Workshop on NC and Gen. Geom. in Physics 09/2024

Sheaf Topos Theory as a Setting for Physics

based on

"Field Theory via Higher geometry I: Smooth sets of Fields"

arXiv:2312.16301 (~120 p.)

– w Hisham Sati

and soon

Part II (synthetic/infinitesimal), Part III (fermionic/super)

Part IV (higher)

2311.11026 short intro

– w Hisham Sati, Urs Schreiber.

see also

[2403.16456, 2406.11304, 2408.09921]

Problem #1

Lagrangian Field Theory (Variational problems)

- In (bosonic) classical field theory:

(i) "smooth space" of fields

$$F := C^\infty(M, N) \quad \text{(generally } \Gamma_M(F) \text{ etc.)}$$

(ii) "smooth" (local) action functional

$$S : F \longrightarrow \mathbb{R}$$
$$\phi \longmapsto \int_M L(\phi) = \int_M L(\phi, \partial\phi, \dots, \partial^k\phi) \cdot dx^1 \cdots dx^n$$

with extrema $\phi_0 \in F_0 \hookrightarrow F$

calc. of Variations

s.t. $\frac{\partial}{\partial t}(S \circ \phi_t)\Big|_{t=0} = 0$ (⇔ .... $EL(\phi_0) = 0$)

$\forall$ "smooth paths"

$\underbrace{\phi_t : \mathbb{R}^1 \longrightarrow F}$ through $\phi_0 \in F$.

⚠ usually taken to be $\phi_t \in C^\infty(M \times \mathbb{R}^1, N)$

Similarly $\phi^k \in C^\infty(M \times \mathbb{R}^k, N)$

Needs

(i) "Smooth" structure on mapping spaces $\mathcal{F}$
($\phi_t : \mathbb{R}^1 \to \mathcal{F}$ smooth)

(ii) Local action functionals
$S : \mathcal{F} \longrightarrow \mathbb{R}$ are smooth
($\Rightarrow S \circ \phi_t : \mathbb{R} \to \mathbb{R}$ smooth)

Category of Gen. Smooth Spaces.

(iii) Rigorous Calc. of Variations
(Infinite jet bundles)

(iv) Extrema $\mathcal{F}_0 \hookrightarrow \mathcal{F}$ form smooth subspace.
($\Rightarrow$ smooth local observables on classical configurations)

problem #2

Fermionic fields

- In physical field theories, there exist "odd" "fermionic" fields ψ

  s.t. $\boxed{\psi^\alpha \cdot \psi^\beta = -\psi^\beta \cdot \psi^\alpha}$

  [Dirac electron field $\int_M \{\bar{\psi} \gamma^\mu \partial_\mu \psi\}$]

- **Example:** Fermionic Particle on a line $\mathbb{R}$ (time)

$$S_f = \int_{\mathbb{R}} \psi \cdot \frac{d\psi}{dt} \cdot dt$$

EL

$$EL(\psi) = \frac{d\psi}{dt} = 0$$

Note: If ψ not fermionic $\Rightarrow$ $S_f = \int \frac{d}{dt}(\psi^2) \cdot dt$

trivial!

Q: What is the field space F_{ferm} ?

Attempts:

(i) $C^\infty(\mathbb{R}, \mathbb{R})$? ✗ not anticom, action trivial

(ii) $\mathrm{Hom}_{SMan}(\mathbb{R}^1, \mathbb{R}_{odd}) \cong \{0\}$ ✗ trivial

SM

$\mathrm{Hom}_{sAlg}(\mathcal{O}(\mathbb{R}_{odd}), \mathcal{O}(\mathbb{R}^1))$

(##) $\mathrm{Hom}_{\mathrm{SMan}}(\mathbb{R}^1 \times \underbrace{\mathbb{R}^{Aux}_{odd}}_{\theta}, \mathbb{R}_{odd})$ $\quad$ [c.f. $\Phi_t \in C^\infty(M \times \mathbb{R}_t, N)$]

