B-type anomaly coefficients of holographic defects

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based on my work with M. de Leeuw, C. Kristjansen and M. Volk, PLB **846** (2023) 138235 [arxiv:2307.10946], as well as work in progress

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Section 1

Introduction

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- Weak/strong coupling dilemma: gauge and the string theory couplings are inversely proportional... the two perturbative regimes are disconnected from each other... testing AdS/CFT is practically impossible!

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which contains, for all values of the coupling constant λ, the scaling dimensions Δ of any local gauge invariant operator of $\mathcal{N} = 4$, SYM...

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- According to the *dictionary* of the AdS/CFT duality, the above operators of $\mathcal{N} = 4$, SYM are dual to type IIB string theory states in $\mathrm{AdS}_5 \times \mathrm{S}^5$...

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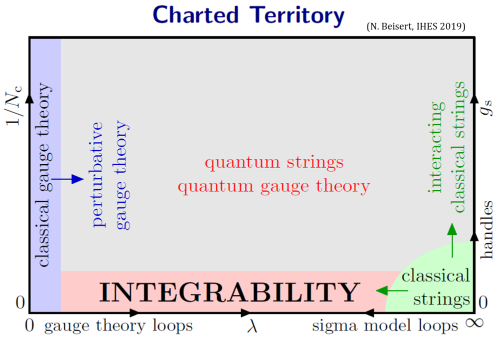

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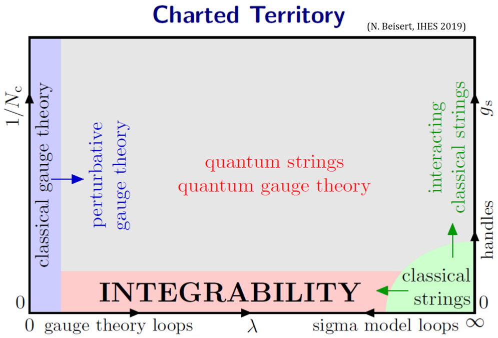

- Ideally, we would like to solve the theory… not only its spectrum… where by *solve* we mean the calculation of the theory's observables: spectrum, correlation functions, scattering amplitudes, Wilson loop expectation values, etc…

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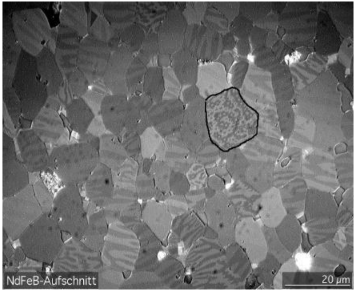

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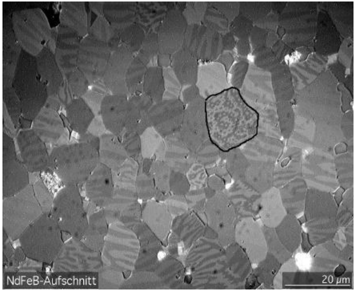

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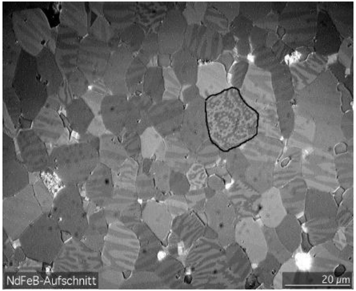

- The real-world gauge theories we would like to study at strong coupling (such as QCD) are neither finite, nor supersymmetric, nor integrable, (or holographic?)... In other words, we need less symmetry!

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- Let us first see how AdS/dCFT is obtained from AdS/CFT...

The AdS/CFT correspondence

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- Spectral problem solved (Gromov-Kazakov-Leurent-Volin, 2013)... solution of full planar theory by computing all observables (correlators, scattering amplitudes, Wilson loops, etc) underway...
- Half-BPS boundary conditions in $\mathcal{N}=4$ SYM were studied by Gaiotto-Witten (2008)...

The AdS/CFT correspondence

The AdS_5/CFT_4 correspondence states that:

$$\mathcal{N}=4,\ \mathfrak{su}(N_c) \text{ super Yang-Mills theory in 4d} \ \Leftrightarrow\ \text{Type IIB superstring theory on } AdS_5\times S^5$$

Maldacena (1997)

Type IIB superstring theory on $AdS_5 \times S^5$ is described by a nonlinear σ-model on a supercoset:

$$AdS_5\times S^5=\frac{SO(4,2)}{SO(4,1)}\times\frac{SO(6)}{SO(5)}\subseteq\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}.$$

Green-Schwarz superstring action on $AdS_5 \times S^5$ is a WZW sigma model (Metsaev-Tseytlin, 1998):

$$S=-\frac{T_2}{2}\int \ell^2 \mathrm{str}\left[J^{(2)}\wedge\star J^{(2)}+J^{(1)}\wedge J^{(3)}\right],\qquad J\equiv\mathfrak{g}^{-1}d\mathfrak{g},\qquad T_2\equiv\frac{1}{2\pi\alpha'}=\frac{\sqrt{\lambda}}{2\pi\ell^2}.$$

