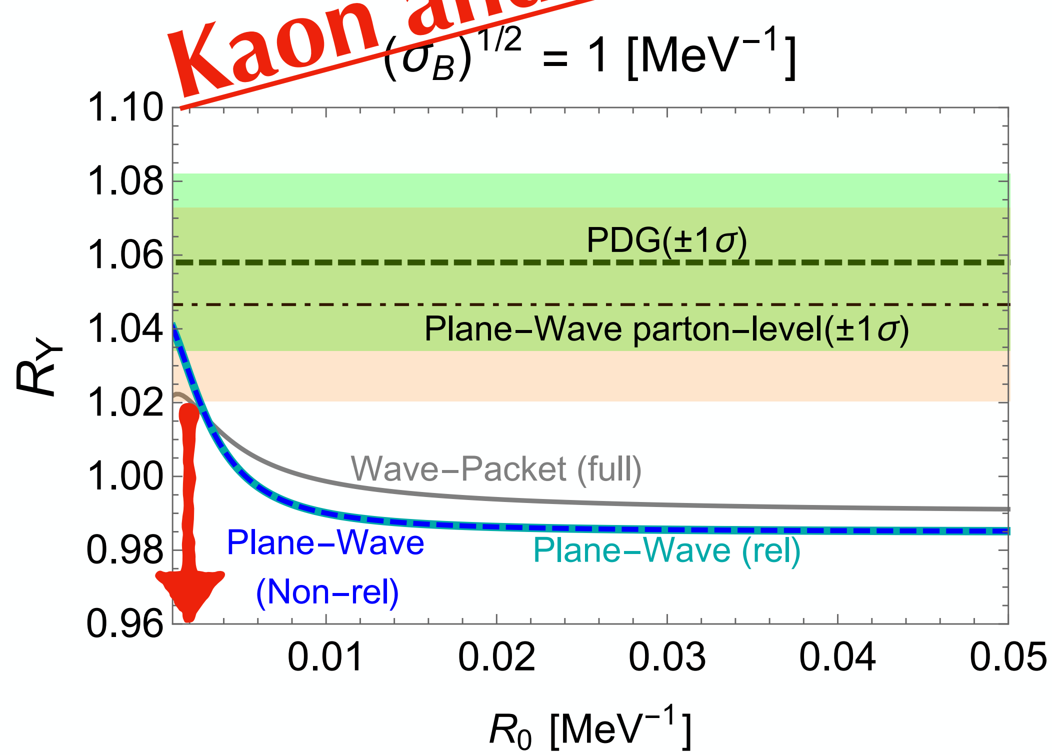
- For Kaons:

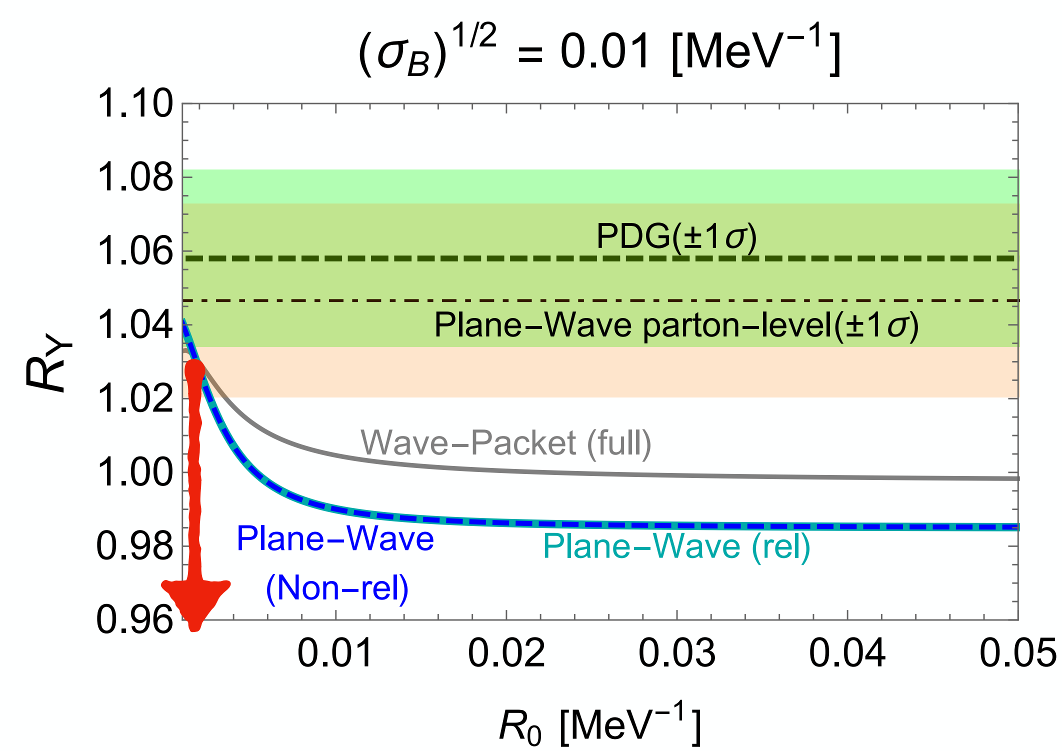




- For B-mesons:

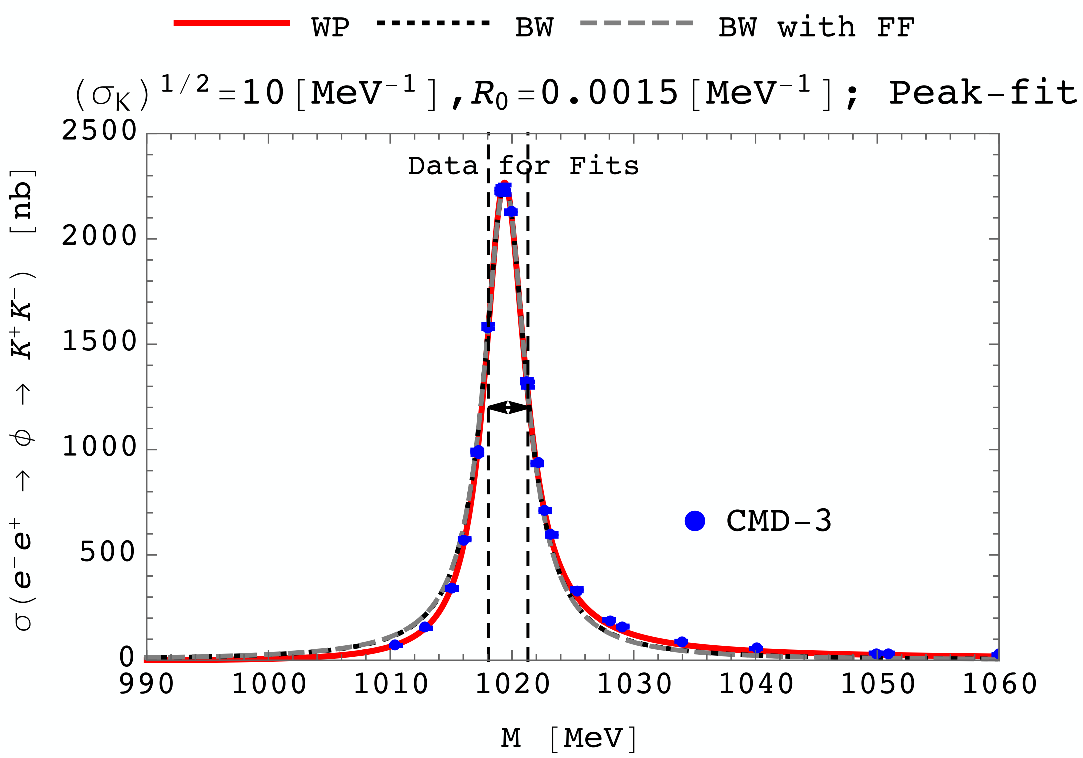
Kaon and B-meson Cases also work fine.

