Courant bracket twisted simultaneously by a 2-form B and a bi-vector θ

Ilija Ivanišević

Group for Gravitation, Particles and Fields
Institute of Physics Belgrade

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Outline

- Introduction: Courant bracket from symmetry generators
- Different twisted Courant brackets
- Courant bracket twisted simultaneously by B and θ: motivation and computation

Symmetries of bosonic string

- Action for classical closed bosonic string

$$\mathcal{S} = \kappa \int_{\Sigma} d\sigma d\tau \Big[B_{\mu\nu}(x) + \frac{1}{2} G_{\mu\nu}(x) \Big] \partial_+ x^\mu \partial_- x^\nu \,, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma$$

- Hamiltonian

$$\mathcal{H} = \frac{1}{2\kappa} \pi_\mu (G^{-1})^{\mu\nu} \pi_\nu + \frac{\kappa}{2} x'^\mu G^E_{\mu\nu} x'^\nu - 2 x'^\mu B_{\mu\rho} (G^{-1})^{\rho\nu} \pi_\nu$$

$$\pi_\mu = \kappa G_{\mu\nu} \dot{x}^\nu - 2\kappa B_{\mu\nu} x'^\nu \,, \quad G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$$

- Symmetry generators

$$\mathcal{H}_{(G,B)} + \{\mathcal{G}, \mathcal{H}_{(G,B)}\} = \mathcal{H}_{(G+\delta G, B+\delta B)}$$

- Diffeomorphisms:

$$\mathcal{G}_\xi = \int_0^{2\pi} d\sigma \xi^\mu(x) \pi_\mu , \qquad \delta_\xi G = \mathcal{L}_\xi G \,, \quad \delta_\xi B = \mathcal{L}_\xi B$$

- Local gauge transformations:

$$\mathcal{G}_\lambda = \int_0^{2\pi} d\sigma \lambda_\mu(x) \kappa x'^\mu , \qquad \delta_\lambda G = 0 \,, \quad \delta_\lambda B = d\lambda$$

T-duality

- T-duality is a string feature which connects apparently different theories that produce the same observable results.
- The simplest example - closed bosonic string with one dimension compactified to a circle

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4}\,, \quad R \leftrightarrow \frac{l_s^2}{R}\,, \quad m \leftrightarrow n$$

- T-duality relations between canonical variables

$$\kappa x'^{\mu} \cong \pi_{\mu}$$

- T-duality relates diffeomorphisms and local gauge transformations
- T-dual background fields

$$\begin{aligned} {}^\star G^{\mu\nu} &= (G_E^{-1})^{\mu\nu}\,, \quad G^E_{\mu\nu} = G_{\mu\nu} - 4B_{\mu\rho}(G^{-1})^{\rho\sigma}B_{\sigma\nu} \\ {}^\star B^{\mu\nu} &= \frac{\kappa}{2}\theta^{\mu\nu}\,, \quad \theta^{\mu\nu} = -\frac{2}{\kappa}(G^{-1})^{\mu\rho}B_{\rho\sigma}(G_E^{-1})^{\sigma\nu}\,. \end{aligned}$$

Courant bracket

- Generalized vectors are elements of the smooth section of the generalized tangent bundle $T\mathcal{M} \oplus T^\star\mathcal{M}$.
- Double symmetry parameter and double canonical variable:

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}, \quad X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix}$$

- Generator can be expressed in terms of $O(D,D)$ invariant inner product

$$\mathcal{G}_\Lambda = \mathcal{G}_\xi + \mathcal{G}_\lambda = \int d\sigma \langle \Lambda, X \rangle$$

$$\langle \Lambda, X \rangle = \Lambda^M \eta_{MN} X^N \quad \eta_{MN} = \begin{pmatrix} 0 & \delta^\mu_\nu \\ \delta^\mu_\nu & 0 \end{pmatrix}$$

- Poisson bracket algebra of generators gives rise to the Courant bracket

$$\left\{ \mathcal{G}_{\Lambda_1}, \mathcal{G}_{\Lambda_2} \right\} = -\mathcal{G}_{[\Lambda_1,\Lambda_2]_\mathcal{C}}$$

$$[\Lambda_1,\Lambda_2]_\mathcal{C} = [\xi_1,\xi_2]_L \oplus \left(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) \right)$$