The $AdS_5 \times S^5$ supercoset is a semi-symmetric space, i.e. its elements afford a $\mathbb{Z}_4$ decomposition:

$$J=J^{(0)}+J^{(1)}+J^{(2)}+J^{(3)},\qquad \Omega\left[J^{(n)}\right]=i^nJ^{(n)},\qquad \Omega(M)=-\mathcal{K}M^{\mathrm{st}}\mathcal{K}^{-1},\quad \mathcal{K}=\begin{bmatrix}\gamma_{13} & 0\\ 0 & \gamma_{13}\end{bmatrix}.$$

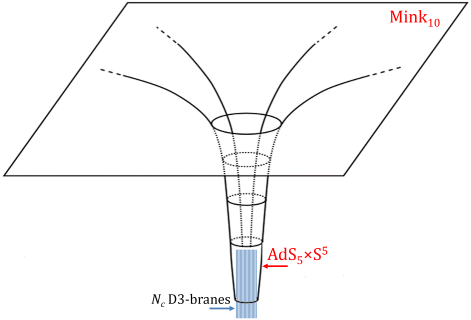
Nonlinear sigma models on semi-symmetric spaces are classically integrable (Bena-Polchinski-Roiban, 2003)...

Section 2

Probe-brane defect systems

The D3-D5 system: bulk geometry

Type IIB string theory on $AdS_5 \times S^5$ is encountered very close to a system of N_c coincident D3-branes:



The D3-branes extend along x_1, x_2, x_3...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						

The D3-D5 system: bulk geometry

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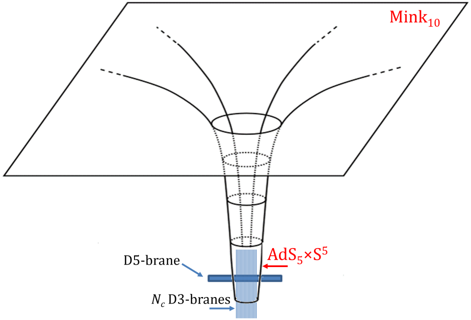

Now insert a single (probe) D5-brane at $x_3 = x_7 = x_8 = x_9 = 0$...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D5	•	•	•		•	•	•			

The D3-D5 system: bulk geometry

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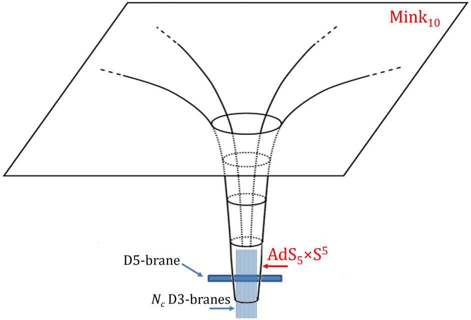
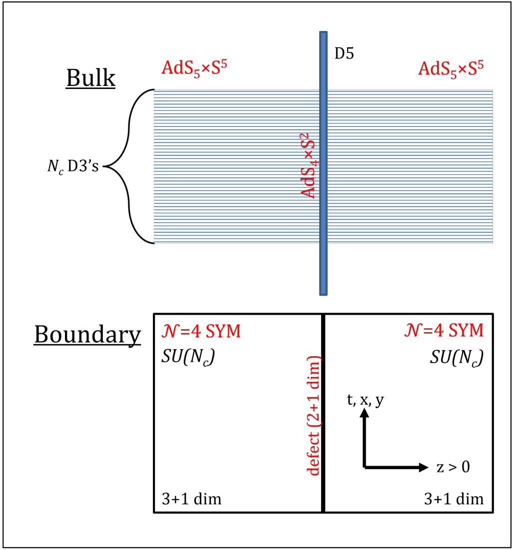


Now insert a single (probe) D5-brane at $x_3 = x_7 = x_8 = x_9 = 0$...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D5	•	•	•		•	•	•			

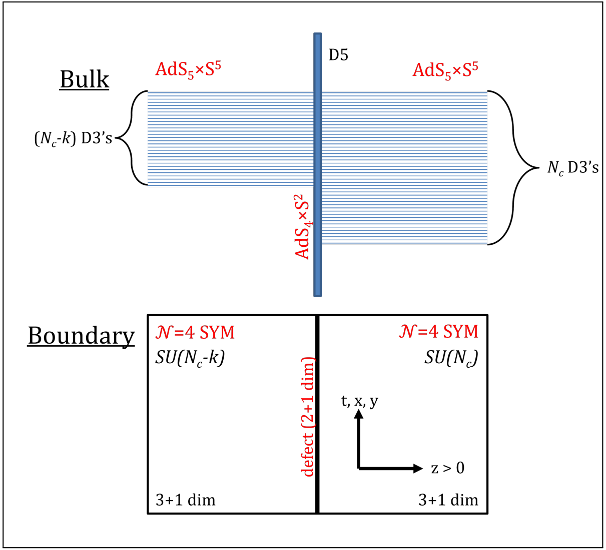
... its geometry will be $AdS_4 \times S^2$ (Karch-Randall, 2001b)...