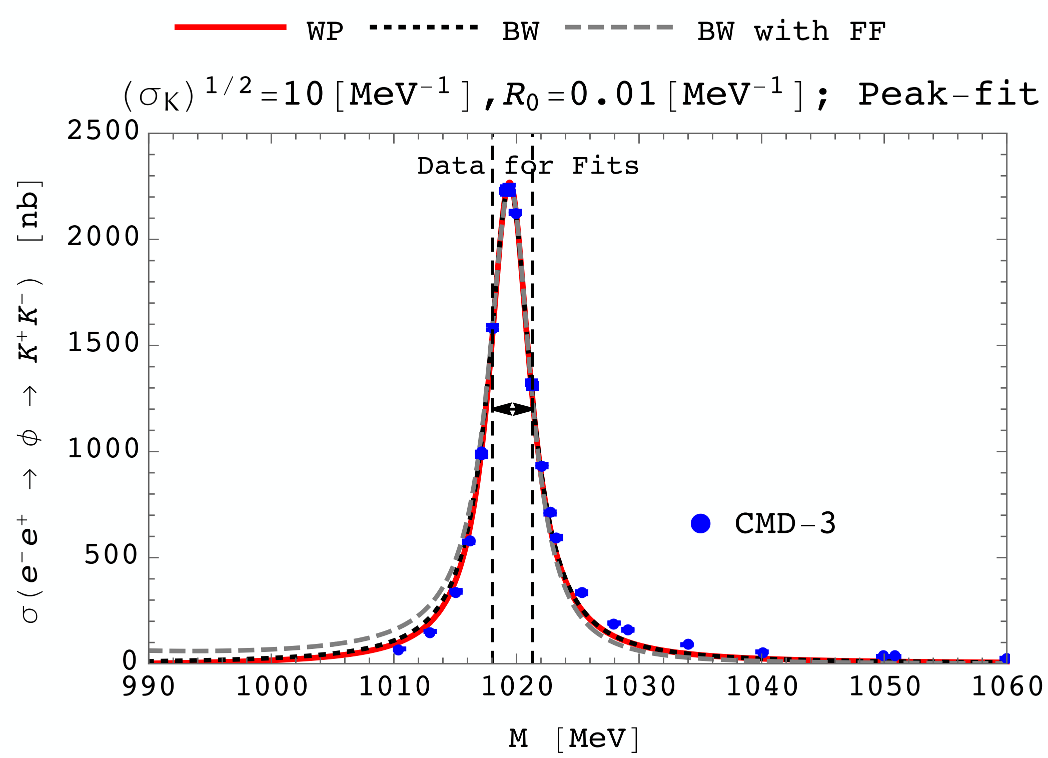


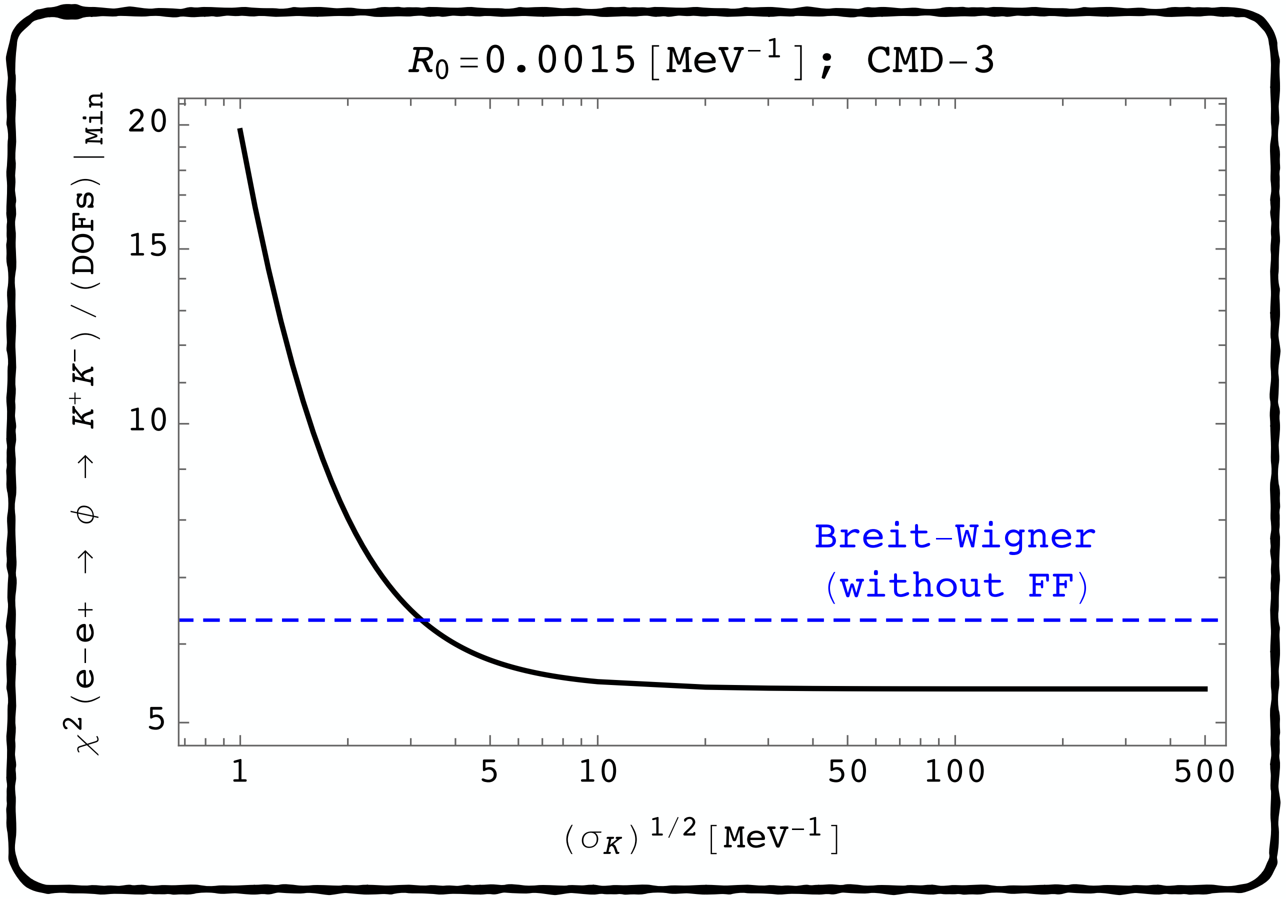


Constraint via Resonant shape

- via $e^- e^+ \to \phi \to K^+ K^-$



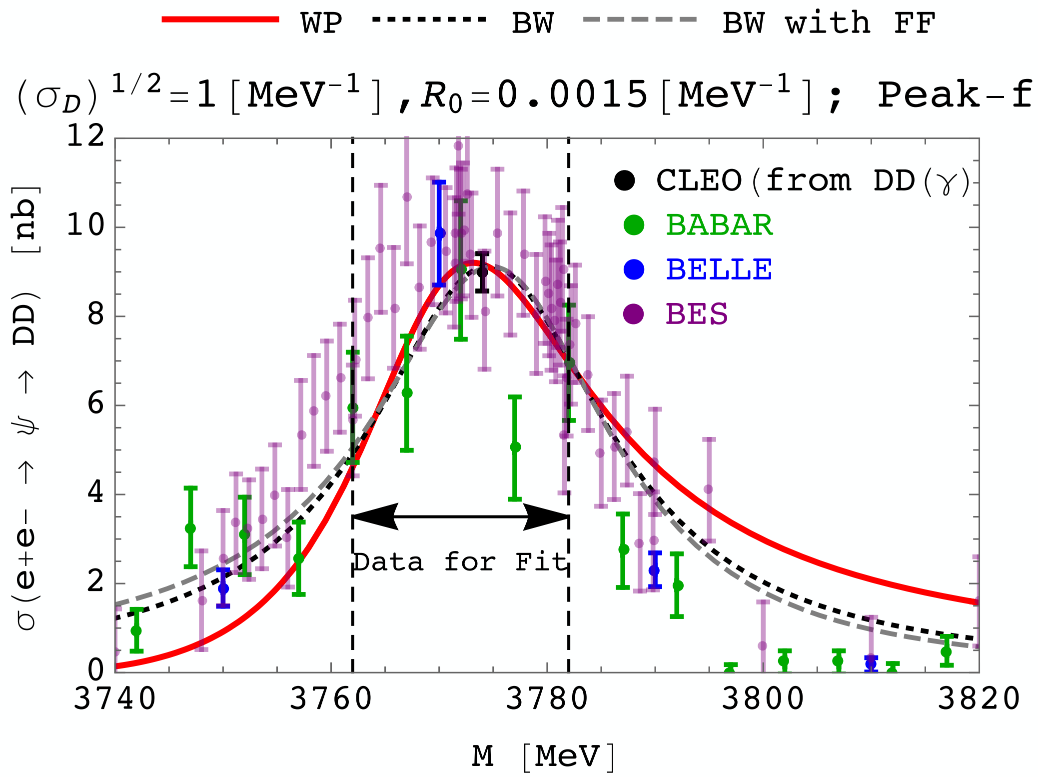


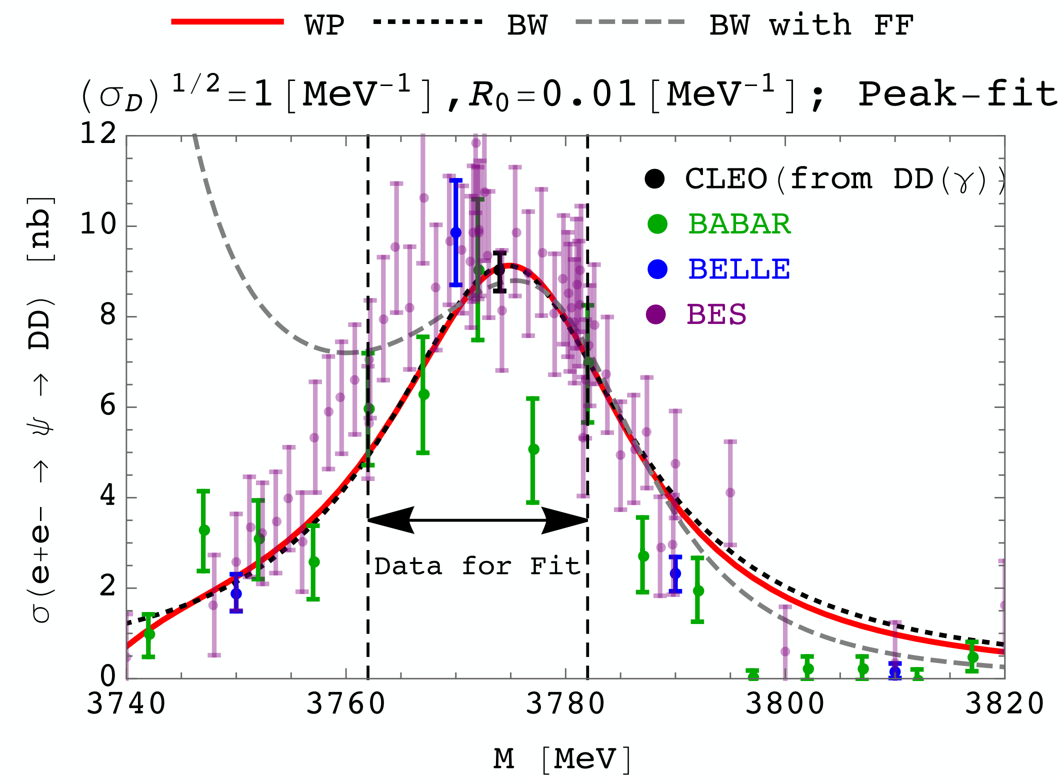


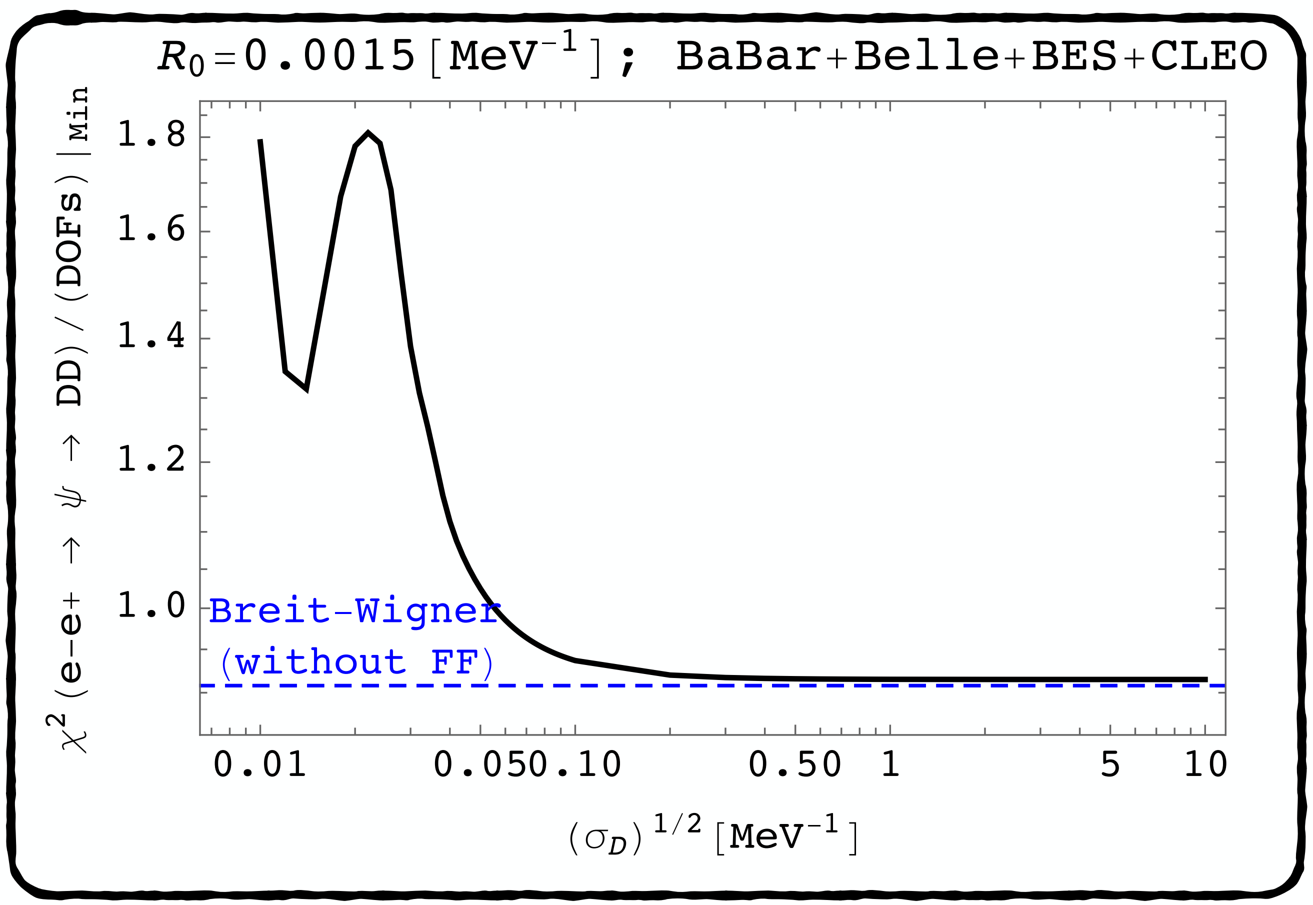
Note: $\sigma(e^-e^+ \to \Phi)$ and m_Φ are determined by statistical fits.

Constraint via Resonant shape

- via $e^- e^+ \to \psi \to 2D\,(D^+ D^- \text{ and } D^0 \overline{D^0})$







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Summary & Discussion

1. The S-matrix in Gaussian wave packet contains **full information** of the **quantum particles. ➜ More informative** & **regularised**.
2. Characterising S-matrix, in particular, **“bulk”** and **“boundary”**.
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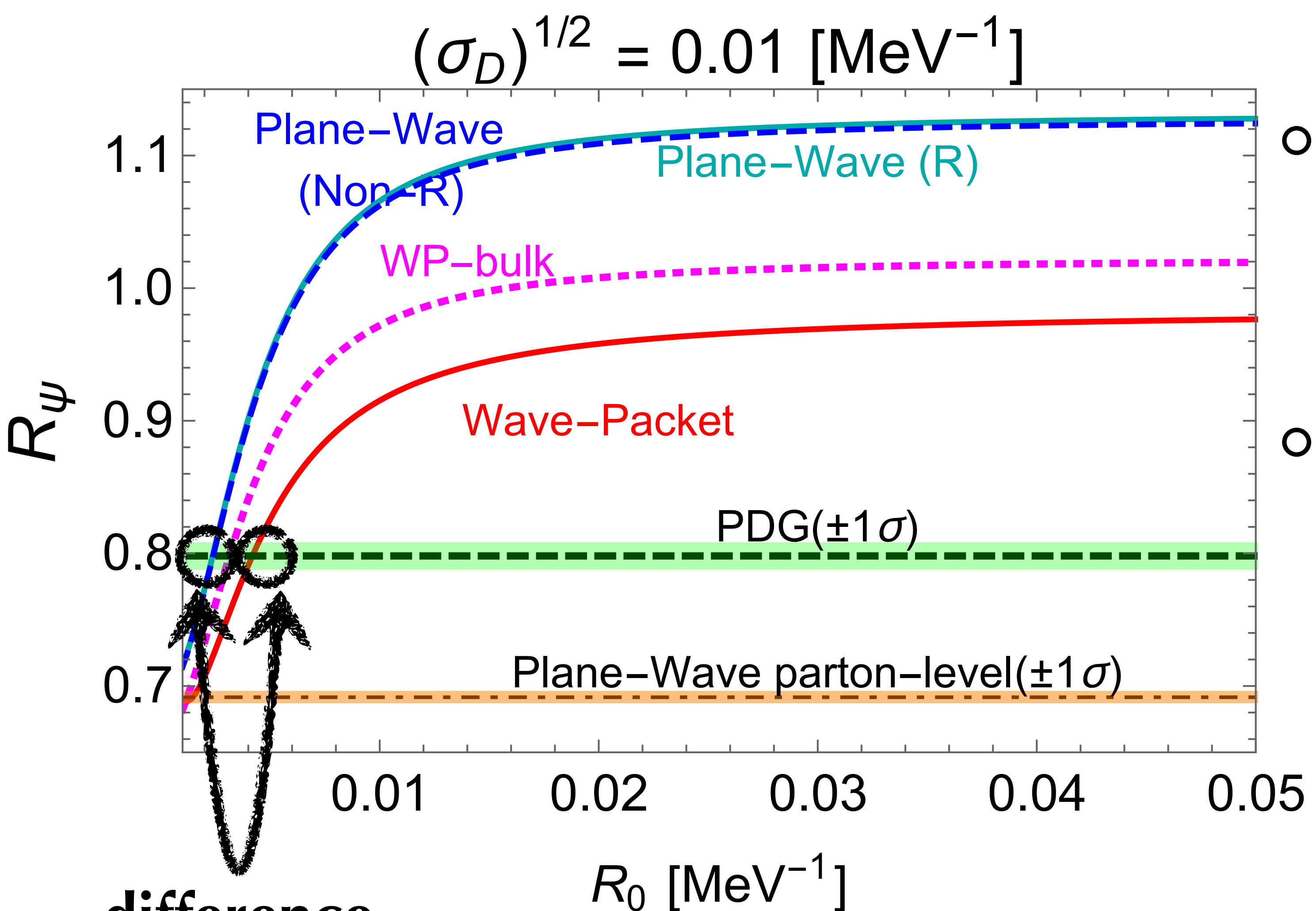

difference

- Considering the form factor appropriately, ➜ R_0 around the QCD scale gives us good fits.
- Can we **distinguish** the “wave-packet” correction from the plane-wave part (in this variable or others)?
➜ Further discussion is necessary.