Twisted Courant bracket

- Change of basis by transformation e^T that is from $O(D,D)$

$$\hat{X}^M = (e^T)^M{}_N \, X^N \,, \quad \hat{\Lambda}^M = (e^T)^M{}_N \, \Lambda^N$$
$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X \rangle = \int d\sigma \langle e^T \Lambda, e^T X \rangle = \int d\sigma \langle \hat{\Lambda}, \hat{X} \rangle = \mathcal{G}^{(T)}_{\hat{\Lambda}}$$

- Poisson bracket algebra:

$$\begin{aligned}
\Big\{ \mathcal{G}_{\Lambda_1}, \mathcal{G}_{\Lambda_2} \Big\} &= -\mathcal{G}_{[\Lambda_1, \Lambda_2]_{\mathcal{C}}} \\
\Big\{ \mathcal{G}^{(T)}_{\hat{\Lambda}_1}, \mathcal{G}^{(T)}_{\hat{\Lambda}_2} \Big\} &= -\int d\sigma \langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X \rangle = -\int d\sigma \langle [e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}, e^{-T}\hat{X} \rangle \\
&= -\int d\sigma \langle e^T [e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}, \hat{X} \rangle = -\mathcal{G}^{(T)}_{[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}} \,,
\end{aligned}$$

- Twisted Courant bracket

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = e^T [e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}} \,.$$

B-twisted Courant brackets

- B-shifts $e^{\hat{B}}$

$$\hat{B} = \begin{pmatrix} 0 & 0 \\ 2B & 0 \end{pmatrix}, \quad e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix}$$

- Action of B-shifts on double canonical variable

$$\hat{X}^M = (e^{\hat{B}})^M{}_N X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix}$$

- Auxiliary currents algebra gives rise to the H-flux

$$\begin{aligned} \{i_\mu(\sigma), i_\nu(\bar{\sigma})\} &= -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar{\sigma}) \\ B_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \end{aligned}$$

- B-twited Courant bracket

$$\left\{ \mathcal{G}^{\hat{B}}_{\Lambda_1}, \mathcal{G}^{\hat{B}}_{\Lambda_2} \right\} = -\mathcal{G}^{\hat{B}}_{[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\hat{B}}}}$$

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\hat{B}}} = [\xi_1, \xi_2]_L \oplus \left(\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1 - \frac{1}{2} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1) + dB(\xi_1, \xi_2, .) \right)$$

θ-twisted Courant brackets

- θ-transformations $e^{\hat{\theta}}$

$$\hat{\theta} = \begin{pmatrix} 0 & \kappa\theta \\ 0 & 0 \end{pmatrix}, \quad e^{\hat{\theta}} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix}$$

- Action of θ-transformations on double canonical variable

$$\hat{X}^M = (e^{\hat{\theta}})^M{}_N X^N = \begin{pmatrix} \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} k^\mu \\ \pi_\mu \end{pmatrix}$$

- Auxilliary current algebra gives rise to Q and R-flux

$$\{k^\mu(\sigma), k^\nu(\bar{\sigma})\} = -\kappa Q_\rho{}^{\mu\nu} k^\rho \delta(\sigma - \bar{\sigma}) - \kappa^2 R^{\mu\nu\rho}\pi_\rho \delta(\sigma - \bar{\sigma})$$
$$Q_\rho{}^{\mu\nu} = \partial_\rho \theta^{\mu\nu}, \qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$$

- Auxiliary currents in case of B-twisted and θ-twisted Courant brackets relate via T-duality

$$i_\mu = \pi_\mu + 2B_{\mu\nu}\kappa x'^\nu \xleftrightarrow{\pi_\mu \cong \kappa x'^\mu \quad B_{\mu\nu} \cong \frac{\kappa}{2}\theta^{\mu\nu}} \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu$$

Motivation

- We would like to construct a basis of currents such that all fluxes appear, and that the basis is invariant under T-duality.
- First idea - composition of B-shifts and θ-transformations

$$e^{\hat{\theta}} e^{\hat{B}} = \begin{pmatrix} 1 + 2\kappa\theta B & \kappa\theta \\ 2B & 1 \end{pmatrix}$$

- Two set of currents

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \,, \quad \hat{k}^\mu = \kappa x'^\mu + \kappa \theta^{\mu\nu} i_\nu$$