The D3-D5 system: description



- The defect reduces the total bosonic symmetry of the system from $SO(4,2) \times SO(6)$ to $SO(3,2) \times SO(3) \times SO(3)$. The corresponding superalgebra $\mathfrak{psu}(2,2|4)$ becomes $\mathfrak{osp}(4|4)$. Supersymmetry studied by Domokos-Royston (2022)...
- The D3-D5 system describes IIB string theory on $\text{AdS}_5 \times \text{S}^5$ bisected by a D5 brane with worldvolume geometry $\text{AdS}_4 \times \text{S}^2$.
- The D5-brane is stable... the tachyonic instability in the fluctuations of ψ does not violate the BF bound (Karch-Randall, 2001b)...
- The probe D5-brane is classically integrable... i.e. infinite conserved charges for open strings with D5-brane BCs (Dekel-Oz, 2011)...
- The dual field theory is still $SU(N_c)$, $\mathcal{N} = 4$ SYM in $3+1$ dimensions, that interacts with a CFT living on the $2+1$ dimensional defect: $S = S_{\mathcal{N}=4} + S_{2+1}$ (DeWolfe-Freedman-Ooguri, 2001).
- $\mathcal{N} = 4$ spin chain not modified by the presence of the defect... open spin chain ending on defect fields remains integrable (DeWolfe-Mann, 2004)...

The $(\text{D3-D5})_k$ dSCFT



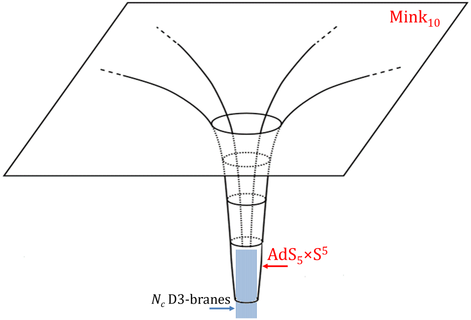
- Despite stability, add $k \neq 0$ units of background magnetic flux over S^2... brane geometry $AdS_4 \times S^2$...
- D5-brane with flux preserves classical integrability of open strings (Zarembo-GL, 2021)...
- The SCFT gauge group $SU(N_c) \times SU(N_c)$ breaks to $SU(N_c - k) \times SU(N_c)$...
- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs (Karch-Randall, 2001b)... dCFT correlators = Higgs condensates of gauge-invariant operators of $\mathcal{N} = 4$ SYM (Nagasaki-Yamaguchi, 2012)...
- Matrix product states... overlaps with Bethe states... Scalar one-point functions (de Leeuw, Kristjansen, Zarembo, 2015)... closed-form det formulas... integrable quench criteria satisfied (Piroli, Pozsgay, Vernier, 2017; de Leeuw-Kristjansen-GL, 2018)...
- Two-point functions of (spin-2) stress tensor, displacement operator, anomaly coefficients (de Leeuw-Kristjansen-GL-Volk 2023)... More below!
- Strong-coupling computations were recently set up (Georgiou-GL-Zoakos, 2023)...

Subsection 2

The D3-D7 probe-brane system

The D3-D7 system: bulk geometry

IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is encountered very close to a system of N_c coincident D3-branes:

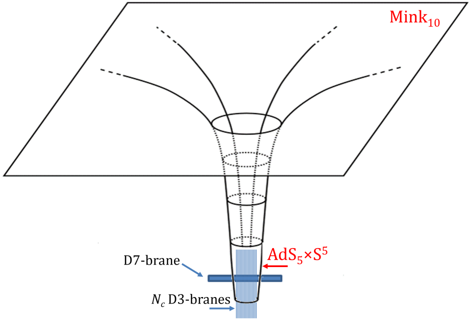


The D3-branes extend along x_1, x_2, x_3...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						

The D3-D7 system: bulk geometry

IIB string theory on $AdS_5 \times S^5$ is encountered very close to a system of N_c coincident D3-branes:

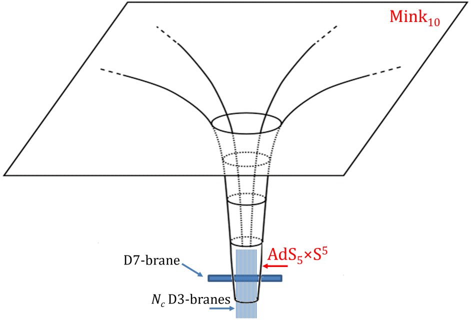


Now insert a single D7-brane at $x_3 = x_9 = 0$...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D7	•	•	•		•	•	•	•	•	

The D3-D7 system: bulk geometry

IIB string theory on $AdS_5 \times S^5$ is encountered very close to a system of N_c coincident D3-branes:

