Gauge Theories on quantum Minkowski spaces: ρ versus κ

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Content

- Quantum Minkowski space-times in short
- Algebra for ρ-Minkowski from group algebra and Weyl quantization
- ρ-deformed Poincaré algebra
- Scalar field theory on ρ-Minkowski
- A gauge theory model on ρ-Minkowski
- 1-loop tadpole: ρ versus κ

• Coordinates algebra ($\kappa > 0$, $[\kappa] = 1$) of κ-Minkowski space:

$$[x_0, x_i] = \frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0, \quad i, j = 1, \cdots, d. \qquad (1)$$

Informally, κ-Minkowski $\sim$ universal enveloping algebra of $\mathfrak{g}$
Acted on by a deformation of the Poincaré algebra

Pioneering papers:
J. Lukierski, H. Ruegg, A. Nowicki, V. N. Tolstoï, Phys. Lett. B264 (1991) 331.
J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B293 (1992) 344.

S. Majid and H. Ruegg, Phys. Lett. B334 (1994) 348.

• Subject of a huge literature since $\sim$ 3 decades:
algebra + phenomenological impacts + scalar field theories

Reviews: See for instance
J. Lukierski, "kappa-Deformations: Historical Developments and Recent Results", J. Phys. Conf. Ser. 804, 012028 (2017)
J. Kowalski-Glikman, "Introduction to Doubly Special Relativity", Lect. Notes Phys. 669 (2005) 131 [1]

[1] (1) belongs to a wider class of deformations of the form $[x_\mu, x_\nu] = i(a_\mu x_\nu - a_\nu x_\mu)$, $a_\mu a_\mu = \pm 1, 0$, see J. Lukierski, V. Lyakhovsky, M. Mozrzymas, Phys. Lett. B538 (2002) 375.

Another deformation of Minkowski: ρ-Minkowski space-time

• Coordinates algebra $\mathfrak{g}$ ($\rho > 0$, $[\rho] = -1$) of ρ-Minkowski space: Euclidean algebra $\mathfrak{e}(2)$ plus central coordinate x_3

$$[x_0, x_1] = i\rho x_2, \quad [x_0, x_2] = -i\rho x_1, \; [x_1, x_2] = 0,$$
$$[x_3, x_i] = 0, \; i = 0, 1, 2$$

– If Exchange $x_0 \longleftrightarrow x_3$: "commutative time". Does not alter the following conclusions.

• Considered closely more recently from various viewpoints:
– NC field theory[2],
– Black hole physics [3],
– Localisation and quantum observers[4]
– AdS/CFT: gauge dual of Yang-Baxter deformation[5] of $AdS_5 \times S_5$ superstring

• Appeared a long ago (see J. Lukierski, M. Woronowicz, Phys. Lett. B 633 (2006) 116)

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[2] M. Dimitrijević Ćirić, N. Konjik, M. A. Kurkov, F. Lizzi and P. Vitale, Phys. Rev. D **98** (2018) 085011.

[3] M. Dimitrijević Ćirić, N. Konjik and A. Samsarov, Class. Quant. Grav. **35** (2018) 175005; Phys. Rev. D **101** (2020) 116009

[4] F. Lizzi, L. Scala and P. Vitale, Phys. Rev. D 106 (2022) 025023; F. Lizzi, P. Vitale, Phys. Lett. B **818** (2021) 136372

[5] T. Meier, S. J. van Tongeren, Phys. Rev. Lett. 131 (2023) 121603, JHEP12 (2023) 045.

Star product, involution, trace from group algebra and Weyl quantization

• ρ-Minkowski $\longleftarrow (\mathcal{A}, \star_\rho, \dagger)$
• Extend the old construction of von Neumann to formalize the Weyl quantization of phase space $\longrightarrow$ Moyal product

J. von Neumann, Math. Ann. **104** (1931) 570. H. Weyl, Zeitschrift für Physik **46** (1927) 1.

• Exploit main properties of convolution algebras for the group $\mathcal{G}$ related to the coordinates algebra $\mathfrak{g}$ combined with Weyl quantization map

Yields star-product as twisted convolution, natural involution plus natural trace

$\longrightarrow$ In the following, corresponding $*$-algebra denoted by $\mathcal{M}_\rho$

• Convenient to construct star-products when the noncommutativity of space is of "Lie algebra type"

B. Durhuus, A. Sitarz, J. Noncommut. Geom. **7** (2013) 605. J.-C. Wallet, Nucl. Phys. B912 (2016) 354. T. Poulain, J.-C. Wallet, Phys. Rev. D 98 (2018) 025002.