- All fluxes do appear

$$\begin{aligned}
\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} &= -2B_{\mu\nu\rho} \hat{k}^\rho \delta(\sigma - \bar{\sigma}) - \mathcal{F}_{\mu\nu}^{\ \ \rho}\, i_\rho \delta(\sigma - \bar{\sigma}) \\
\{\hat{k}^\mu(\sigma), \hat{k}^\nu(\bar{\sigma})\} &= -\kappa \mathcal{Q}_\rho^{\ \mu\nu} \hat{k}^\rho \delta(\sigma - \bar{\sigma}) - \kappa^2 \mathcal{R}^{\mu\nu\rho} i_\rho \delta(\sigma - \bar{\sigma}) \\
\{i_\mu(\sigma), \hat{k}^\nu(\bar{\sigma})\} &= \kappa \delta_\mu^\nu \delta'(\sigma - \bar{\sigma}) + \mathcal{F}_{\mu\rho}^{\ \ \nu}\, \hat{k}^\rho \delta(\sigma - \bar{\sigma}) - \kappa \mathcal{Q}_\mu^{\ \nu\rho} i_\rho \delta(\sigma - \bar{\sigma})
\end{aligned}$$

- However, there is no invariance under T-duality. This is the consequence of non-commutativity of two transformations

$$e^{\hat{\theta}} e^{\hat{B}} \neq e^{\hat{B}} e^{\hat{\theta}}$$

Twisting transformations

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- Second idea - twist by $e^{\breve{B}}$

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta \\ 2B & 0 \end{pmatrix}$$

- In this instance, all contributions of Taylor's expansion are non-zero

$$\breve{B}^2 = \begin{pmatrix} 2\kappa(\theta B)^\mu{}_\nu & 0 \\ 0 & 2\kappa(B\theta)_\mu{}^\nu \end{pmatrix}, \quad \breve{B}^3 = \begin{pmatrix} 0 & 2\kappa^2(\theta B\theta)^{\mu\nu} \\ 4\kappa(B\theta B)_{\mu\nu} & 0 \end{pmatrix}$$

$$\breve{B}^{2n} = \begin{pmatrix} (\alpha^n)^\mu{}_\nu & 0 \\ 0 & ((\alpha^T)^n)_\mu{}^\nu \end{pmatrix}, \quad \breve{B}^{2n+1} = \begin{pmatrix} 0 & \kappa(\alpha^n\theta)^{\mu\nu} \\ 2(B\alpha^n)_{\mu\nu} & 0 \end{pmatrix}$$

- α is symmetric parameter

$$\alpha = 2\kappa\theta B\,, \quad (\alpha)^\mu{}_\nu = (\alpha^T)_\nu{}^\mu$$

- Full transformation

$$e^{\breve{B}} = \begin{pmatrix} \left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\right)^\mu{}_\nu & \kappa\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\mu{}_\rho \theta^{\rho\nu} \\ 2B_{\mu\rho}\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\rho{}_\nu & \left(\sum_{n=0}^\infty \frac{(\alpha^T)^n}{(2n)!}\right)_\mu{}^\nu \end{pmatrix}$$

Auxiliary currents

- We can rewrite the twisting transformation in the following way:

$$e^{\breve{B}} = \begin{pmatrix} \mathcal{C} & \kappa \mathcal{S}\theta \\ 2B\mathcal{S} & \mathcal{C}^T \end{pmatrix}, \quad \mathcal{C} = \cosh(\sqrt{\alpha}), \quad \mathcal{S} = \frac{\sinh(\sqrt{\alpha})}{\sqrt{\alpha}}$$

- Currents

$$\breve{X}^M = (e^{\breve{B}})^M{}_N \, X^N = \begin{pmatrix} \breve{k}^\mu \\ \breve{\imath}_\mu \end{pmatrix}$$

$$\breve{k}^\mu = \kappa \mathcal{C}^\mu{}_\nu x'^\nu + \kappa(\mathcal{S}\theta)^{\mu\nu}\pi_\nu = \mathcal{C}^\mu{}_\nu(\kappa x'^\mu + \kappa\breve{\theta}^{\mu\nu}\pi_\nu), \quad \breve{\theta}^{\mu\nu} = (\mathcal{S}\mathcal{C}^{-1})^\mu{}_\rho \theta^{\rho\nu}$$

$$\breve{\imath}_\mu = 2\kappa(B\mathcal{S})_{\mu\nu}x'^\nu + (\mathcal{C}^T)_\mu{}^\nu \pi_\nu = (\mathcal{C}^T)_\mu{}^\nu(\pi_\nu + 2\kappa \breve{B}_{\nu\rho}x'^\rho), \quad \breve{B}_{\mu\nu} = B_{\mu\rho}(\mathcal{S}\mathcal{C}^{-1})^\rho{}_\nu$$

- Parameter α is self T-dual

$$\alpha = (2\kappa\theta B) \cong (2\kappa B\theta) = \alpha^T$$
$$\mathcal{C} \cong \mathcal{C}^T, \quad \mathcal{S} \cong \mathcal{S}^T$$