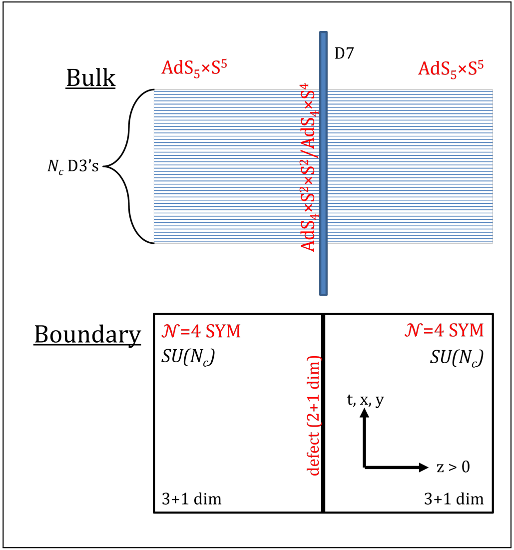


Now insert a single D7-brane at $x_3 = x_9 = 0$... its geometry will be either $AdS_4 \times S^4$ or $AdS_4 \times S^2 \times S^2$...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D7	•	•	•		•	•	•	•	•	

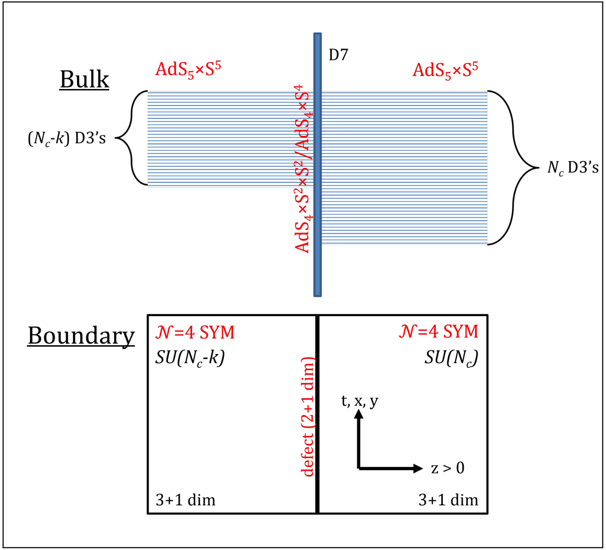
(Davis-Kraus-Shah, 2008; Myers-Wapler, 2008; Bergman-Jokela-Lifschytz-Lippert, 2010)...

The D3-D7 system: description



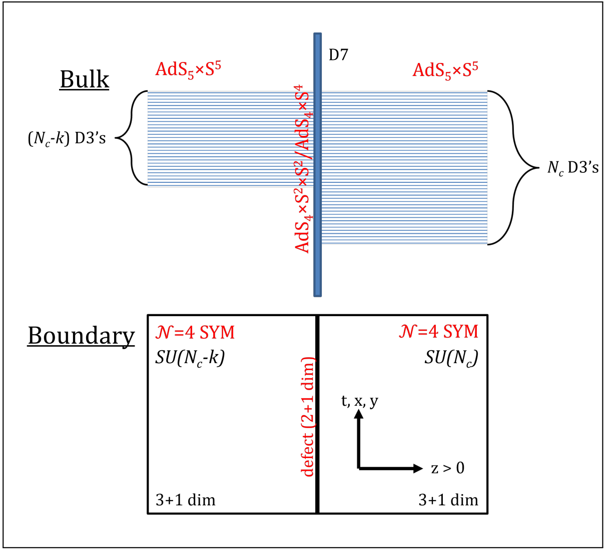
- The defect reduces the total bosonic symmetry of the system from $SO(4,2) \times SO(6)$ to either $SO(3,2) \times SO(5)$ or $SO(3,2) \times SO(3) \times SO(3)$... All susy broken! (relative brane codimension in flat space: $\#_{\mathrm{ND}} = 6 \to$ no unbroken susy)...
- The D3-D7 system describes IIB string theory on $AdS_5 \times S^5$ bisected by a D7-brane with worldvolume geometry $AdS_4 \times S^4$ or $S^2 \times S^2$... maximal S^4 & $S^2 \times S^2$ sit on the equator of S^5...
- The D7-branes are unstable: tachyonic instabilities in fluctuations violate the BF bound (Davis-Kraus-Shah, 2008; Bergman-Jokela-Lifschytz-Lippert, 2010)... S^4 and $S^2 \times S^2$ "slip-off" (either side of) the S^5 equator, collapsing to points...
- Various ways to lift the instability... embed D7 in full D3-brane geometry instead of near-horizon (Davis-Kraus-Shah, 2008)... impose an AdS cutoff Λ (Kutasov-Lin-Parnachev, 2011; Mezzalira-Parnachev, 2015)... add instanton flux on S^4 (Myers-Wapler, 2008), and magnetic flux on $S^2 \times S^2$ (Bergman-Jokela-Lifschytz-Lippert, 2010)...
- The dual field theory is still $SU(N_c)$, $\mathcal{N} = 4$ SYM in $3+1$ dimensions, that interacts with a CFT living on the $2+1$ dimensional defect: $S = S_{\mathcal{N}=4} + S_{2+1}$... boundary degrees of freedom are fermions (Rey, 2009)...