- Start from non trivial part of coordinate algebra $\mathfrak{e}_2$. Relevant group $\mathcal{G}$ is

$$\mathcal{G} = SO(2) \ltimes_\phi \mathbb{R}^2, \;\; \phi(x) = ax, \;\; a \in SO(2), \;\; x \in \mathbb{R}^2$$

$$(a_1, x_1)(a_2, x_2) = (a_1 a_2, x_1 + a_1 x_2), \;\; (a, x)^{-1} = (a^{-1}, -a^{-1}x), \mathbb{I}_\mathcal{G} = (\mathbb{I}_H, 0)$$

- Convolution algebra $\mathbb{C}[\mathcal{G}] = (L^1(\mathcal{G}), \circ, ^\star)$ – For $F, G \in L^1(\mathcal{G}), s, t \in \mathcal{G}$

$$(F \circ G)(s) = \int_\mathcal{G} d\mu(t) F(st) G(t^{-1}), \;\; F^\star(s) = \overline{F}(s^{-1}) \Delta(s^{-1})$$

– $d\mu((a, x)) = d\mu_{\mathbb{R}^2}(x) \; d\mu_{SO(2)}(a) \; |\det(a)|^{-1} = d\mu_{\mathbb{R}^3}$ (Lebesgues)
– $\Delta((a, x)) = \Delta_{\mathbb{R}^2}(x) \; \Delta_{SO(2)}(a) \; |\det(a)|^{-1} = 1$ (unimodular)

– $F = \mathcal{F} f$, i.e interpreted as functions on momentum space: $F(p_0, p_i)$

- Use Weyl quantization $Q(f) = \pi(\mathcal{F} f)$, Q *-morphism of algebra

$$\pi : \mathbb{C}[\mathcal{G}] \to \mathcal{B}(\mathcal{H}), \;\; \pi(F) = \int_\mathcal{G} d\mu(s) F(s) \pi_U(s)$$

- Combine Q with $\pi(F \circ G) = \pi(F)\pi(G)$, $\pi(F)^\dagger = \pi(F^*)$ to obtain:

$$f \star g = \mathcal{F}^{-1}(\mathcal{F} f \circ \mathcal{F} g), \;\; f^\dagger = \mathcal{F}^{-1}(\mathcal{F}(f)^*)$$

• Star-product and involution for $\mathcal{M}_\rho$

$$(f \star_\rho g)(x_0, \vec{x}, x_3) = \int \frac{dp_0}{2\pi}\, dy_0\, e^{-ip_0 y_0} f(x_0 + y_0, \vec{x}) g(x_0, R(-\rho p_0)\vec{x}, x_3),$$
$$f^\dagger(x_0, \vec{x}, x_3) = \int \frac{dp_0}{2\pi}\, dy_0\, e^{-ip_0 y_0} \bar{f}(x_0 + y_0, R(-\rho p_0)\vec{x}, x_3), ,$$

$R(-\rho p_0) \in SO(2)$

• Properties.

$$\int d^4x\, (f^\dagger \star_\rho g)(x) = \int d^4x\, \bar{f}(x) g(x),\ \int d^4x\, (f \star_\rho g)(x) = \int d^4x\, (g \star_\rho f)(x),$$

– Lebesgue integral defines a trace w.r.t. $\star_\rho$; cyclicity holds. $\star_\rho$ "almost" closed (K. Hersent, J.-C. Wallet, JHEP 07 (2023) 031).

– $\star_\rho$ different from star-product already used in literature[6]. Does not derive from a twist.

[6] M. Dimitrijević Ćirić, N. Konjik, M. A. Kurkov, F. Lizzi and P. Vitale, Phys. Rev. D **98** (2018) 085011.

ρ-Poincaré algebra

V. Maris, J.-C. Wallet, JHEP 2024, 119 (2024)

- ρ-Minkowski space $\mathcal{M}_\rho$ acted on by a deformation of Poincaré algebra $\mathcal{P}_\rho$.