- T-duality between currents

$$\breve{k}^\mu = \mathcal{C}^\mu{}_\nu\left(\kappa x'^\nu + \kappa\breve{\theta}^{\nu\rho}\pi_\rho\right) \xleftrightarrow{\pi_\mu \cong \kappa x'^\mu \quad B_{\mu\nu} \cong \frac{\kappa}{2}\theta^{\mu\nu}} \breve{\imath}_\mu = (\mathcal{C}^T)_\mu{}^\nu\left(\pi_\nu + 2\breve{B}_{\nu\rho}\kappa x'^\rho\right)$$

Courant bracket simultaneously twisted by B and θ

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- Expression for the full bracket: $[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\breve{B}}} = \Lambda\,, \quad \Lambda_i = \xi_i \oplus \lambda_i$

$$\begin{aligned}\xi \quad &= \quad [\xi_1, \xi_2]_{\hat{L}} - \kappa\breve{\theta}\Big(\hat{\mathcal{L}}_{\xi_1}\lambda_2 - \hat{\mathcal{L}}_{\xi_2}\lambda_1 - \frac{1}{2}\hat{d}(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\ &+[\xi_1, \kappa\breve{\theta}(\lambda_2)]_{\hat{L}} - [\xi_2, \kappa\breve{\theta}(\lambda_1)]_{\hat{L}} + \frac{\kappa^2}{2}[\breve{\theta}, \breve{\theta}]_{\hat{S}}(\lambda_1, \lambda_2, .) \\ &+2\kappa\breve{\theta}\ \hat{d}\hat{B}(., \xi_1, \xi_2) - 2\wedge^2 \kappa\breve{\theta}\ \hat{d}\hat{B}(., \lambda_1, \xi_2) + 2\wedge^2 \kappa\breve{\theta}\ \hat{d}\hat{B}(., \lambda_2, \xi_1) + 2\wedge^3 \kappa\breve{\theta}\ \hat{d}\hat{B}(\lambda_1, \lambda_2, .)\end{aligned}$$

$$\begin{aligned}\lambda \quad &= \quad \hat{\mathcal{L}}_{\xi_1}\lambda_2 - \hat{\mathcal{L}}_{\xi_2}\lambda_1 + \frac{1}{2}\hat{d}(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + \kappa[\lambda_1, \lambda_2]_{\breve{\theta}} \\ &+2\hat{d}\hat{B}(\xi_1, \xi_2, .) - 2\kappa\breve{\theta}\ \hat{d}\hat{B}(\lambda_2, ., \xi_1) + 2\kappa\breve{\theta}\ \hat{d}\hat{B}(\lambda_1, ., \xi_2) + 2\wedge^2 \kappa\breve{\theta}\ \hat{d}\hat{B}(\lambda_1, \lambda_2, .)\end{aligned}$$

- Twisted Lie bracket: $\mathcal{C}\Big([\xi_1, \xi_2]_{\hat{L}}\Big) = \mathcal{C}^{-1}[\mathcal{C}\xi_1, \mathcal{C}\xi_2]_L$
- Twisted Koszul bracket

$$[\lambda_1, \lambda_2]_{\breve{\theta}} = \hat{\mathcal{L}}_{\breve{\theta}(\lambda_1)}\lambda_2 - \hat{\mathcal{L}}_{\breve{\theta}(\lambda_2)}\lambda_1 - \hat{d}(\breve{\theta}(\lambda_1, \lambda_2))$$

- Twisted Schouten-Nijenhuis bracket

$$[f, g]_{\hat{S}} = 0\,, \quad [\xi, f]_{\hat{S}} = \mathcal{L}_{\mathcal{C}\xi}(f)\,, \quad [\xi_1, \xi_2]_{\hat{S}} = [\xi_1, \xi_2]_{\hat{L}}$$

$$[\theta_1, \theta_2 \wedge \theta_3]_{\hat{S}} = [\theta_1, \theta_2]_{\hat{S}} \wedge \theta_3 + (-1)^{(p-1)q}\theta_2 \wedge [\theta_1, \theta_3]_{\hat{S}}\,, \quad [\theta_1, \theta_2]_{\hat{S}} = -(-1)^{(p-1)(q-1)}[\theta_2, \theta_1]_{\hat{S}}$$