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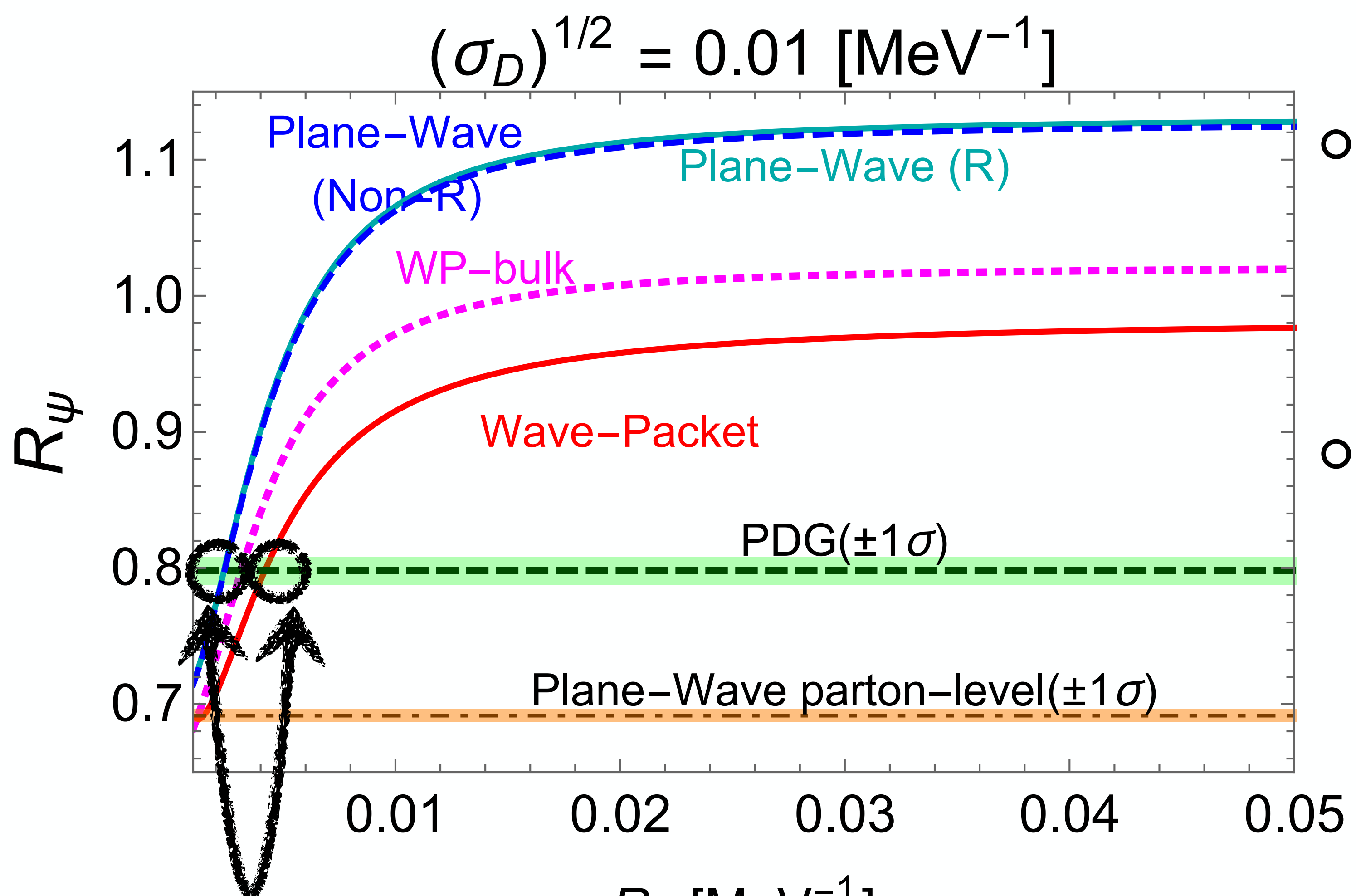

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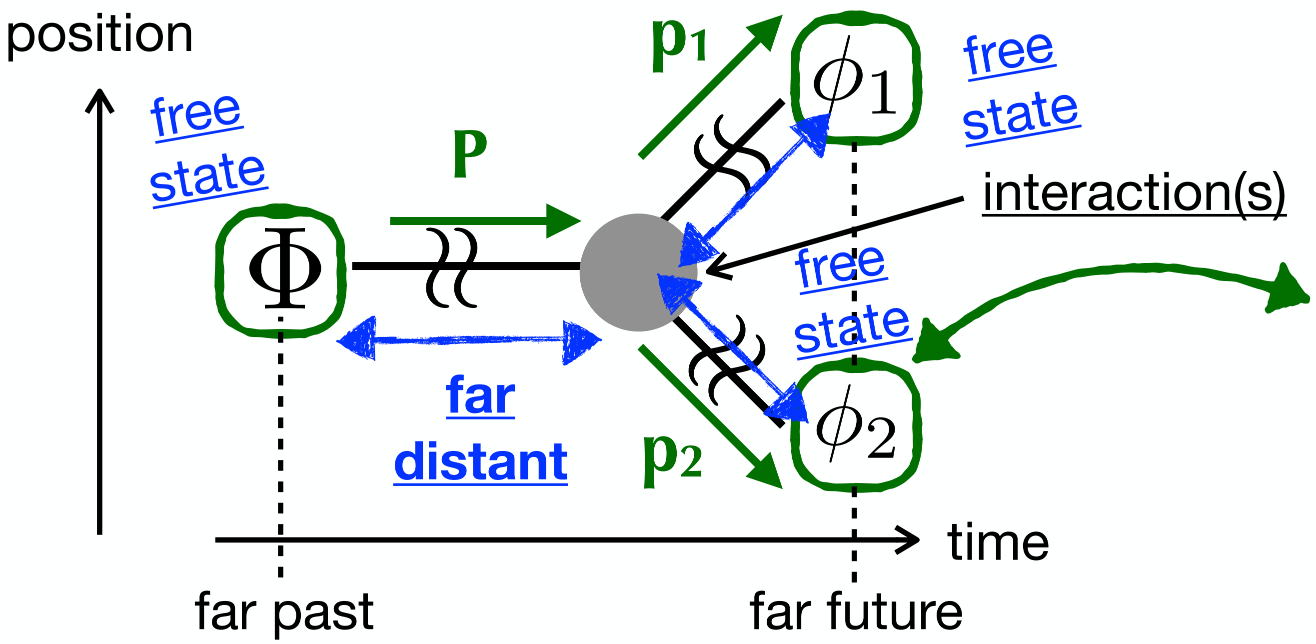
Backup Slides

Review on plane-wave amplitude

- We focus on the (1→2)-body relativistic transition/decay: $\Phi \to \phi_1\phi_2$



momentum eigenstates

(external free: also mass eigenstates; $E_i^2 = \mathbf{p}_i^2 + m_i^2$)

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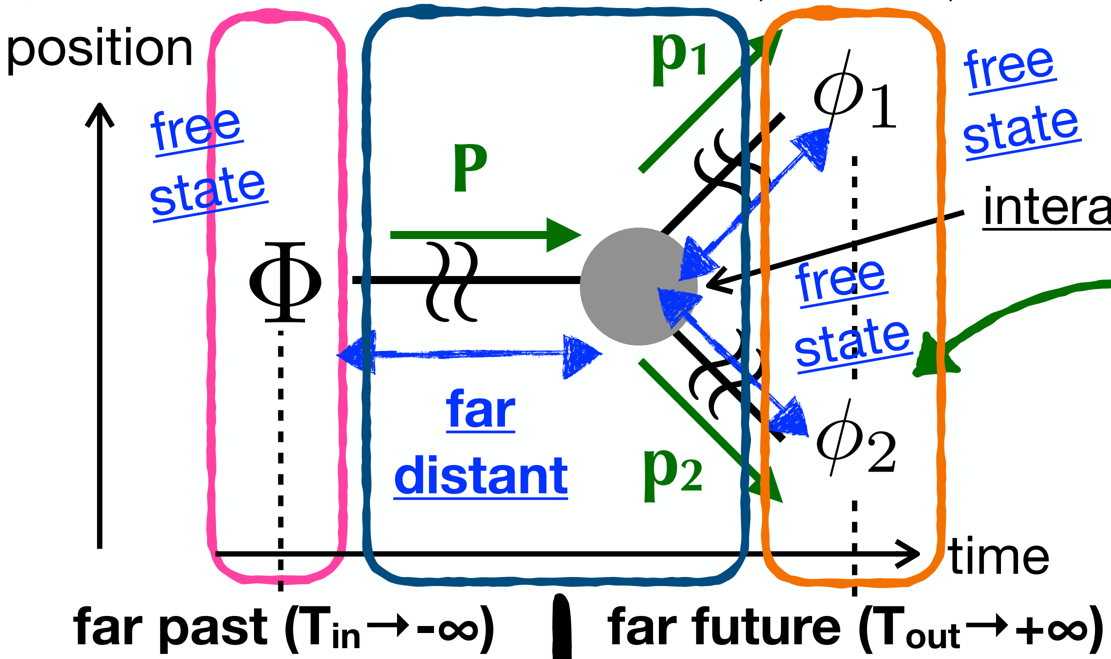

momentum eigenstates
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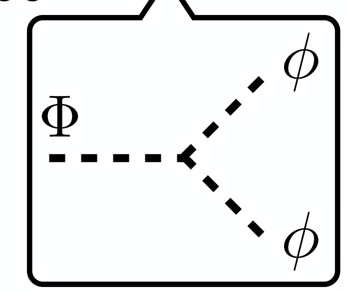
transition amplitude
(S-matrix)

(in interaction picture) interaction Hamiltonian density

○ $\widehat{H}^{(I)}_{\rm int} = \int_{-\infty}^{\infty} d^3\boldsymbol{x}\ \widehat{\mathcal{H}}^{(I)}_{\rm int}(t,\boldsymbol{x})$



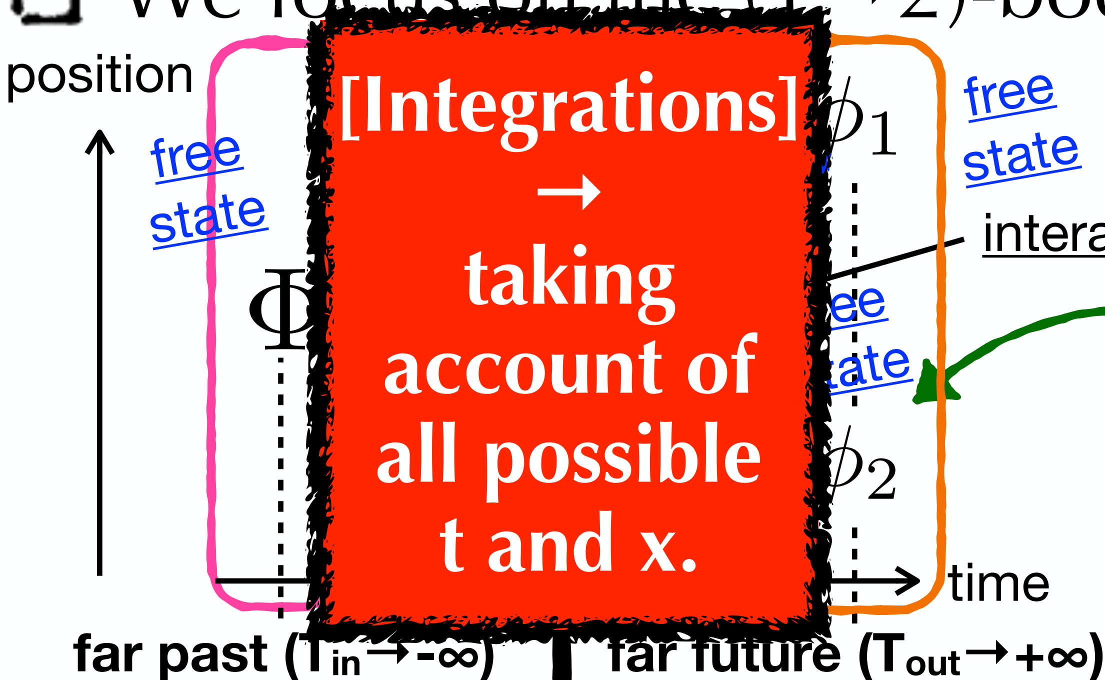
$$S_{\rm PW} = \underset{\text{out free state}}{\langle \boldsymbol{p}_1, \boldsymbol{p}_2|} \mathrm{T}e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \underset{\text{in free state}}{|\boldsymbol{P}_0\rangle}$$

$$= \underbrace{(2\pi)^4\delta^4(P_{\rm out} - P_{\rm in})}_{\text{manifest energy-momentum conservation (due to translation invariance)}} \times \underbrace{(iM_{\rm PW})}_{\text{(factorised) amplitude}}$$

○ ('T' represents the time-ordered product for relativistic process.)