– One can check $\mathcal{M}_\rho$ is a left module algebra over $\mathcal{P}_\rho$

– $\mathcal{M}_\rho$ dual to $\mathcal{T}_\rho\{P_0, P_3, P_\pm\}$ deformed translations

$$[M_i, N_j] = i\epsilon_{ijk}N_k, \quad [M_i, M_j] = i\epsilon_{ijk}M_k, \quad [N_i, N_j] = -i\epsilon_{ijk}M_k, \quad [N_i, P_0] = iP_i$$
$$[N_i, P_j] = i\delta_{ij}P_0, \quad [M_i, P_j] = i\epsilon_{ijk}P_k, \quad [P_\mu, P_\nu] = [M_j, P_0] = 0,$$

$$\Delta(M_\pm) = M_\pm \otimes \mathbb{I} + \mathcal{E}_\mp \otimes M_\pm, \ \Delta(N_\pm) = N_\pm \otimes \mathbb{I} + \mathcal{E}_\mp \otimes N_\pm$$
$$\Delta(M_3) = M_3 \otimes \mathbb{I} + \mathbb{I} \otimes M_3, \ \Delta(N_3) = N_3 \otimes \mathbb{I} + \mathbb{I} \otimes N_3$$
$$\Delta(P_{0,3}) = P_{0,3} \otimes \mathbb{I} + \mathbb{I} \otimes P_{0,3}, \ \Delta(P_\pm) = P_\pm \otimes \mathbb{I} + \mathcal{E}_\mp \otimes P_\pm,$$
$$\Delta(\mathcal{E}_\pm) = \mathcal{E}_\pm \otimes \mathcal{E}_\pm, \ \mathcal{E}_\pm = e^{\pm i\rho P_0}$$
$$\epsilon(P_\mu) = 0, \quad \epsilon(\mathcal{E}) = 1, \ \epsilon(M_j) = \epsilon(N_j) = 0, \ j = \pm, 3,$$

$$S(P_0) = -P_0, \ S(P_3) = -P_3, \ S(P_\pm) = -\mathcal{E}_\mp P_\pm, \quad S(\mathcal{E}) = \mathcal{E}^{-1}$$
$$S(M_j) = -M_j, \quad S(N_j) = -N_j, \ j = \pm, 3,$$

Action of $\mathcal{P}_\rho$:

$$(P_\mu \triangleright f)(x) = -i\partial_\mu f(x), \ (M_j \triangleright f)(x) = (\epsilon^l_{jk} x^k P_l \triangleright f)(x),$$
$$(N_j \triangleright f) = ((x_0 P_j - x_j P_0) \triangleright f)(x)$$

ρ-Poincaré invariance

• One can check:

$$h \blacktriangleright \int d^4x \, \mathcal{L} := \int d^4x \; h \triangleright \mathcal{L} = \epsilon(h) \int d^4x \, \mathcal{L},$$

for any $h \in \mathcal{P}_\rho$, $\mathcal{L} \in \mathcal{M}_\rho$.

• Accordingly, the scalar and gauge actions we will consider will be ρ-Poincaré invariant.

1-loop exploration of ϕ^4 scalar field theory on ρ-Minkowski. UV/IR mixing? K. Hersent, J.-C. Wallet, JHEP 07 (2023) 031

$$S(\phi,\overline{\phi}) = \int d^4x \, (\partial_\mu \overline{\phi} \partial_\mu \phi + m^2 \overline{\phi}\phi) + S_{int},$$

$$S^{O}_{int} = g \int d^4x \; \phi^\dagger \star_\rho \phi \star_\rho \phi^\dagger \star_\rho \phi; \quad S^{NO}_{int} = g \int d^4x \; \phi^\dagger \star_\rho \phi^\dagger \star_\rho \phi \star_\rho \phi$$

- 2-point function:

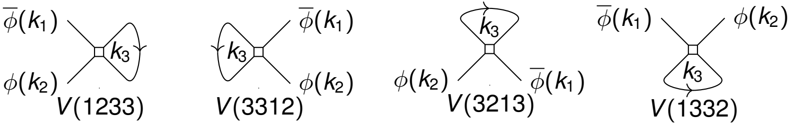


Figure: Diagrams contributing to the 2-point function - Orientable interaction

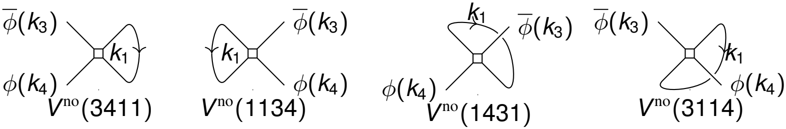
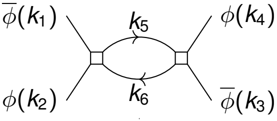