The $(\text{D3-D7})_k$ system



- To stabilize the D7-brane, we add a (non-abelian) instanton bundle through its S^4 component (Myers-Wapler, 2008) and an (abelian) magnetic flux through each S^2 (Bergman-Jokela-Lifschytz-Lippert, 2010)...
- This forces exactly k (flux units) of the N_c D3-branes ($N_c \gg k$) to end on the D7-brane...
- The homogeneous instanton flux is non-abelian... study of classical string integrability hard in the $SO(5)$ symmetric case... the $SU(2) \times SU(2)$ symmetric system is most probably not integrable...
- On the gauge theory side, gauge group $SU(N_c) \times SU(N_c)$ breaks to $SU(N_c) \times SU(N_c - k)$...
- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs... dCFT correlators = Higgs condensates of gauge-invariant operators of $\mathcal{N} = 4$ SYM...
- Matrix product states... overlaps with Bethe states... scalar one-point functions (de Leeuw-Kristjansen-GL, 2016)... integrable quench criteria satisfied in the $SO(5)$ symmetric case (Piroli, Pozsgay, Vernier, 2017; de Leeuw-Kristjansen-GL, 2018)...

The (D3-D7)$_k$ system

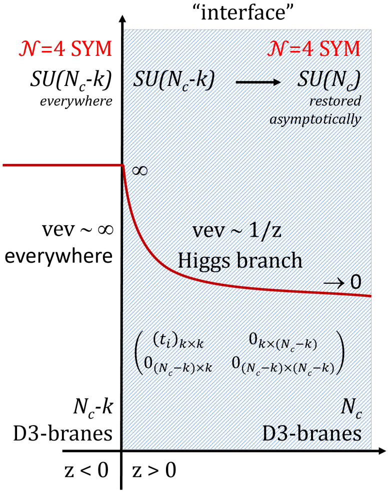


- Yet another sign of integrability of the $SO(5)$ symmetric system are closed-form determinant formulas which have been found for all scalar one-point functions (de Leeuw-Gombor-Kristjansen-GL-Pozsgay, 2019)...
- Weak-coupling analysis also provides evidence of non-integrability for the $SU(2) \times SU(2)$ symmetric system (de Leeuw-Kristjansen-Vardinghus, 2019)...
- Two-point functions of the (spin-2) stress tensor, displacement operator, anomalies... More below...
- Strong-coupling computations were recently set up (Georgiou-GL-Zoakos, 2023)...

Subsection 3

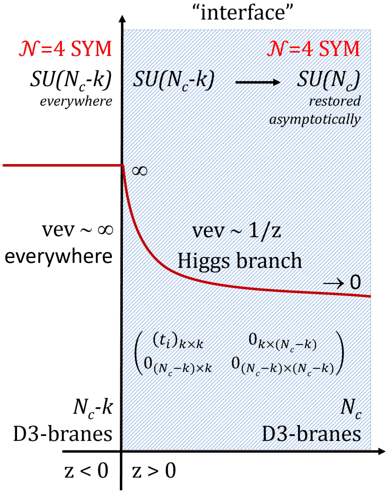
One-point functions

The D3-D5 interface: $SU(2) \times SU(2)$ symmetry



- An interface is a wall between two (different/same) QFTs...
- It can be described by means of classical solutions that are known as "fuzzy-funnel" solutions (Constable-Myers-Tafjord, 1999 & 2001)...

The D3-D5 interface: $SU(2) \times SU(2)$ symmetry



- An interface is a wall between two (different/same) QFTs...
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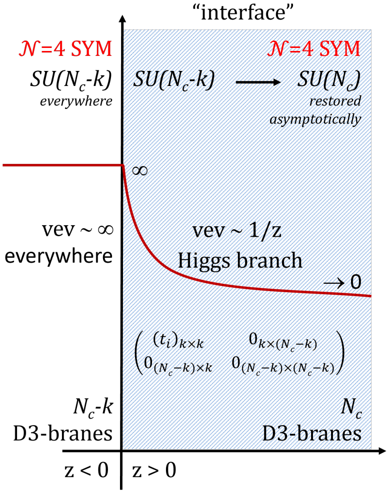
- Here, an interface (situated at $z = 0$) separates the $SU(N_c)$ and $SU(N_c - k)$ regions of the $(\text{D3-D5})_k$ dCFT...

The D3-D5 interface: $SU(2) \times SU(2)$ symmetry



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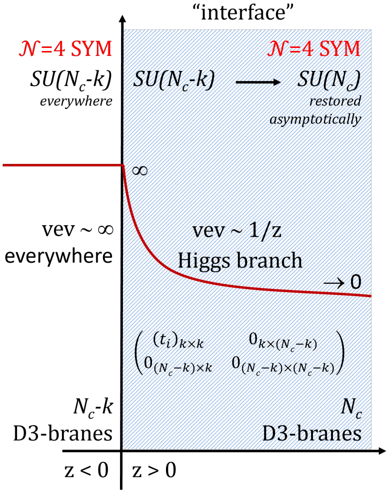
- Here, an interface (situated at $z = 0$) separates the $SU(N_c)$ and $SU(N_c - k)$ regions of the $(\text{D3-D5})_k$ dCFT...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = \left[\varphi_j, \left[\varphi_j, \varphi_i\right]\right], \quad i, j = 1, \ldots, 6.$$