Review on plane-wave amplitude

- We focus on the (1→2)-body relativistic transition/decay: $\Phi \to \phi_1\phi_2$



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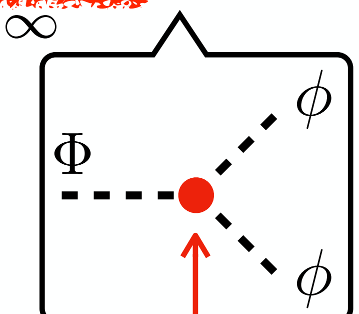

@ (t,**x**)

$$S_{\rm PW} = \underset{\text{out free state}}{\langle \boldsymbol{p}_1, \boldsymbol{p}_2|}\ \mathrm{T}e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)}\ \underset{\text{in free state}}{|\boldsymbol{P}_0\rangle}$$

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plane-wave basis

[QFT textbooks]

Plane wave — the **standard tool** for describing **particles**:

- Basis (@ Schrödinger Pic.): $\boxed{\langle \boldsymbol{x} | \boldsymbol{p} \rangle \propto e^{i\,\boldsymbol{p}\cdot\boldsymbol{x}}}$

  (plane wave: the **eigenstate** of $\boldsymbol{p}$) $\longleftrightarrow$ $\boldsymbol{x}$ **completely undetermined** (**non-normalisable** mode)

- Expansion of Scalar operator (in Int. Pic.):

  - $$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3(2E_{\boldsymbol{p}})}} \left[\boxed{e^{+ip\cdot x}}\, \boxed{\hat{a}_{\boldsymbol{p}}} + \text{h.c.} \right] \qquad \left(E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m_\phi^2} \right)$$

    Wave function of plane wave

    Annihilation op. for momentum-**p** state

  - $$|\boldsymbol{p}\rangle = \hat{a}^\dagger_{\boldsymbol{p}} |0\rangle$$

    the one-particle state

    (ignoring the overall factor e^{-iEt})

$$\langle x | \boldsymbol{p} \rangle \propto e^{i\,p\cdot x} \Big|_{p^0 = E_{\boldsymbol{p}}}$$

- $x = \big(x^0 \,(= t),\, \boldsymbol{x} \big)$ — 4d position
- $\langle x | = \langle \boldsymbol{x} | e^{-i\hat{H}_{\text{free}} t}$ — $\langle x |$: Int. Pic.; $\langle \boldsymbol{x} |$: Sch. Pic.

Gaussian basis

[Ishikawa, Shimomura (0508303), Ishikawa, Oda (1809.04285)]

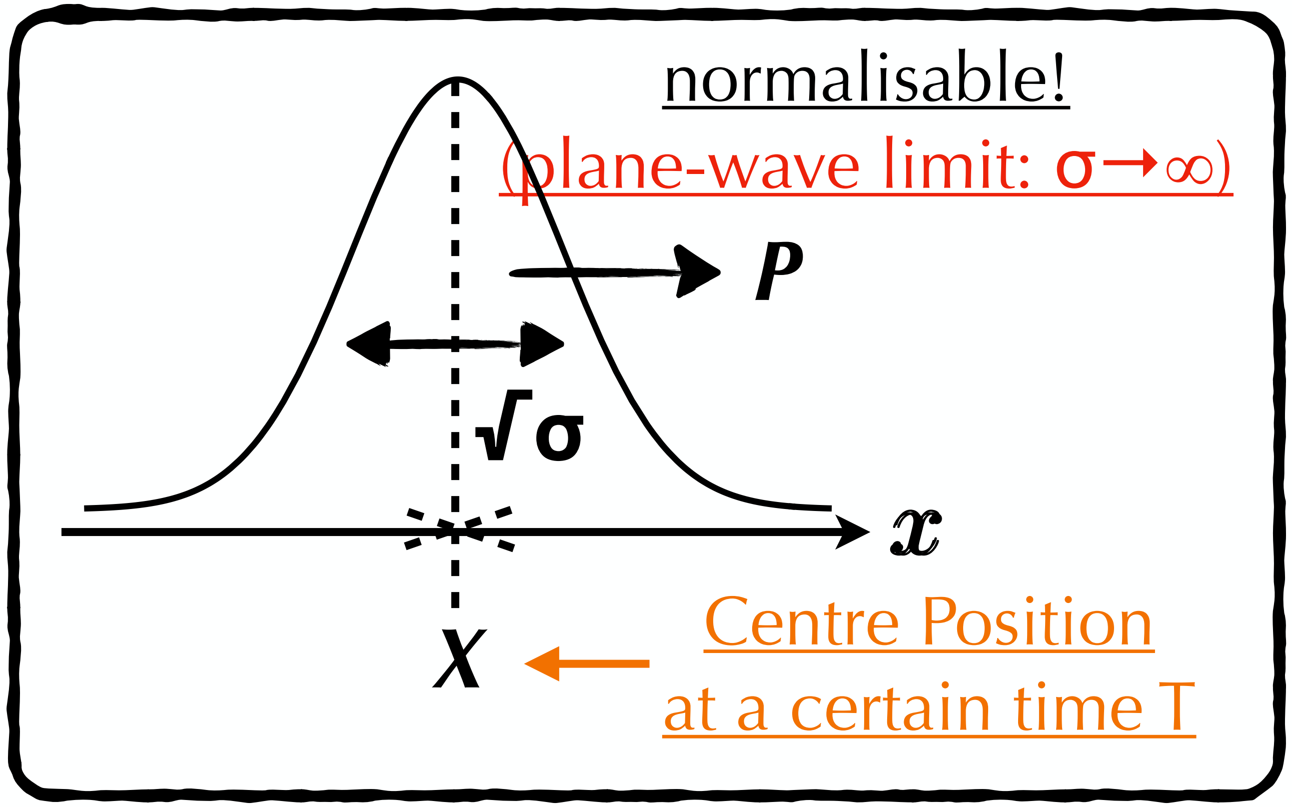
- **Key**: Fields can be expanded in any complete sets of bases.
  → Perturbations under **normalised** bases are possible. ➜ **Gaussian!**

- Gaussian basis $\boxed{\langle \boldsymbol{x} | \sigma, \boldsymbol{X}, \boldsymbol{P} \rangle}$

  Form (@ Schrödinger Pic.):

$$\boxed{\simeq e^{i\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{X}) - \frac{(\boldsymbol{x}-\boldsymbol{X})^2}{2\sigma}}}$$

(a coherent state) (when T=0)



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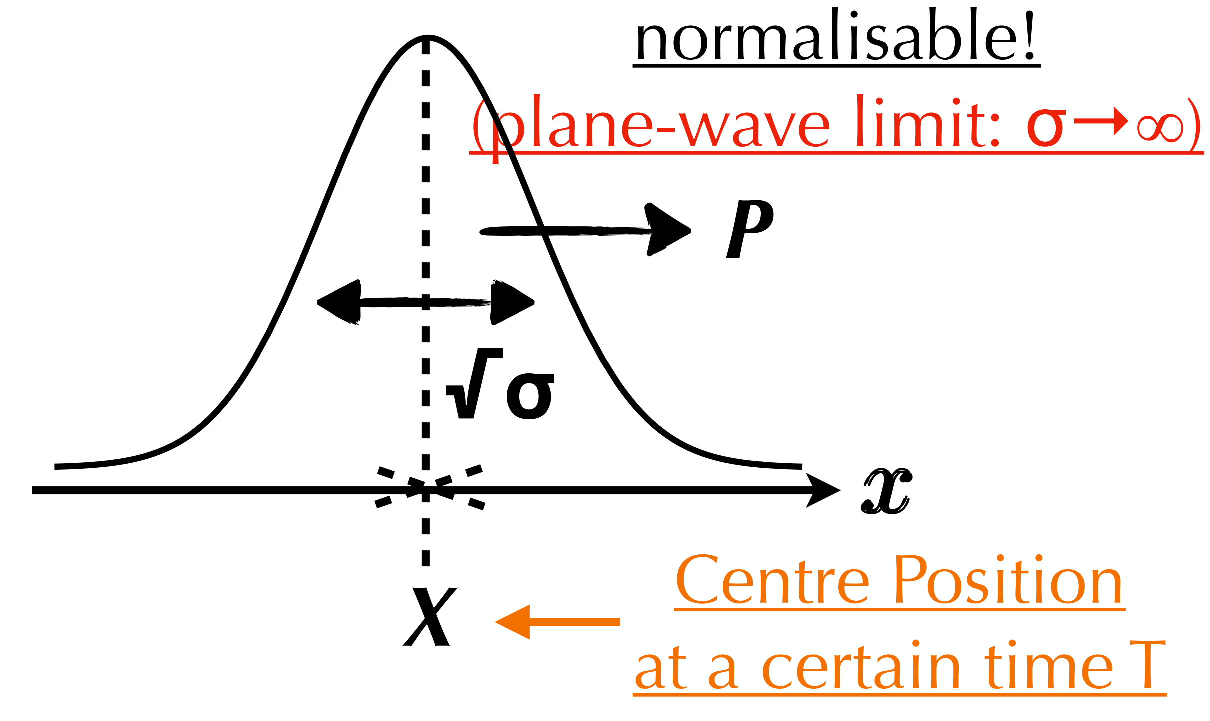


- **Expansion of Scalar operator (in Int. Pic.):**

$$\circ\ \hat{\phi}(x) = \int \frac{\mathrm{d}^3\boldsymbol{X}\,\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3} \left[\boxed{f_{\sigma,X,\boldsymbol{P}}(x)}\, \boxed{\hat{A}(\sigma, X, \boldsymbol{P})} + \text{h.c.} \right]$$

Wave function of Gaussian wave packet

(**X** is defined @ T)

Annihilation op.
for the corresponding wave-packet state

$$\circ\ |\mathcal{P}\rangle = \hat{A}^\dagger(\mathcal{P})\,|0\rangle\,, \quad \left[\mathcal{P} = \{\sigma, \underbrace{X^0(=T), \boldsymbol{X}}_{=:X}, \boldsymbol{P}\}\right]$$

the one-particle state

Gaussian wavefunction

[Ishikawa, Oda (1809.04285)]

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\boldsymbol{X}\,\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3}\left[\boxed{f_{\sigma,X,\boldsymbol{P}}(x)}\,\hat{A}(\sigma, X, \boldsymbol{P}) + \mathrm{h.c.}\right]$$

Wave function of Gaussian wave packet

(***X*** is defined @ T)

- $$f_{\sigma,X,\boldsymbol{P}}(x) := \int \frac{d^3\boldsymbol{p}}{\sqrt{2E_{\boldsymbol{p}}}}\overset{\text{Int. Pic.}}{\langle x|\boldsymbol{p}\rangle}\underset{\text{Gaussian state}}{\langle \boldsymbol{p}|\sigma, X, \boldsymbol{P}\rangle}$$

$$= \left(\frac{\sigma}{\pi}\right)^{3/4}\int \frac{d^3\boldsymbol{p}}{\sqrt{2p^0}(2\pi)^{3/2}}e^{ip\cdot(x-X)-\frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2}\Bigg|_{p^0=E_{\boldsymbol{p}}}$$