Figure: Diagrams contribution to the 2-point function – Non orientable interaction

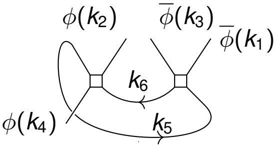
1-loop exploration of ϕ^4 scalar field theory on ρ-Minkowski

K. Hersent, J.-C. Wallet, JHEP 07 (2023) 031

• 4-point function – Orientable case:



$V(1256)V(3465) = V(1256)V(5643)$



$V(5462)V(3615)$

Figure: Typical diagrams among the 12 diagrams for the 4-point function.

	Orientable	Non orientable	UV/IR mixing
2-point function	no IR singularity	IR singularity	YES
4-point function	IR singularity	X	YES

• IR singularities responsible for UV/IR mixing
• $D = 4$ quadratic UV divergence for 2-point function in orientable case
• Conclusions similar to M. Dimitrijević et al., Phys. Rev. D 98 (2018) 085011.
• UV/IR mixing of Lie algebra-type noncommutatitive ϕ^4-theories recently re-investigated in K. Hersent, JHEP 2024 (2024) 23.

- Convenient to use a twisted derivation-based differential calculus.

– Introduced a long ago [7],
– Untwisted versions in NC gauge theories [8]
– Graded and extended versions introduced in [9]

- Start from set of twisted derivations of $\mathcal{T}_\rho$

$$\mathfrak{D} = \{P_\mu : \mathcal{M}_\rho \to \mathcal{M}_\rho,\ \mu = 0, 3, \pm,\ \{P_0, P_3\}_{\mathbb{I}} \oplus \{P_+\}_{\mathcal{E}_+} \oplus \{P_-\}_{\mathcal{E}_-}\},$$

$$\mathcal{E}_\pm = e^{i\pm\rho P_0}$$

$$P_{0,3}(f \star g) = P_{0,3}(f) \star g + f \star P_{0,3}(g),$$

$$P_\pm(f \star g) = P_\pm(f) \star g + \mathcal{E}_\mp(f) \star P_\pm(g)$$

[7] Michel Dubois-Violette. Lectures on graded differential algebras and noncommutative geometry. 2000. arXiv: math/9912017 [math.QA].

[8] E. Cagnache, T.Masson, and J.-C. Wallet, J. Noncommut. Geom. 5 (2011) 39

[9] A. de Goursac, T. Masson, J.-C. Wallet, J. Noncommut. Geom. 6 (2012) 343

• One checks: $\mathfrak{D}$ is a graded abelian Lie algebra and $\mathcal{Z}(\mathcal{M}_\rho)$-bimodule:

$$(z.P_\mu)(f) = z \star P_\mu(f) = P_\mu(f) \star z := (P_\mu.z)(f)$$

Grading defined by $\tau(P_{0,3}) = 0, \ \tau(P_\pm) = \pm 1$:

$$\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$$

• Exterior algebra of differential forms defined as usual from the spaces of $\mathcal{Z}(\mathcal{M}_\rho)$-multilinear antisymmetric maps $\omega : \mathfrak{D}^n \to \mathcal{M}_\rho$.
One has $\omega(P_1, P_2, ..., P_n) \in \mathcal{M}_\rho$ and $\omega(P_1, P_2, ..., P_n.z) = \omega(P_1, P_2, ..., P_n) \star z$.

$(\Omega^\bullet(\mathcal{M}_\rho) = \bigoplus_{n=0}^4 \Omega^n(\mathcal{M}_\rho), \times, \ d)$ graded differential algebra can be constructed with differential d with $d^2 = 0$.