The D3-D5 interface: $SU(2) \times SU(2)$ symmetry



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$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = [\varphi_j, [\varphi_j, \varphi_i]], \quad i, j = 1, \ldots, 6.$$

- A manifestly $SO(3) \simeq SU(2)$ symmetric solution is given by $(z > 0)$:

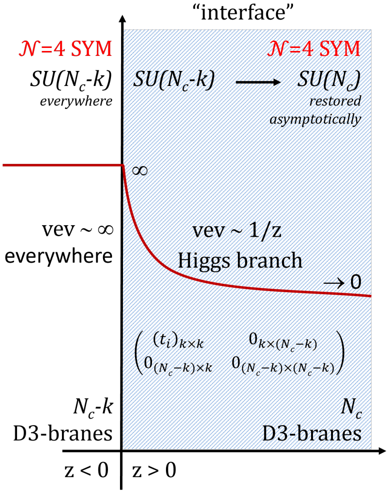
$$\varphi_{2i-1}(z) = \frac{1}{z}\begin{bmatrix} (t_i)_{k\times k} & 0_{k\times(N_c-k)} \\ 0_{(N_c-k)\times k} & 0_{(N_c-k)\times(N_c-k)} \end{bmatrix} \quad \& \quad \varphi_{2i} = 0,$$

Diaconescu (1996), Giveon-Kutasov (1998)

where the matrices t_i furnish a k-dimensional representation of $\mathfrak{su}(2)$:

$$[t_i, t_j] = i\epsilon_{ijk} t_k.$$

The D3-D5 interface: $SU(2) \times SU(2)$ symmetry



- An interface is a wall between two (different/same) QFTs...
- It can be described by means of classical solutions that are known as "fuzzy-funnel" solutions (Constable-Myers-Tafjord, 1999 & 2001)...
- Here, an interface (situated at $z = 0$) separates the $SU(N_c)$ and $SU(N_c - k)$ regions of the (D3-D5)$_k$ dCFT...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = [\varphi_j, [\varphi_j, \varphi_i]], \qquad i, j = 1, \ldots, 6.$$

- A manifestly $SO(3) \simeq SU(2)$ symmetric solution is given by ($z > 0$):

$$\varphi_{2i-1}(z) = \frac{1}{z}\begin{bmatrix} (t_i)_{k\times k} & 0_{k\times(N_c-k)} \\ 0_{(N_c-k)\times k} & 0_{(N_c-k)\times(N_c-k)} \end{bmatrix} \quad \& \quad \varphi_{2i} = 0,$$

Diaconescu (1996), Giveon-Kutasov (1998)

- The solution also satisfies the Nahm equations:

$$\frac{d\varphi_i}{dz} = \frac{i}{2}\epsilon_{ijk}[\varphi_j, \varphi_k],$$

as expected for a half-BPS interface (Gaiotto-Witten, 2008)...

One-point functions

Following Nagasaki & Yamaguchi (2012), the one-point functions of local gauge-invariant scalar operators,

$$\langle \mathcal{O}(\mathrm{z}, \mathbf{x}) \rangle = \frac{\mathcal{C}}{\mathrm{z}^{\Delta}}, \qquad \mathrm{z} > 0,$$

can be calculated within the D3-D5 defect CFT from the corresponding fuzzy-funnel solution, for example:

$$\boxed{\mathcal{O}(z, \mathbf{x}) = \Psi^{\mu_1 \cdots \mu_L} \mathrm{tr}\left[\varphi_{2\mu_1 - 1} \ldots \varphi_{2\mu_L - 1}\right] \xrightarrow[\text{interface}]{SU(2)} \frac{1}{z^L} \cdot \Psi^{\mu_1 \cdots \mu_L} \mathrm{tr}\left[t_{\mu_1} \ldots t_{\mu_L}\right]}$$

where $\Psi^{\mu_1 \cdots \mu_L}$ is an $SO(6)$ symmetric tensor and the constant $\mathcal{C}$ is given by (MPS="*matrix product state*"),

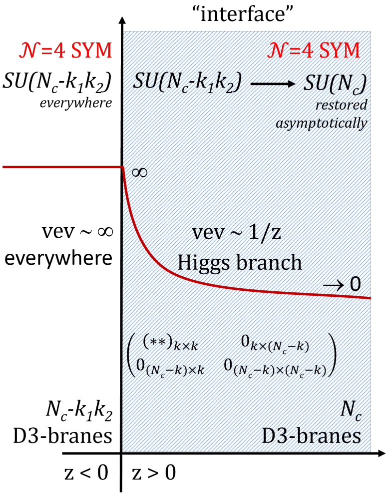
$$\mathcal{C} = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \qquad \left\{ \begin{array}{ll} \langle \mathrm{MPS} | \Psi \rangle \equiv \Psi^{\mu_1 \cdots \mu_L} \mathrm{tr}\left[t_{\mu_1} \ldots t_{\mu_L}\right] & (\text{"overlap"}) \\ \langle \Psi | \Psi \rangle \equiv \Psi^{\mu_1 \cdots \mu_L} \Psi_{\mu_1 \cdots \mu_L} & \end{array} \right\},$$