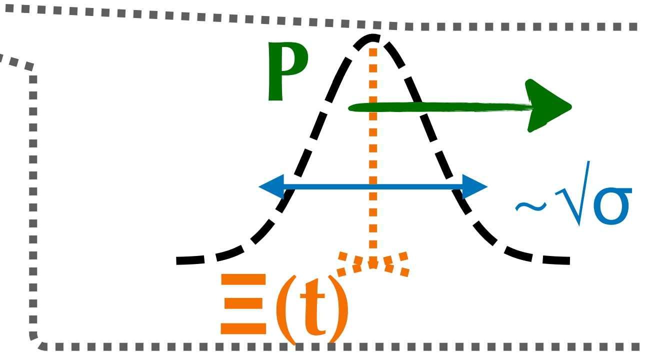
saddle-point approx. for a large **σ**

$$\left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2}\frac{1}{\sqrt{2P^0}(2\pi)^{3/2}}e^{iP\cdot(x-X)-\frac{1}{2\sigma}(\boldsymbol{x}-\Xi(t))^2}\Bigg|_{P^0=E_{\boldsymbol{P}}}$$

on-shell condition

$$\Xi(t) := \boldsymbol{X} + \boldsymbol{V}(\boldsymbol{P})(t-T),\ \boldsymbol{V}(\boldsymbol{P}) := \boldsymbol{P}/E_{\boldsymbol{P}})$$

Position of Centre of
the Gaussian peak at the time (t)



(some details on Gaussian state)

- **Normalisable:** $\langle \sigma, \boldsymbol{X}, \boldsymbol{P} | \sigma, \boldsymbol{X}, \boldsymbol{P} \rangle = 1$
- Coherent: $\delta x_i^2 = \dfrac{\sigma}{2},\ \delta p_i^2 = \dfrac{1}{2\sigma} \quad (i = x, y, z)$
- Non-orthogonal:

$$\langle \sigma, \boldsymbol{X}, \boldsymbol{P} | \sigma', \boldsymbol{X}', \boldsymbol{P}' \rangle = \left(\frac{\sigma_I}{\sigma_A} \right)^{3/4} e^{-\frac{1}{4\sigma_A}(\boldsymbol{X}-\boldsymbol{X}')^2 - \frac{\sigma_I}{4}(\boldsymbol{P}-\boldsymbol{P}')^2 + \frac{1}{2\sigma_I}(\sigma\boldsymbol{P}+\sigma'\boldsymbol{P}')\cdot(\boldsymbol{X}-\boldsymbol{X}')}$$

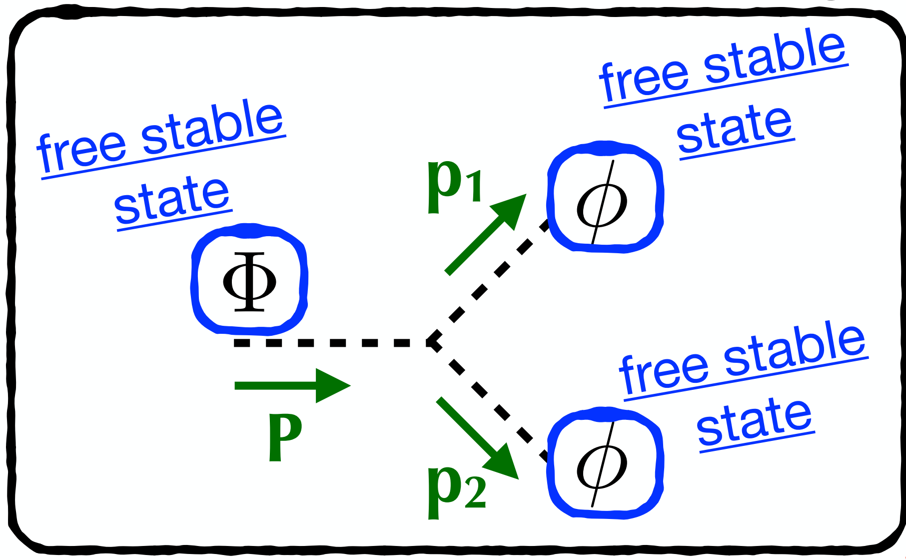
$$\left(\sigma_A := \frac{\sigma + \sigma'}{2},\ \sigma_I^{-1} := \frac{\sigma^{-1} + \sigma'^{-1}}{2} \right)$$

- Over-complete: $\displaystyle\int \frac{d^3\boldsymbol{X}\, d^3\boldsymbol{P}}{(2\pi)^3} |\sigma, \boldsymbol{X}, \boldsymbol{P}\rangle\langle \sigma, \boldsymbol{X}, \boldsymbol{P}| = \hat{1}$

- Algebra of Creation/Annihilation operators:
  - $\left[\hat{A}(\sigma, T, \boldsymbol{X}, \boldsymbol{P}),\ \hat{A}^\dagger(\sigma', T, \boldsymbol{X}', \boldsymbol{P}') \right] = \langle \sigma, T, \boldsymbol{X}, \boldsymbol{P} | \sigma', T, \boldsymbol{X}', \boldsymbol{P}' \rangle$
  - (others) $= 0$

Two contributions in P

- Technically, it is straightforward to derive the form of P (full prob.).



[Ishikawa, Oda (1809.04285)]

$$P = \int |\mathcal{S}|^2 \frac{d^3\boldsymbol{X}_1 d^3\boldsymbol{p}_1}{(2\pi)^3}\frac{d^3\boldsymbol{X}_2 d^3\boldsymbol{p}_2}{(2\pi)^3}$$

$$\simeq \Gamma(T_{\rm out} - T_{\rm in}) + \Delta P$$

for large limit of σ's

proportional to (T_{out} - T_{in}), 'Fermi's Golden rule'

Constant in (T_{out} - T_{in})

(averaged frequency) × (time interval)

"correction" to "averaging"

Understandable naturally.
We try to see the structure lying beneath.

NOTE: Hereafter in the appendix, the Initial state is taken as **free (non-decaying)**.

S-matrix of the simplest 1→2: Φ(free) → φφ

[Ishikawa, Oda (1809.04285)]

- When $\hat{H}_{\rm int}(t) = \int d^3\boldsymbol{x}\ \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite $T_{\rm in}$ & $T_{\rm out}$, S becomes

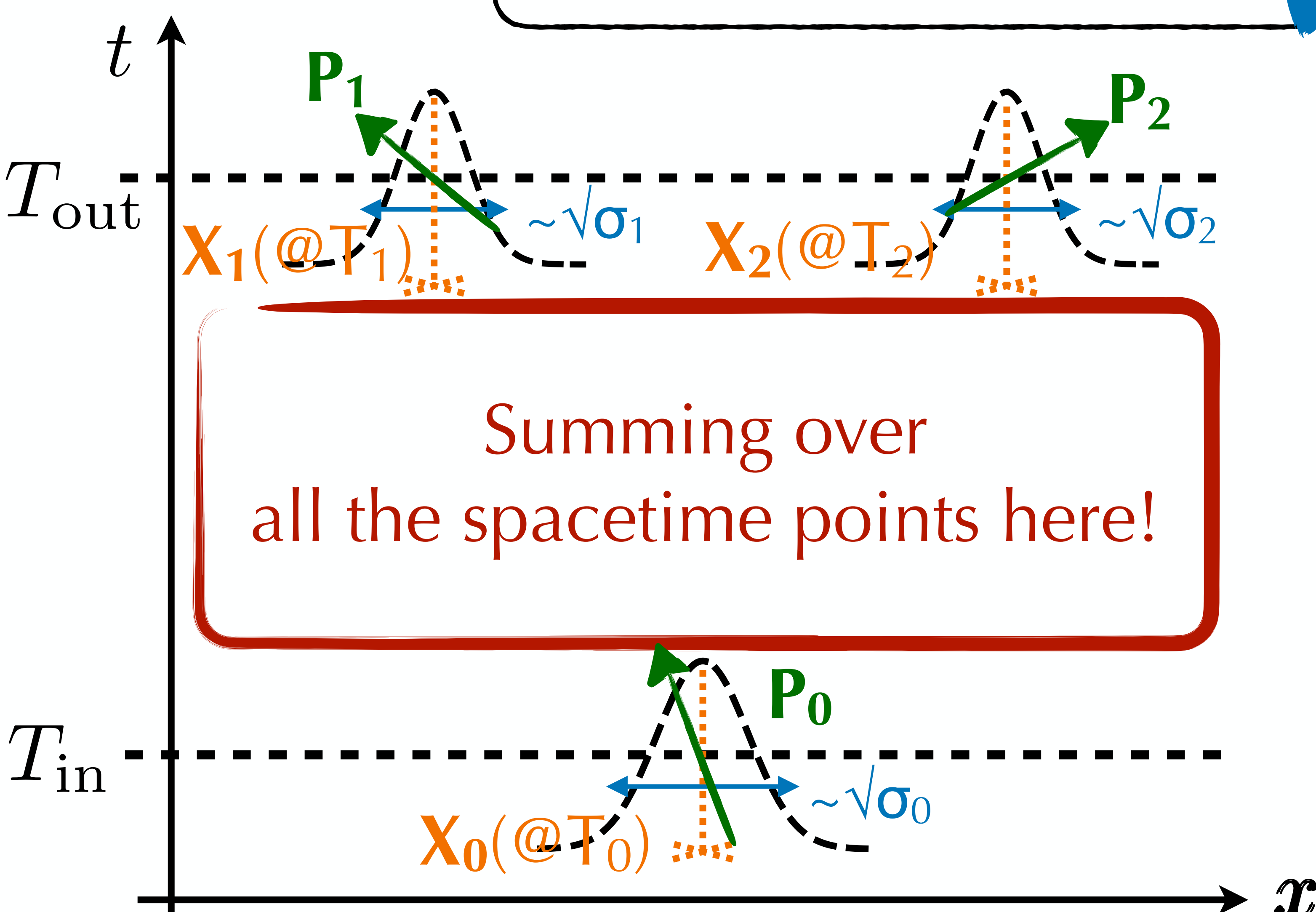
$$\mathcal{S} := \overbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}^{\text{free out-state}} \mathrm{T}e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \overbrace{|\mathcal{P}_0\rangle}^{\text{free in-state}}$$

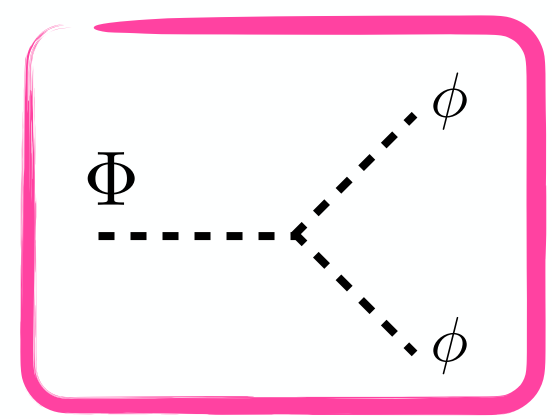
$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO)

$$\longrightarrow\ -\frac{i\kappa}{\sqrt{2}}\int_{T_{\rm in}}^{T_{\rm out}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\ f^*_{\phi,\sigma_2;\Pi_2}(x)\ f_{\Phi,\sigma_0;\Pi_0}(x)$$

"Wave-packet Feynman Rule"





(Wick contraction for on-shell part])

[Ishikawa, Oda (1809.04285)]

$$\left(\Pi_i = \{\underbrace{X_i^0, \boldsymbol{X}_i}_{X_i}, \boldsymbol{P}_i\} \right)$$

$$\circ \ \hat{A}_{\sigma_3}(\Pi_3)\,\hat{\phi}(x) = \int \mathrm{d}^6\boldsymbol{\Pi} f^*_{\sigma;\Pi}(x) \left[\hat{A}_{\sigma_3}(\Pi_3)\,, \hat{A}^\dagger_\sigma(\Pi)\right]$$

$$= \int \mathrm{d}^6\boldsymbol{\Pi} \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E_\phi(\boldsymbol{p})}} \langle \sigma; \Pi \,|\, \phi, \boldsymbol{p}\rangle \langle \phi, \boldsymbol{p} \,|\, \phi, x\rangle \langle \sigma_3; \Pi_3 \,|\, \phi, \sigma; \Pi\rangle$$

$$= \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E_\phi(\boldsymbol{p})}} \langle \sigma_3; \Pi_3 \,|\, \phi, \boldsymbol{p}\rangle \langle \phi, \boldsymbol{p} \,|\, \phi, x\rangle$$

$$= f^*_{\sigma_3;\Pi_3}(x)$$

for a final state

S-matrix of the simplest 1→2: Φ(free) → ϕϕ

[Ishikawa, Oda (1809.04285)]

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[Reminder]

[Details of **Gaussian (on-shell) wave functions**]

$$f_{\Psi,\sigma;\Pi}(x) = \left(\frac{\sigma}{\pi}\right)^{3/4} \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2p^0}\,(2\pi)^{3/2}} e^{ip\cdot(x-X)-\frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2}\Big|_{p^0=E_\Psi(\boldsymbol{p})}$$

saddle-point approx. for a large σ

$$\left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2} \frac{1}{\sqrt{2P^0}\,(2\pi)^{3/2}} e^{iP\cdot(x-X)-\frac{(\boldsymbol{x}-\boldsymbol{\Xi}(t))^2}{2\sigma}}\Big|_{P^0=E_\Psi(\boldsymbol{P})}$$