- For any $\omega \in \Omega^p(\mathcal{M}_\rho)$, $\eta \in \Omega^q(\mathcal{M}_\rho)$, $\omega \times \eta \in \Omega^{p+q}(\mathcal{M}_\rho)$

$$(\omega \times \eta)(P_1, \ldots, P_{p+q}) \\ = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{(\sigma)} \omega(P_{\sigma(1)}, \ldots, P_{\sigma(p)}) \star \eta(P_{\sigma(p+1)}, \ldots, P_{\sigma(p+q)}),$$

$$\mathrm{d}\omega(P_1, \ldots, P_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} P_j \triangleright \left(\omega(P_1, \ldots, \vee_j, \ldots, P_{p+1})\right),$$

$$\mathrm{d}^2 = 0$$

- Differential satisfies a twisted Leibniz rule:

$$\mathrm{d}(\omega \times \eta) = d\omega \times \eta + (-1)^{\delta(\omega)} \omega \times_{\mathcal{E}} d\eta$$

for any $\omega, \eta \in \Omega^\bullet(\mathcal{M}_\rho)$
– $\delta(.)$: degree of form
– $\times_{\mathcal{E}}$: indicates that a twist acts on the 1st factor depending on the actual derivation acting on the 2nd factor

- Pick right hermitian module over $\mathcal{M}_\rho$, says, $\mathbb{E}$.
- Twisted connection defined as $\nabla : \mathfrak{D}_i \times \mathbb{E} \to \mathbb{E}, \ i = 0, \pm$

$$\nabla_{P_\mu + P'_\mu}(m) = \nabla_{P_\mu}(m) + \nabla_{P'_\mu}(m), \ \forall (P_\mu, P'_\mu) \in \mathfrak{D}_i \times \mathfrak{D}_i, \ i = 0, \pm$$

$$\nabla_{z.P_\mu}(m) = \nabla_{P_\mu}(m) \star z, \ \forall P_\mu \in \mathfrak{D}, \forall z \in Z(\mathcal{M}_\rho),$$

$$\nabla_{P_\mu}(m \triangleleft f) = \nabla_{P_\mu}(m) \triangleleft f + \beta_{P_\mu}(m) \triangleleft P_\mu(f), \ \forall P_\mu \in \mathfrak{D}, \qquad (2)$$

((2) holds for linear combinations of P_μ's homogeneous in twist degree

- Assume now: $\mathbb{E} \simeq \mathcal{M}_\rho, \ \ m \triangleleft f = m \star f, h(m_1, m_2) = m_1^\dagger \star m_2$

$$\nabla_{P_\mu}(m \triangleleft f) = \nabla_{P_\mu}(m) \triangleleft f + \mathcal{E}_\mu(m) \triangleleft P_\mu(f), \ \forall P_\mu \in \mathfrak{D}.$$

$$\mathcal{E}_\mu = \mathbb{I}, \mathbb{I}, e^{\pm i\rho P_0}, \ \mu = 0, 3, \pm$$

- The curvature $\mathcal{F}(P_\mu, P_\nu) := \mathcal{F}_{\mu\nu} : \mathbb{E} \to \mathbb{E}$ ($\mathcal{F}_{\mu\nu}(m \star f) = \mathcal{F}_{\mu\nu}(m) \star f$) is

$$\mathcal{F}_{\mu\nu} := \mathcal{E}_\nu \nabla_\mu \mathcal{E}_\nu^{-1} \nabla_\nu - \mathcal{E}_\mu \nabla_\nu \mathcal{E}_\mu^{-1} \nabla_\mu, \ \ \mu, \nu = 0, 3, \pm$$

• Set: $A_\mu = \nabla_{P_\mu}(\mathbb{I}),\ \nabla_\mu := \nabla_{P_\mu},\ \mathcal{F}_{\mu\nu}(\mathbb{I}) = F_{\mu\nu}$

• Gauge transformations: automorphisms of $\mathbb{E}$ preserving the hermitian structure $h(m_1, m_2)$

$$\mathcal{U} = \{g \in \mathbb{E} \simeq \mathcal{M}_\rho,\ \ g^\dagger \star g = g \star g^\dagger = 1\}.$$

• Twisted gauge transformations for the connection:

$$\nabla_\mu^g(.) = (\mathcal{E}_\mu \triangleright g^\dagger) \star \nabla_\mu(g \star .)$$

$$A_\mu^g = (\mathcal{E}_\mu \triangleright g^\dagger) \star A_\mu \star g + (\mathcal{E}_\mu \triangleright g^\dagger) \star P_\mu g$$