which ensures that the 2-point function will be normalized to unity ($\mathcal{O} \to (2\pi)^L \left(L\lambda^L\right)^{-1/2} \cdot \mathcal{O}$):

$$\langle \mathcal{O}(\mathrm{x}_1) \, \mathcal{O}(\mathrm{x}_2) \rangle = \frac{1}{|\mathrm{x}_1 - \mathrm{x}_2|^{2\Delta}},$$

within $SU(N_c)$, $\mathcal{N} = 4$ SYM (i.e. without the defect). Once more, we set $\mathrm{x}_i \equiv (\mathrm{z}_i, \mathbf{x}_i)$, where $\mathbf{x}_i \equiv \{\mathrm{x}_i^{(0,1,2)}\}$.

The D3-D7 interface: $SU(2) \times SU(2)$ symmetry



- To compute correlation functions in the dCFT that is dual to the $SU(2) \times SU(2)$ symmetric D3-D7 system, we set up the corresponding interface...
- The interface (placed at $z = 0$) separates the $SU(N_c)$ and $SU(N_c - k_1 k_2)$ regions of the $(\text{D3-D7})_{k_1 k_2}$ dCFT... It will be described by a fuzzy funnel solution...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = [\varphi_j, [\varphi_j, \varphi_i]], \quad i, j = 1, \dots, 6.$$

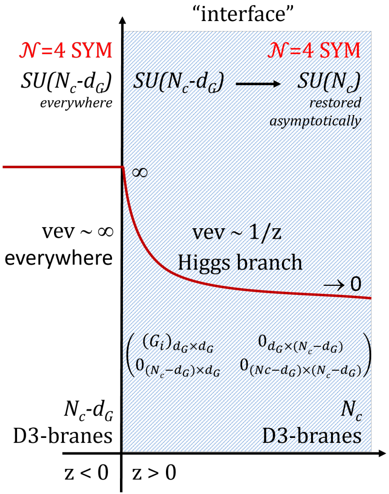
- The wanted $SU(2) \times SU(2) \subset SU(3,2) \times SU(2) \times SU(2)$ solution is:

$$\varphi_i(z) = -\frac{1}{z} \times \begin{cases} \left[(t_i)_{k_1} \otimes \mathbb{1}_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 1, 2, 3 \\ \left[\mathbb{1}_{k_1} \otimes (t_i)_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 4, 5, 6. \end{cases}$$

Kristjansen-Semenoff-Young (2012)

- The defect CFT is not supersymmetric so that the interface does not satisfy the Nahm equations...

The D3-D7 interface: $SO(5)$ symmetry



- The interface for the dCFT that is dual to the $SO(5)$ symmetric D3-D7 system (placed at $z = 0$) separates the $SU(N_c)$ and $SU(N_c - d_G)$ regions of the $(\text{D3-D7})_{d_G}$ dCFT... It will be described by a fuzzy funnel solution...
- For no vectors/fermions, we solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = [\varphi_j, [\varphi_j, \varphi_i]], \quad i, j = 1, \ldots, 6.$$

- A manifestly $SO(5) \subset SO(3,2) \times SO(5)$ symmetric solution is given by:

$$\varphi_i(z) = \frac{G_i \oplus 0_{(N_c-d_G)\times(N_c-d_G)}}{\sqrt{8}\, z}, \quad i = 1, \ldots, 5, \qquad \varphi_6 = 0\,.$$

Kristjansen-Semenoff-Young (2012)

- Once more, the defect CFT is not supersymmetric so that the interface does not satisfy the Nahm equations...
- The five $d_G \times d_G$ matrices G_i are known as the "fuzzy" S^4 matrices...

The fuzzy S^4 G-matrices

The five $d_G \times d_G$ fuzzy S^4 matrices (G-matrices) G_i are given by:

$$G_i \equiv \Bigg[\underbrace{\overbrace{\gamma_i \otimes \mathbb{1}_4 \otimes \ldots \otimes \mathbb{1}_4}^{n \text{ factors}} + \mathbb{1}_4 \otimes \gamma_i \otimes \ldots \otimes \mathbb{1}_4 + \ldots + \mathbb{1}_4 \otimes \ldots \otimes \mathbb{1}_4 \otimes \gamma_i}_{n \text{ terms}}\Bigg]_{\text{sym}} \qquad (i = 1, \ldots, 5),$$

Castelino-Lee-Taylor (1997)

where γ_i are the five 4×4 Euclidean Dirac matrices:

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

and σ_i are the three Pauli matrices. The ten commutators of the five G-matrices,

$$G_{ij} \equiv \frac{1}{2}\left[G_i, G_j\right],$$

furnish a d_G-dimensional (anti-hermitian) irreducible representation of $\mathfrak{so}(5) \simeq \mathfrak{sp}(4)$:

$$\left[G_{ij}, G_{kl}\right] = 2\left(\delta_{jk} G_{il} + \delta_{il} G_{jk} - \delta_{ik} G_{jl} - \delta_{jl} G_{ik}\right).$$

The fuzzy S^4 G-matrices

The dimension of the G-matrices is equal to the instanton number $d_G = (n+1)(n+2)(n+3)/6$:

n	1	2	3	4	5	6	7	8	9	10	...
d_G	4	10	20	35	56	84	120	165	220	286	...