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[Reminder]

$$\Xi(t) := \boldsymbol{X} + \boldsymbol{V}_\Psi(\boldsymbol{P})\,(t-T)$$

Uniform linear motion of the centre (= Peak!)

$$\boldsymbol{V}_\Psi(\boldsymbol{P}) := \boldsymbol{P}/E_\Psi(\boldsymbol{P})$$

$$E_\Psi(\boldsymbol{P}) := \sqrt{\boldsymbol{P}^2 + m_\psi^2}$$

$$f_{\Psi,\sigma;\Pi}(x) \simeq \left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2}\frac{1}{\sqrt{2P^0}\,(2\pi)^{3/2}}\, e^{iP\cdot(x-X)-\frac{(\boldsymbol{x}-\Xi(t))^2}{2\sigma}}\Bigg|_{P^0=E_\Psi(\boldsymbol{P})}$$

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[Ishikawa, Oda (1809.04285)]

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$$\mathcal{S} := \underbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}_{\text{free out-state}} \mathrm{T}e^{-i\int_{T_{\text{in}}}^{T_{\text{out}}} dt\, \hat{H}_{\text{int}}^{(I)}(t)} \underbrace{|\mathcal{P}_0\rangle}_{\text{free in-state}}$$

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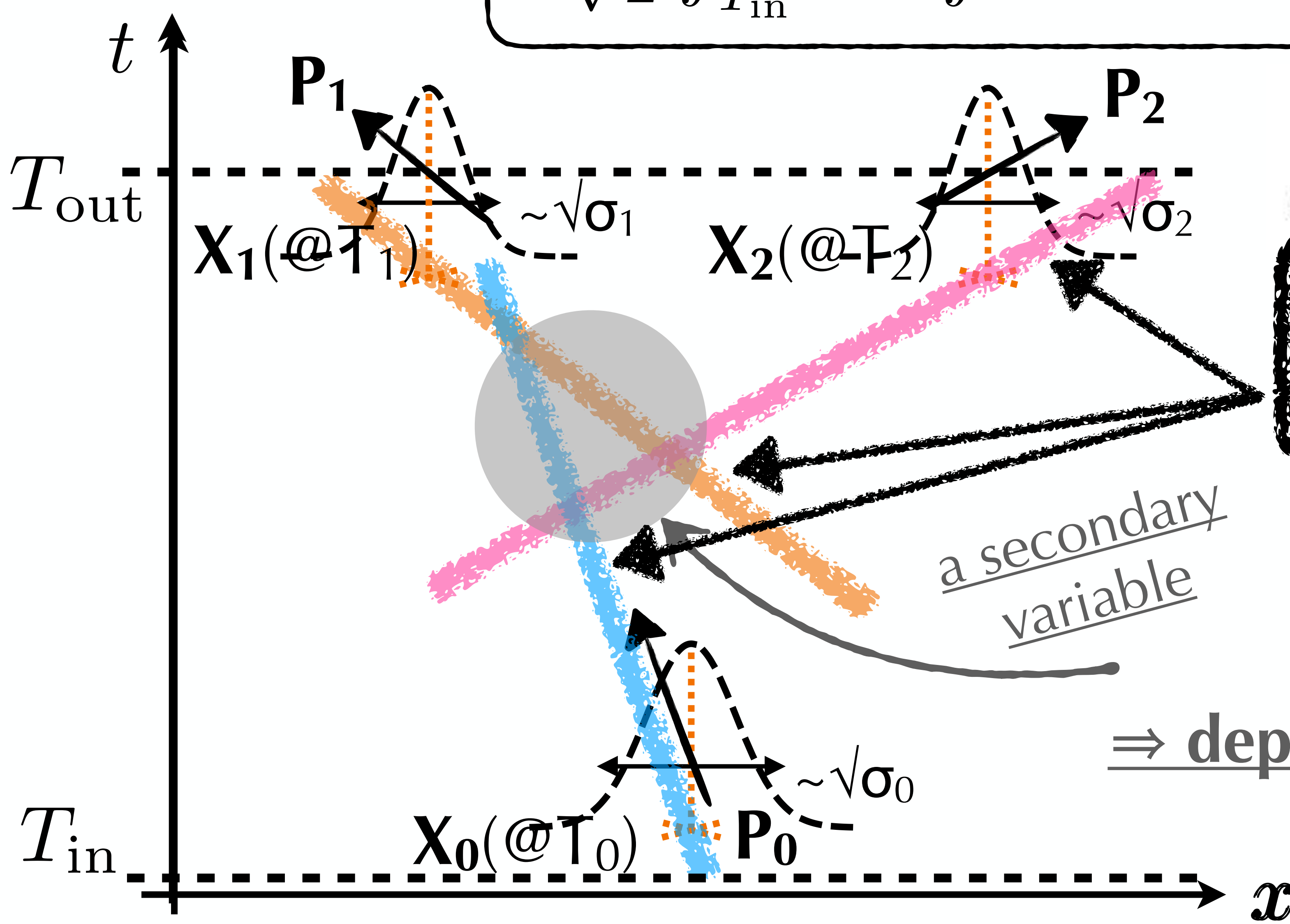
$$\longrightarrow \quad -\frac{i\kappa}{\sqrt{2}}\int_{T_{\text{in}}}^{T_{\text{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\; f^*_{\phi,\sigma_2;\Pi_2}(x)\; f_{\Phi,\sigma_0;\Pi_0}(x)$$



"(classical) trajectories"
⇒ charactering the S-matrix

"overlap domain of the wave packets"
⇒ depending on the trajectories

Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \boxed{\left(\prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right)} \boxed{e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}} G(\mathfrak{T})$$

an exact form

normalisation factors of Gaussians

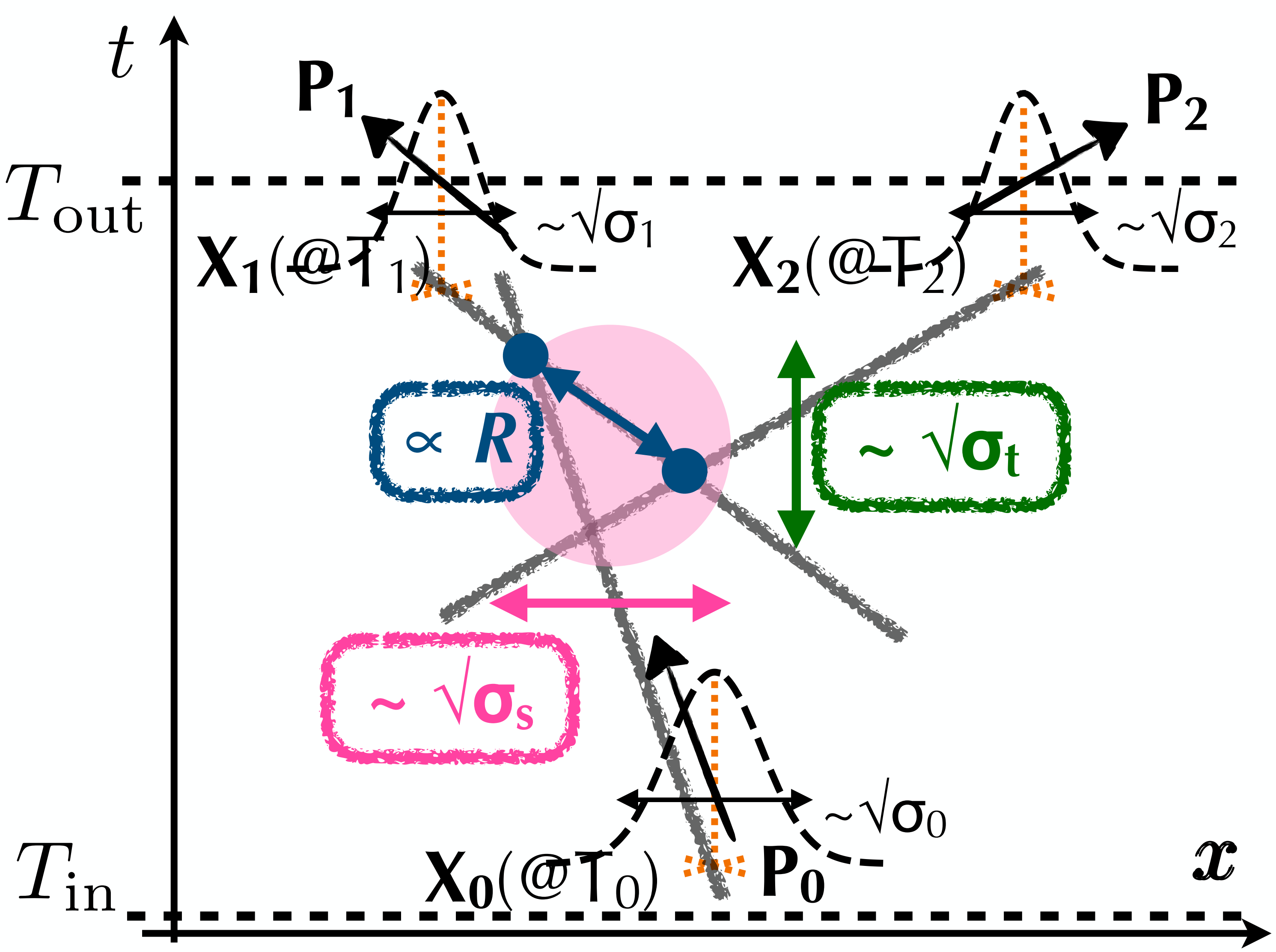
overlaps of the wave packets (including approximated Energy-Momentum conservation)

Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta \boldsymbol{P})^2-\frac{\mathcal{R}}{2}}(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

- **Feature 1 : Geometrical variables characterise *S*.**

$(\delta\omega \sim \delta E := E_{\rm out} - E_{\rm in},\ \delta \boldsymbol{P} := \boldsymbol{P}_{\rm out} - \boldsymbol{P}_{\rm in})$



Result of S(Φ(free) → ϕϕ)

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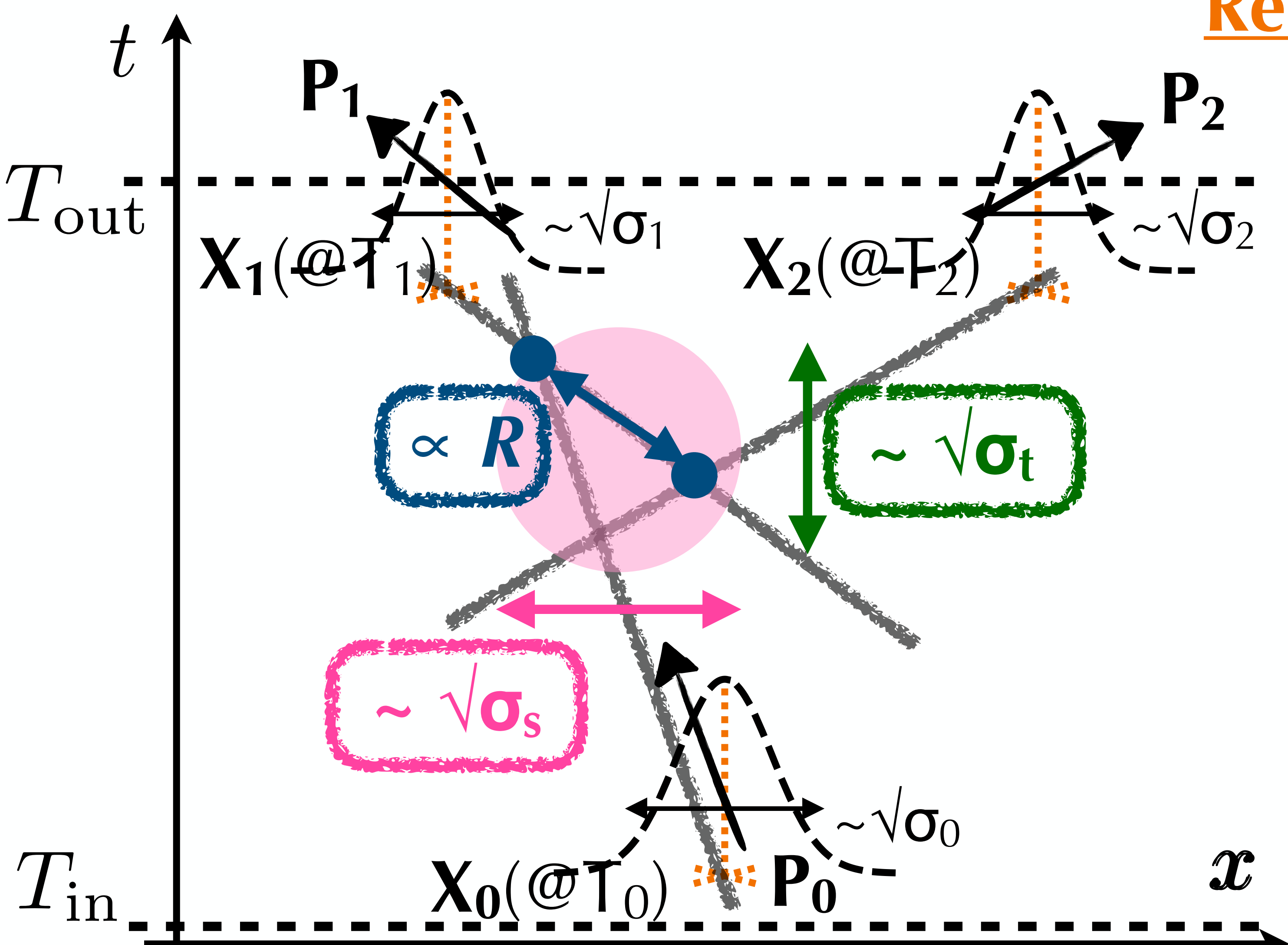
- **Feature ①:** Geometrical variables characterise S.