$$\underset{Set}{\cong} \mathrm{Hom}_{sAlg}\left(\mathcal{O}(\mathbb{R}_{odd}), \mathcal{O}(\mathbb{R}^1 \times \underset{\theta}{\mathbb{R}^{Aux}_{odd}})\right)$$

$$\cong \mathrm{Hom}_{sAlg}\Big(\underset{\substack{\mathbb{R}\oplus\mathbb{R}\cdot b \\ S^1\text{weet}}}{\mathbb{R}[\overset{odd}{b}]}, C^\infty(\mathbb{R}^1)\otimes\mathbb{R}[\theta]\Big)$$

$$b \longmapsto \theta \cdot f(t)$$

$$\cong \Big\{ f(t)\cdot \underbrace{\theta}_{odd} \;\Big|\; f(t) \in C^\infty(\mathbb{R}^1) \Big\}$$

$$\underbrace{\qquad}_{\text{" } \varphi_{orz} \text{ "}}$$

$$\cong C^\infty(\mathbb{R}^1)\cdot\theta$$ non-trivial and "odd"

but then $\varphi_{orz} \cdot \frac{d}{dt}\varphi_{orz} = f\,\frac{df}{dt}\cdot\theta^2 = 0$

$\Longrightarrow$ $S_f = 0.$ trivial

(iv) Similarly, $\mathrm{Hom}_{\mathrm{SMan}}(\mathbb{R}^1 \times \underbrace{\mathbb{R}^{2\ \mathrm{Aux}}_{\mathrm{odd}}}_{\theta_1, \theta_2}, \mathbb{R}_{\mathrm{odd}})$

$$\cong C^\infty(\mathbb{R}^1)\cdot\theta_1 \oplus C^\infty(\mathbb{R}^1)\cdot\theta_2$$

So $\psi_{0|2} = f_1(t)\cdot\theta_1 + f_2(t)\cdot\theta_2$

$S(\psi^{0|2}) = \cdots = \int \left(f_1(t)\cdot\frac{df_2}{dt} - f_2\cdot\frac{df_1}{dt}\right)\theta_1\cdot\theta_2$

$\neq 0, \neq d(\cdots)$

- Takeaway: – S_f $\mathcal{O}(2)$ – polynomial)
  - ⇝ need 2 "auxilliary coord." $\mathbb{R}^{2\ \mathrm{aux}}_{\mathrm{odd}}$
  - – Higher polynomials
  - ⇝ more auxillary coords !

[Q]: • What is ψ? What is its "field space" $\mathcal{F}_f$?

• Also need smooth structure!

Super Smooth Sets

- **Basic principle** (Topos Theory) [Grothendieck (alg. geom), Lawvere (synthetic logic), Lurie (conversion), Schreiber (∞-version in field th…)]

✗ **Not** set of points + structure

✓ **But** purely operationally, by giving meaning to 'probe the would-be space.'

In particular, • what are smooth paths in field space?
• → $\mathbb{R}^{0|2}$-parametrized elements?
→ $\mathbb{R}^{0|2}$ → →

Definition

A **super smooth set** $\mathcal{X}$ is a (pre) sheaf

$$\mathcal{X} : \mathrm{SCart}^{op}_{f.d.} \longrightarrow \mathrm{Set}$$

on SCart (w.r.t. the open coverage (Diff. good)).

$\mathbb{R}^{p|q}$
 w/ $\mathcal{O}(\mathbb{R}^{p|q}) := C^\infty(\mathbb{R}^p) \otimes \mathbb{R}[\theta_1, \ldots, \theta_q]$.

Intuition on Super Smooth Sets

On objects

(i) For $\Sigma \in \mathrm{SCart}$, (probe)

$X(\Sigma) \in \mathrm{Set}$

the "(smooth) Σ-shaped plots in X."

$$\phi_\Sigma \in X(\Sigma) \equiv \mathrm{Plots}(\Sigma, X)$$

$$\leftrightsquigarrow$$

$$\text{"}\Sigma \xrightarrow[\Sigma\text{-plot}]{\phi_\Sigma} X\text{'}$$

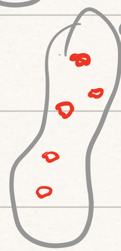
e.g.