• Twisted gauge transformations for the field strength

$$F_{\mu\nu}^g = (\mathcal{E}_\mu \mathcal{E}_\nu \triangleright g^\dagger) \star F_{\mu\nu} \star g$$

(no summation over indices μ, ν in the RHS)

• Gauge invariant action ($[G]=0$)

$$S_\rho := \frac{1}{4G^2}\langle F_{\mu\nu}, F_{\mu\nu}\rangle = \frac{1}{4G^2}\int d^4x\ F^\dagger_{\mu\nu}\star F_{\mu\nu} = \frac{1}{4G^2}\int d^4x\ \overline{F_{\mu\nu}}(x)F_{\mu\nu}(x),$$

also ρ-Poincaré invariant. (Assume A_μ, $\mu=0,1,2,3$ real-valued).
– Gauge-invariance: use $((\mathcal{E}_\mu\mathcal{E}_\nu)\triangleright g)^\dagger = (\mathcal{E}_\mu\mathcal{E}_\nu)\triangleright g,\ \mu,\nu=0,3,\pm$ plus cyclicity of the integral w.r.t $\star_\rho$

• Comments (classical level):
– "dimension" $d\geq 3$
– Kinetic operator = the one for usual QED (comes from differential calculus)
– "photon" has self-interactions as expected (ρ^{-1} large)
– comparison with similar construction for κ-Minkowski[10]: gauge invariance is obtained for a unique value of dimension: $d=5$
Lost of cyclicity of the integral w.r.t $\star_\kappa\rightarrow$ KMS weight "twisted trace"

[10] Ph. Mathieu, J.-C. Wallet, JHEP 05(2020) 115, JHEP 03 (2021) 209

• BRST gauge-fixing – BRST symmetry is twisted as for κ-Minkowski case[11]

$$S = S_\rho + s\int d^4x\,(\overline{C}^\dagger \star (\overline{P_\mu}(A_\mu)) + c.c.,\ \ (s^2 = 0)$$

– Lorentz-type gauge : $\overline{P_\mu}(A_\mu) = P_0A_0 + P_3A_3 + P_+A_- + P_-A_+$
– BRST operation: $sA_\mu = P_\mu C + A_\mu \star C - (\mathcal{E}_\mu \triangleright C) \star A_\mu$, $s\overline{C}^\dagger = b^\dagger$, $sb^\dagger = 0$

• Result: Tadpole vanishes! There is no 1-point function ($\Gamma^1(A) = 0$)

• Non-vanishing tadpole frequent in gauge theories on quantum spaces[12]
– Effective action not gauge-invariant ("gauge symmetry radiatively broken")
– Vacuum not stable against quantum fluctuations
– non zero tadpole in gauge theory on κ-Minkowski[13] obtained similarly to S_ρ (computation done for a whole family of gauge functions)

$\longrightarrow$ comparison of the computations for ρ and κ suggests that the structure of differential calculus strongly matters for the vanishing of tadpole (within the considered type of differential calculus)

[11] P. Mathieu, J.-C. Wallet, Phys. Rev. D 103, 086018 (2021)
[12] K. Hersent, P. Mathieu, J.-C. Wallet, Physics Reports 1014 (2023) 1-83
[13] K. Hersent, P. Mathieu, J.-C. Wallet, Phys. Rev. D 105 (2022) 106013

Outlook

• Delicate but routine computations for ρ-Minkowski gauge theory: Perform computation of tadpole in a family of gauge functions [14], plus search for UV/IR mixing (freedom?) [15].

• Explore other available deformations of Minkowski plus gauge theories on them (in progress) $\longrightarrow$ more NC gauge theory prototypes

Note: all these present (and forthcoming) gauge theories are "only" NC versions of Yang-Mills theory.

• One step ahead: NC analogs of Einstein-Hilbert action using notion of NC (quantum) analog of tangent bundle[16], with κ-Minkowski space as NC analog of tangent space. Exploration of classical properties of resulting actions in progress...
to be continued...

THANK YOU !

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[14] V. Maris, J.-C. Wallet (2024)
[15] A. Besson, V. Maris, J.-C. Wallet (2024)
[16] K. Hersent, J.-C. Wallet (2024)