$(\delta\omega \sim \delta E := E_{\rm out} - E_{\rm in},\ \delta\boldsymbol{P} := \boldsymbol{P}_{\rm out} - \boldsymbol{P}_{\rm in})$



- **Feature ②:**

The limit ($\sigma_s \to \infty$ and $\sigma_t \to \infty$) ⇒ Recovery of the energy-momentum conservation

Note:

$$\left(\sqrt{\frac{\sigma}{2\pi}}e^{-\frac{\sigma}{2}(p-p_0)^2}\underbrace{\longrightarrow}_{\sigma\to\infty}\delta(p-p_0)\right)$$

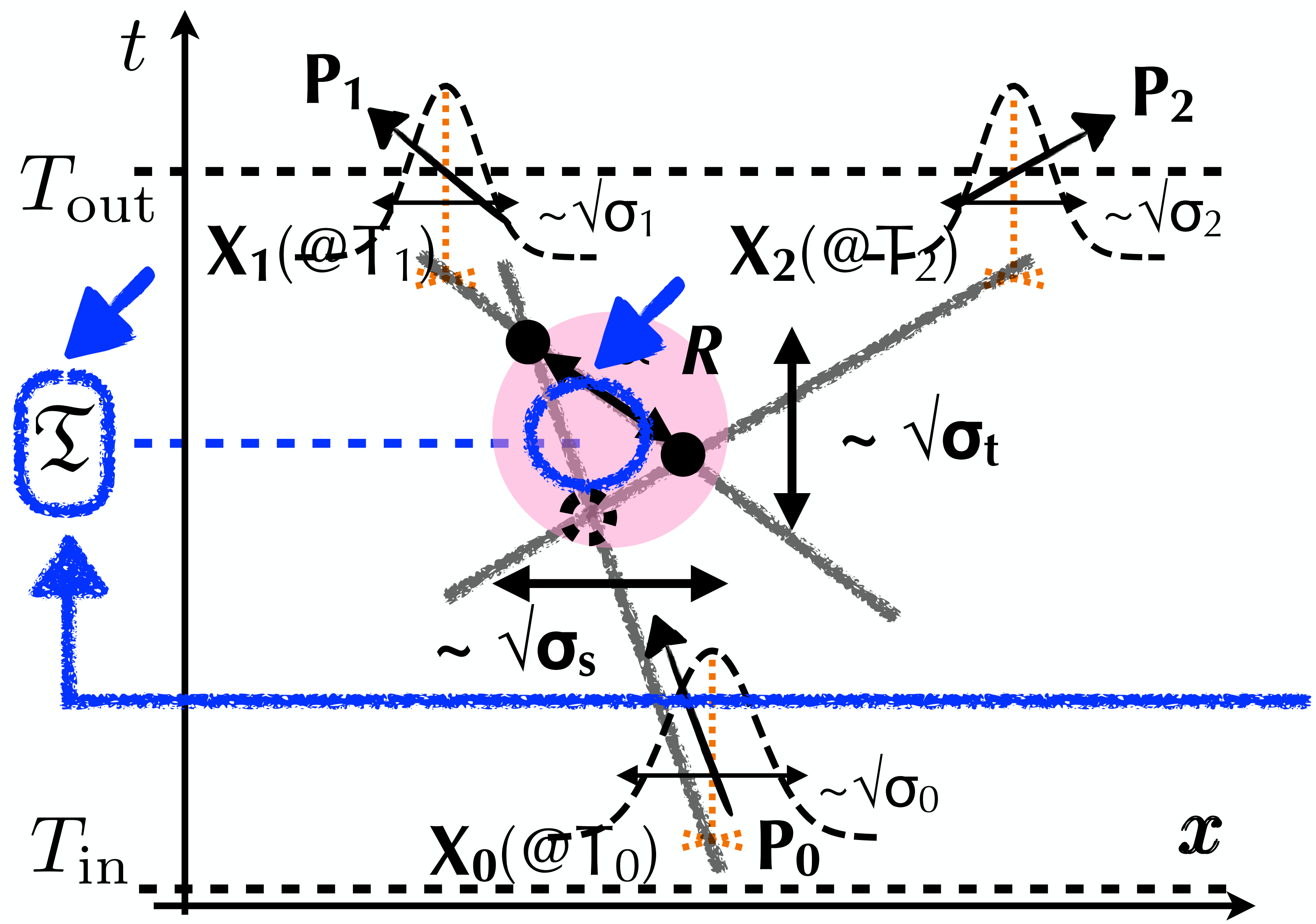
Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left(\prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

- **Feature ③ : Terms are classified into "bulk" and "boundary".**

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

"window function"



determined by the trajectories (configurations of external particles)

Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,\boxed{G(\mathfrak{T})}$$

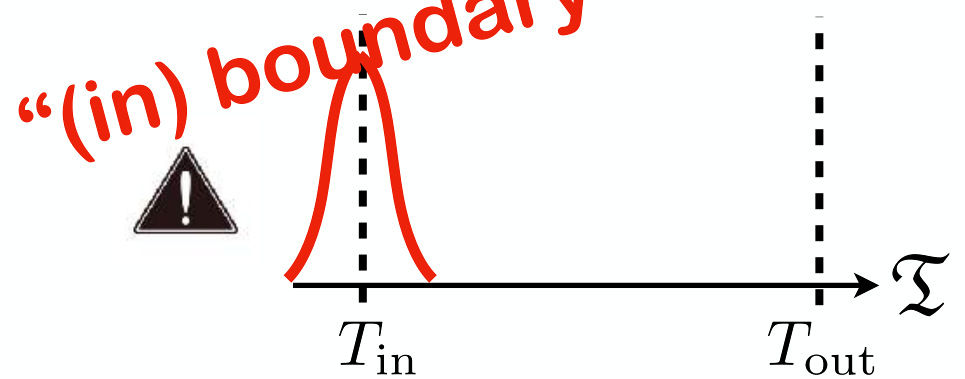
- **Significant Feature:** **Terms are classified** into **"bulk"** and **"boundary"**

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

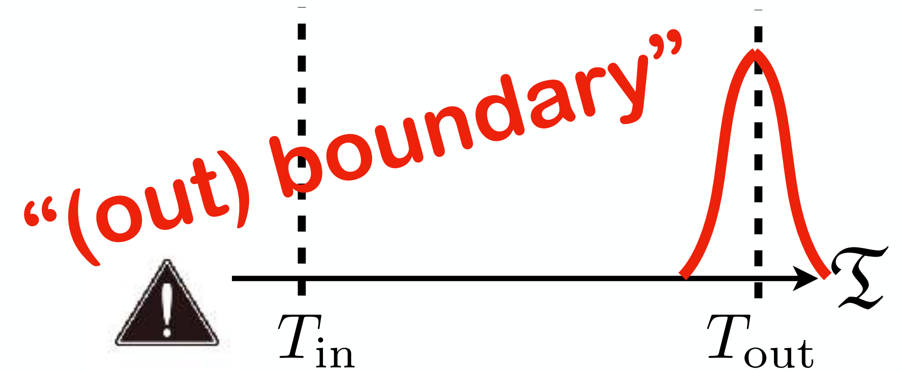
approximately

$$\boxed{G(\mathfrak{T})} \simeq \frac{1}{2}\left[\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)-\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

$$-\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\text{in}})/\sigma_t\right]}$$



$$+\frac{e^{-\frac{(\mathfrak{T}-T_{\text{out}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{out}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\text{out}})/\sigma_t\right]}$$



Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\,\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

- **Significant Feature:** **Terms are classified** into **"bulk"** and **"boundary"**

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

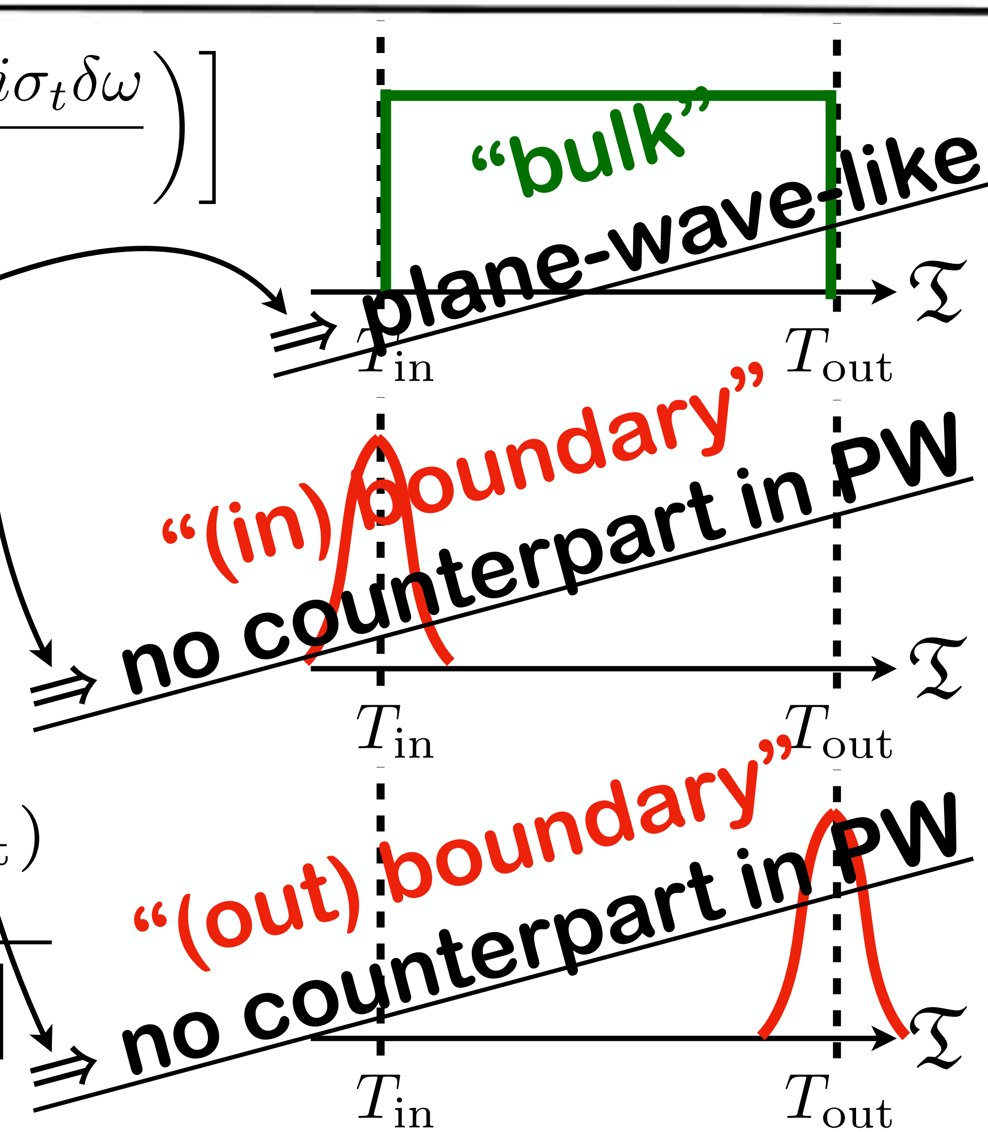
approximately

$$G(\mathfrak{T}) \simeq \frac{1}{2}\left[\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)-\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

[in the causality point of view]

$$-\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\text{in}})/\sigma_t\right]}$$

$$+\frac{e^{-\frac{(\mathfrak{T}-T_{\text{out}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{out}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\text{out}})/\sigma_t\right]}$$



Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left(\prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