Fluxes

arxiv:2312.11268

- We obtain all of the fluxes

$$\{\breve{X}^M, \breve{X}^N\} = -\breve{F}^{MN}{}_P\, \breve{X}^P \delta(\sigma - \bar{\sigma}) + \kappa \eta^{MN} \delta'(\sigma - \bar{\sigma})$$

$$\breve{F}^{MN\rho} = \begin{pmatrix} \kappa^2 \breve{\mathcal{R}}^{\mu\nu\rho} & -\kappa \breve{\mathcal{Q}}_\nu{}^{\mu\rho} \\ \kappa \breve{\mathcal{Q}}_\mu{}^{\nu\rho} & \breve{\mathcal{F}}_{\mu\nu}^{\ \ \rho} \end{pmatrix}, \qquad \breve{F}^{MN}{}_\rho = \begin{pmatrix} \kappa \breve{\mathcal{Q}}_\rho{}^{\mu\nu} & \breve{\mathcal{F}}_{\nu\rho}^{\ \ \mu} \\ -\breve{\mathcal{F}}_{\mu\rho}^{\ \ \nu} & 2\breve{\mathcal{B}}_{\mu\nu\rho} \end{pmatrix}$$

- H-flux $\breve{\mathcal{B}}_{\mu\nu\rho} = (\mathcal{C}^T)_\mu{}^\alpha (\mathcal{C}^T)_\nu{}^\beta (\mathcal{C}^T)_\rho{}^\gamma (\partial_\alpha \breve{B}_{\beta\gamma} + \partial_\beta \breve{B}_{\gamma\alpha} + \partial_\gamma \breve{B}_{\alpha\beta})$
- F-flux $\breve{\mathcal{F}}_{\mu\nu}^{\ \ \rho} = \breve{f}_{\mu\nu}^{\ \ \rho} - 2\kappa \breve{\mathcal{B}}_{\mu\nu\sigma} \breve{\theta}^{\sigma\rho}$, $\quad \breve{f}_{\mu\nu}^{\ \ \rho} = (\mathcal{C}^{-1})^\rho{}_\sigma \left(\hat{\partial}_\mu \mathcal{C}^\sigma{}_\nu - \hat{\partial}_\nu \mathcal{C}^\sigma{}_\mu \right)$
- Q-flux $\breve{\mathcal{Q}}_\rho{}^{\mu\nu} = \breve{Q}_\rho{}^{\mu\nu} + 2\kappa \breve{\theta}^{\mu\alpha} \breve{\theta}^{\nu\beta} \breve{\mathcal{B}}_{\rho\alpha\beta}$, $\quad \breve{Q}_\rho{}^{\mu\nu} = \hat{\partial}_\rho \breve{\theta}^{\mu\nu} + \breve{f}_{\rho\sigma}^{\ \ \mu}\, \breve{\theta}^{\sigma\nu} - \breve{f}_{\rho\sigma}^{\ \ \nu}\, \breve{\theta}^{\sigma\mu}$
- R-flux $\breve{\mathcal{R}}^{\mu\nu\rho} = \breve{R}^{\mu\nu\rho} + 2\kappa \breve{\theta}^{\mu\alpha} \breve{\theta}^{\nu\beta} \breve{\theta}^{\rho\gamma} \breve{\mathcal{B}}_{\alpha\beta\gamma}$
- $\breve{R}^{\mu\nu\rho} = \breve{\theta}^{\mu\sigma} \hat{\partial}_\sigma \breve{\theta}^{\nu\rho} + \breve{\theta}^{\nu\sigma} \hat{\partial}_\sigma \breve{\theta}^{\rho\mu} + \breve{\theta}^{\rho\sigma} \hat{\partial}_\sigma \breve{\theta}^{\mu\nu} - (\breve{\theta}^{\mu\alpha} \breve{\theta}^{\rho\beta} \breve{f}_{\alpha\beta}^{\ \ \nu} + \breve{\theta}^{\nu\alpha} \breve{\theta}^{\mu\beta} \breve{f}_{\alpha\beta}^{\ \ \rho} + \breve{\theta}^{\rho\alpha} \breve{\theta}^{\nu\beta} \breve{f}_{\alpha\beta}^{\ \ \mu})$

Summary

- Courant bracket governs diffeomorphisms and local gauge transformations for closed bosonic string.
- The twisted Courant brackets can be obtained from generator algebra.
- Courant bracket can be simultaneously twisted by B and θ, this way all fluxes are obtained, and the bracket is invariant under T-duality. All fluxes depend on both B and θ.
- It would be interesting to see if there is some important mathematical or physical interpretation of fluxes.
- In addition, it would be interesting to see some characteristics of Courant algebroid associated with the Courant bracket simultaneously twisted by B and θ.

Thank you for your attention