- $\Sigma = *$ $\rightsquigarrow$ $X(*) \equiv \mathrm{Plots}(*, X)$

  the "points in X"

- $\Sigma = \mathbb{R}^1$ $\rightsquigarrow$ $X(\mathbb{R}^1) \equiv \mathrm{Plots}(\mathbb{R}^1, X)$

  the "smooth lines in X"



- $\Sigma = \underset{\ominus}{\mathbb{R}^{0|1}} = \mathbb{R}_{\mathrm{odd}}$ $\rightsquigarrow$ $X(\mathbb{R}^{0|1})$

  the "odd-points in X"
  (lines)
  lines

On morphisms

[skip]

(ii) For $f : \Sigma' \longrightarrow \Sigma$ in SCart,

(smooth map of probes)

$$f^* \equiv \mathcal{X}(f) : \underset{\text{Plots}(\Sigma, \mathcal{X})}{\underset{\shortparallel}{\mathcal{X}(\Sigma)}} \longrightarrow \underset{\text{Plots}(\Sigma, \mathcal{X})}{\mathcal{X}(\Sigma')} \quad \text{in Set}$$

the "precomposition of Σ-plots"

$$\text{`}\Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}\text{'}$$

$$\downarrow$$

$$\text{``}\Sigma' \xrightarrow{f} \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}\text{'}$$

$$\begin{tikzcd} \Sigma' \arrow[r, "f"] \arrow[rr, bend left, dashed, "f^* \Phi_\Sigma"] & \Sigma \arrow[r, "\Phi_\Sigma"] & \mathcal{X} \end{tikzcd}$$

- Functoriality:

a) $(id_\Sigma)^* = id_{\mathcal{X}(\Sigma)}$ b) $(f \circ g)^* = g^* \circ f^*$

$$\Longleftrightarrow$$

Consistent interpretation

$\Sigma \xrightarrow{id_\Sigma} \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}$ $\quad$ $(\Sigma'' \xrightarrow{f} \Sigma' \xrightarrow{g} \Sigma) \xrightarrow{\Phi_\Sigma} \mathcal{X}$

$= \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}$ $\quad$ $= \Sigma'' \xrightarrow{f} (\Sigma' \xrightarrow{g} \Sigma \xrightarrow{\Phi_\Sigma} \mathcal{X}$

Super smooth maps between super smooth sets

SKIP

Definition

(Gen. super smooth spaces).

The category of super smooth sets is

$$sSmSet := Sh(sCart)$$

with morphisms nat. transformations

- A 'smooth' map $P: X \longrightarrow Y$ should map (smooth) Σ-plots of X to Σ-plots of Y:

$$P_\Sigma : \underbrace{X(\Sigma)}_{\text{set}} \longrightarrow \underbrace{Y(\Sigma)}_{\text{set}}$$

for each $\Sigma \in SMan$.

- It should do so consistently with pullback of probes:

$$\begin{tikzcd} X(\Sigma) \arrow[r, "P_\Sigma"] \arrow[d, "f^*_X"'] \arrow[dr, phantom, "\circlearrowright"] & Y(\Sigma) \arrow[d, "f^*_Y"] \\ X(\Sigma') \arrow[r, "P_{\Sigma'}"'] & Y(\Sigma') \end{tikzcd}$$

commutes $\forall f : \Sigma' \to \Sigma$

Example: Super manifolds as super smooth sets

- There is a functor

$$\begin{aligned} y : \mathrm{SMan} &\longrightarrow \mathrm{sSmSet} \\ M &\longmapsto y(M) = \mathrm{Hom}_{\mathrm{sMan}}(-, M) \end{aligned}$$

- **Consistency:** Let $\mathcal{X} \in \mathrm{sSmSet}$ and $\Sigma \in \mathrm{sCart}$.