☑ **In "1→2",**

- **Bulk** part is **"time-universal"**. As expected, we can show

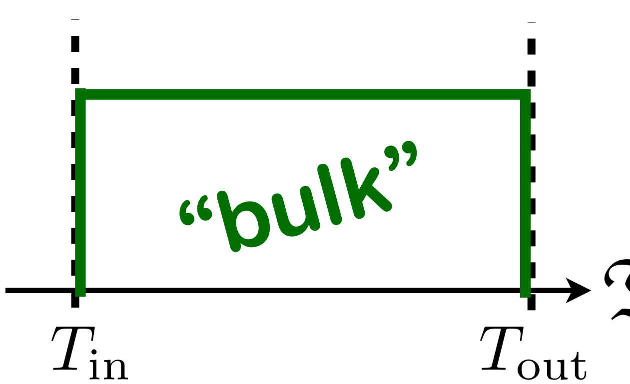
[Marginalised rate per (Volume) & (Time), from S_{bulk} @ $\boldsymbol{P}_0 \to \mathbf{0}$]

$$= \left[\frac{\int d^3 \boldsymbol{X}_{0(=\text{in})}}{V(T_{\text{out}} - T_{\text{in}})} \int \prod_{j=1,2} \frac{d^3 \boldsymbol{X}_j d^3 \boldsymbol{P}_j}{(2\pi)^3} |\mathcal{S}_{\text{bulk}}|^2 \right]_{\boldsymbol{P}_0 \to \mathbf{0}}$$

($\sigma_s \to \infty$ and $\sigma_t \to \infty$: "plane-wave limit")

$\Gamma^{(\text{plane-wave})}_{\Phi \to \phi\phi}$ **(the decay width from $S_{\text{plane-wave}}$)**

$$G(\mathfrak{T}) \supset \frac{1}{2} \left[\operatorname{sgn}\left(\frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) - \operatorname{sgn}\left(\frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) \right]$$



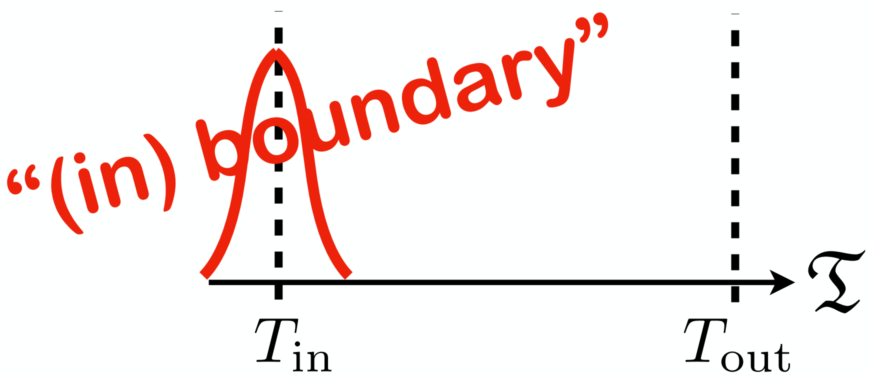
Result of S(Φ(free) → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right) e^{\cancel{-\frac{\sigma_t}{2}(\delta\omega)^2}-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\, (2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

☑ **In "1→2",**

- **No counterpart** of **boundary** terms exists in **S$_{\text{plane-wave}}$**.
- Suppression via energy-non-conservation is **relaxed as "Exponential" → "Power"** [∴**Enhancement**].

$$G(\mathfrak{T}) \supset -\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}\cancel{+\frac{\sigma_t}{2}(\delta\omega)^2}-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\left[\boxed{\delta\omega}-i(\mathfrak{T}-T_{\text{in}})/\sigma_t\right]}$$

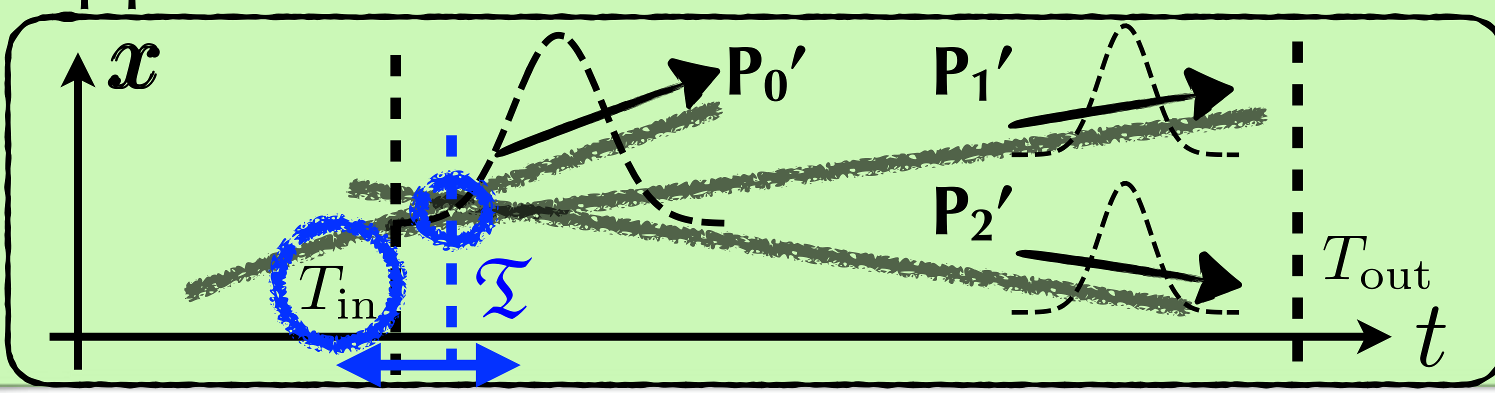


Result of S(Φ(free) → ϕϕ)

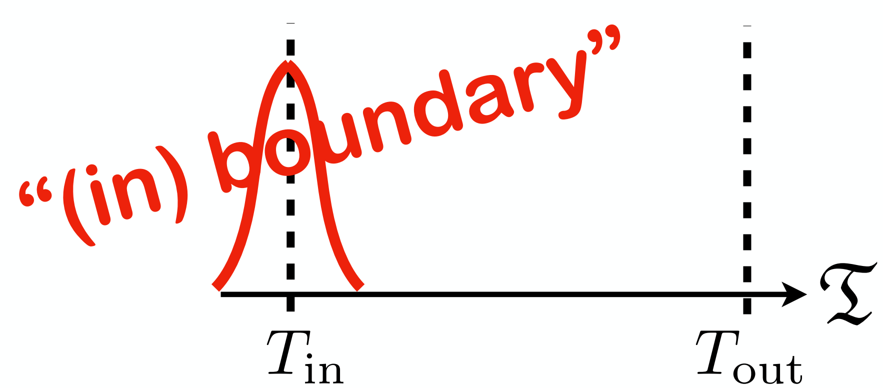
$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4}\,\frac{1}{\sqrt{2E_A}}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

☑ **In "1→2",**

- **No counterpart** of **boundary** terms exists in **S**$_{\textbf{plane-wave}}$.
- Suppression via energy-non-conservation is **relaxed as "Exponential" → "Power"** [∴**Enhancement**].
- Suppression via distances between time domains is **relaxed e.g., in**



$$G(\mathfrak{T}) \supset -\frac{e^{\boxed{-\frac{(\mathfrak{T}-T_{\rm in})^2}{2\sigma_t}}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\rm in})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i\left(\boxed{\mathfrak{T}-T_{\rm in}}\right)/\sigma_t\right]}$$



Result of S(Φ(free) → ϕϕ)

[Ishikawa, Oda (1809.04285)]

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \mathrm{d}x$$

- $$G(\mathfrak{T}) := \int_{T_{\text{in}}}^{T_{\text{out}}} \frac{\mathrm{d}t}{\sqrt{2\pi\sigma_t}} e^{-\frac{1}{2\sigma_t}(t-\mathfrak{T}-i\sigma_t\delta\omega)^2}$$
$$= \frac{1}{2}\left[\mathrm{erf}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{erf}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

- $$G(\mathfrak{T}) = G_{\text{bulk}}(\mathfrak{T}) + G_{\text{in-bdry}}(\mathfrak{T}) + G_{\text{out-bdry}}(\mathfrak{T})$$

$$G_{\text{bdry}}(z) \quad \left(z := \frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)$$

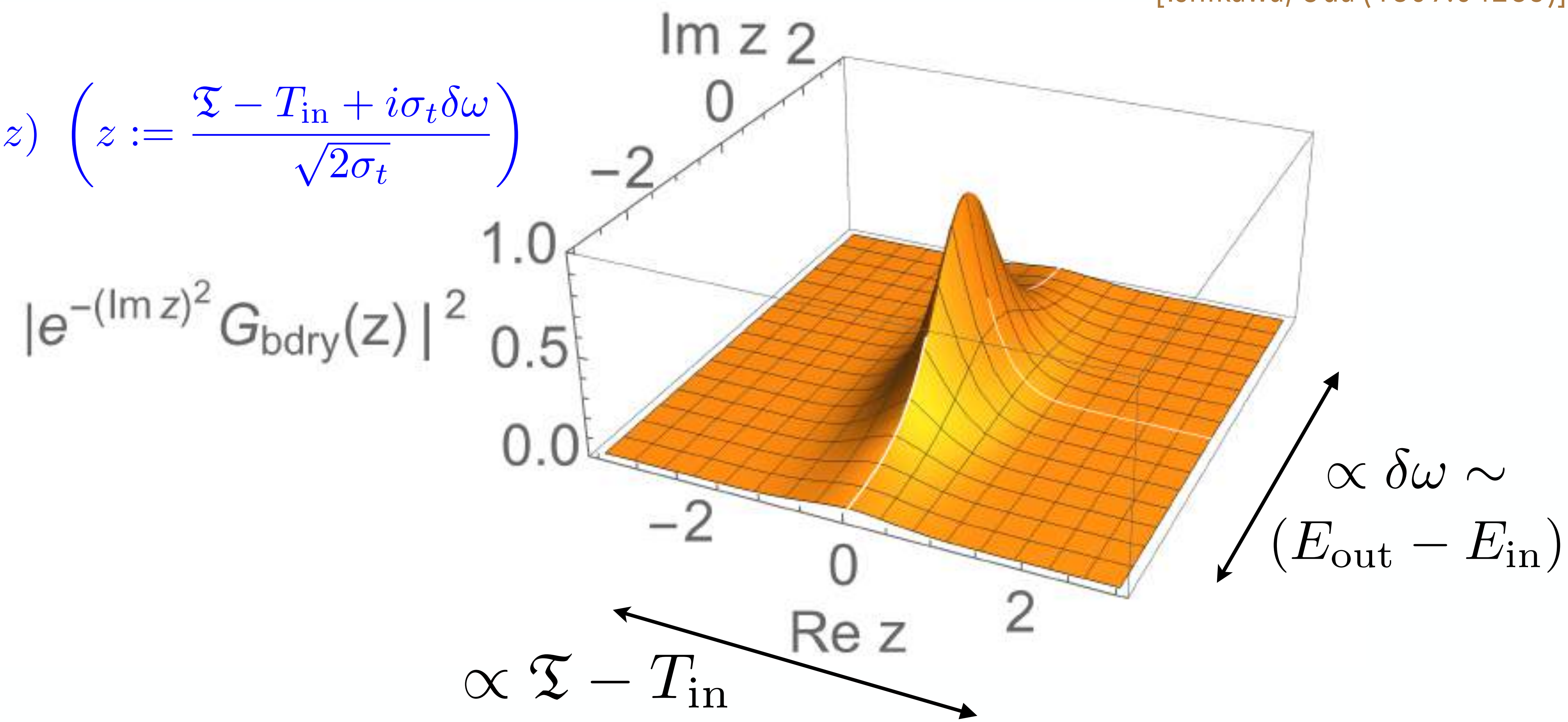
$$G_{\text{bulk}}(\mathfrak{T}) = \begin{cases} 1 & (T_{\text{in}} < \mathfrak{T} < T_{\text{out}}), \\ 0 & (\mathfrak{T} < T_{\text{in}} \text{ or } T_{\text{out}} < \mathfrak{T}), \\ \theta(\delta\omega) & (\mathfrak{T} = T_{\text{in}}), \\ \theta(-\delta\omega) & (\mathfrak{T} = T_{\text{out}}), \end{cases}$$

$$\left\{\begin{aligned} G_{\text{bulk}}(\mathfrak{T}) &:= \frac{1}{2}\left[\mathrm{sgn}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{sgn}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right], \\ G_{\text{in-bdry}}(\mathfrak{T}) &:= \frac{1}{2}\left[\boxed{\mathrm{erf}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{sgn}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)}\right], \\ G_{\text{out-bdry}}(\mathfrak{T}) &:= \frac{1}{2}\left[\mathrm{sgn}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{erf}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]. \end{aligned}\right.$$

Result of S(Φ(free) → ϕϕ)

[Ishikawa, Oda (1809.04285)]

- $G_{\rm bdry}(z) \quad \left(z := \dfrac{\mathfrak{T} - T_{\rm in} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right)$



- $\mathrm{erf}(z) \underbrace{\sim}_{|z| \gg 1} \mathrm{sgn}(z) + e^{-z^2} \left(-\dfrac{1}{\sqrt{\pi} z} \right)$

(We utilised this approximation in the main part.)