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sSmSet}}(y(\Sigma), \mathcal{X}) &\xrightarrow{\sim} \mathcal{X}(\Sigma) \equiv \mathrm{Plot}(\Sigma, \mathcal{X}) \\ \mathcal{P} &\longmapsto \mathcal{P}(\mathrm{id}_\Sigma) \end{aligned}$$

Yoneda lemma

$$\left\ulcorner \begin{aligned} y(M)(M) = \mathrm{Plots}(M, M) &\longmapsto \mathrm{Plots}(M, \mathcal{X}) \\ \mathrm{id}_M &\longmapsto \mathcal{P}_M(\mathrm{id}_M) \end{aligned} \right\lrcorner$$

- In particular, for $\mathcal{X} = y(N)$ then

$$\mathrm{Hom}_{\mathrm{sSmSet}}(y(\Sigma), y(N)) \cong y(N)(\Sigma) := \mathrm{Hom}_{\mathrm{sMan}}(\Sigma, N)$$

$$\boxed{y : \mathrm{sMan} \xhookrightarrow[\text{f.f.}]{} \mathrm{sSmSet}}$$

f.d.

Super Smooth sets of fields

For any $M, N \in \mathrm{sMan}$ [e.g. M, N s.d. manifolds "σ-models"
$\mathbb{R}, \mathbb{R}_{odd}$ fermionic particle

- Want super smooth set $F : \mathrm{sCartSp}^{op} \longrightarrow \mathrm{Set}$.

$\exists$ obvious assignment

$$\mathbb{R}^{p|q} \longmapsto \mathrm{Hom}_{\mathrm{sMan}}(M \times \mathbb{R}^{p|q}, N) \cong \mathrm{Hom}_{\mathrm{sAlg}}(\mathcal{O}(N), \mathcal{O}(M) \otimes \mathcal{O}(\mathbb{R}^{p|q}))$$

(indeed a sheaf.)

Fact

- For any $X, Y \in \mathrm{sSmSet}$ $\exists$ $[X, Y] \in \mathrm{sSmSet}$ "Internal Hom"

- For M, N bosonic.

$$F(\mathbb{R}^1) = C^\infty(M \times \mathbb{R}^1, N) \xhookrightarrow{\text{set}} \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^1, C^\infty(M, N))$$

$$\phi(x, t) \longmapsto \hat{\phi}(t)(-) = \phi(-, t)$$

recovers 'smooth path in F'.

simply as the $\mathbb{R}^1$-plots of field space $[M, N]$

- For $\mathbb{R}$, $\mathbb{R}^{odd}$,

$$\tilde{F}(\mathbb{R}^{0|2}) = \mathrm{Hom}_{\mathrm{SMan}}(\mathbb{R} \times \underbrace{\mathbb{R}^{0|2}}_{\text{"aux" } \theta_1, \theta_2}, \mathbb{R}^{odd})$$

recovers "fermion fields w auxiliary coords"

as $\mathbb{R}^{0|2}$-plots of field space

$[\mathbb{R}, \mathbb{R}^{odd}]$. ! has no "points"

$\Rightarrow$ $S_f = \int \psi \partial_t \psi$ non-trivial as

map of sheafs.

! Takeaway !

Classical field theory naturally takes place in the topos of spaces probeable by $\mathbb{R}^{p|q}$

(*) ∃ Generalizations:

(inf disks) • $D^n(k)$ ⇝ i) $T\mathcal{F} \overset{i/}{=} [\mathbb{D}^1(1), \mathcal{F}] \cong [M, TN] \cong \Gamma_M(VF)$ ≙

ii) $T\mathcal{F}_{EL} \hat{=}$ "Jacobi fields"

iii) Inf. $\mathbb{D}\phi \hookrightarrow \mathcal{F}$

(Simplices) • Δ^k ⇝ super/smooth ∞-groupoids

$$\mathcal{F}(\Delta^1) \equiv \{ \phi \xrightarrow{\sim} \phi' \}$$

(finite) gauge equiv.

$$\mathcal{F}(\Delta^2) \equiv \{ \phi \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \phi' \}$$

(finite) gauge of gauge.