E.g., for $n = 2$, here are the 10×10 G-matrices:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

One-point functions

One-point functions of local gauge-invariant scalar operators,

$$\langle \mathcal{O}(\mathrm{z}, \mathbf{x}) \rangle = \frac{\mathcal{C}}{\mathrm{z}^{\Delta}}, \qquad \mathrm{z} > 0,$$

can again be calculated within the D3-D7 defect CFT from the corresponding fuzzy funnel solution...

$$\boxed{\mathcal{O}(\mathrm{z}, \mathbf{x}) = \Psi^{i_1 \dots i_L} \mathrm{tr}\left[\varphi_{i_1} \dots \varphi_{i_L}\right] \xrightarrow[\text{interface}]{SO(5),\ SO(3)\times SO(3)} \frac{1}{z^L} \cdot \Psi^{i_1 \dots i_L} \mathrm{tr}\left[\tau_{i_1} \dots \tau_{i_L}\right]},$$

where the matrices τ_i are defined in terms of the corresponding fuzzy funnel solution:

$$\tau_i = \begin{cases} \left.\begin{array}{ll} G_i/\sqrt{8}, & i = 1, \dots, 5 \\ 0, & i = 6 \end{array}\right\}, & SO(5) \text{ symmetric interface} \\ \left.\begin{array}{ll} \left[(t_i)_{k_1} \otimes \mathbb{1}_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 1, 2, 3 \\ \left[\mathbb{1}_{k_1} \otimes (t_i)_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 4, 5, 6 \end{array}\right\}, & SO(3) \times SO(3) \text{ symmetric interface.} \end{cases}$$

Again, $\Psi^{i_1 \dots i_L}$ is an $\mathfrak{so}(6)$-symmetric tensor and the constant $\mathcal{C}$ is given by (MPS="*matrix product state*"),

$$\mathcal{C} = \frac{1}{\sqrt{L}} \left(\frac{\pi^2}{\lambda}\right)^{L/2} \cdot \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \qquad \left\{ \begin{array}{ll} \langle \mathrm{MPS} | \Psi \rangle \equiv \Psi^{i_1 \dots i_L} \mathrm{tr}\left[G_{i_1} \dots G_{i_L}\right] & (\text{"overlap"}) \\ \langle \Psi | \Psi \rangle \equiv \Psi^{i_1 \dots i_L} \Psi_{i_1 \dots i_L} & \end{array} \right\}.$$

Section 3

Defect anomaly coefficients

Defect anomalies

Even dimensional CFTs (in curved spacetimes) are afflicted by conformal/Weyl anomalies: the trace of the energy-momentum/stress tensor acquires non-vanishing expectation value that is given by (scheme-independent terms only)...

$$\langle T^{\mu}_{\mu} \rangle^{d=2n} = \frac{4}{d!\,\mathrm{Vol}[S^d]} \times \left[\sum_i c_i\, I_i - (-1)^{d/2} a_d\, E_d \right], \quad n = 1, 2, \ldots$$

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Odd dimensional (compact) spacetimes have no conformal/Weyl (trace) anomalies...

$$\langle T^{\mu}_{\mu} \rangle^{d=2n+1} = 0, \quad n = 1, 2, \ldots$$

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$$\langle T^{\mu}_{\mu} \rangle^{d=2n+1} = \frac{2\delta(z)}{(d-1)!\,\mathrm{Vol}[\mathrm{S}^{d-1}]} \times \left[\sum_j b_j\, \mathrm{I}_j + (-1)^{(d-1)/2} a_d\, \mathring{E}_{d-1} \right], \quad n = 1, 2, \ldots$$

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where E_d, $\mathring{E}_{d-1}$ are the bulk/boundary Euler densities, and $E^{(\mathrm{bry})}$ the boundary term of the Euler characteristic... K_{pq} is the boundary extrinsic curvature, and h_{pq} is the induced metric on the boundary... dimensionalities $d = 5, 6$ not fully classified as of now (no nontrivial CFTs in $d > 6$)...

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$$\hat{K}_{pq} \equiv K_{pq} - \frac{h_{pq}}{d-1} K, \qquad \mathrm{tr}\hat{K}^2 \equiv \mathrm{tr}K^2 - \frac{1}{2} K^2, \qquad \mathrm{tr}\hat{K}^3 \equiv \mathrm{tr}K^3 - K\,\mathrm{tr}K^2 + \frac{2}{9} K^3$$

$$E_4 = \frac{1}{4} \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma}, \qquad E_4^{(\mathrm{bry})} = -4 \delta^{stw}_{pqr} K^p_s \left(\frac{1}{2} R^{qr}_{tw} + \frac{2}{3} K^q_t K^r_w \right)$$

$$h^{\mu\nu} \hat{K}^{\rho\sigma} W_{\mu\nu\rho\sigma} = R^{\nu\rho\sigma}_{\mu} K^{\rho}_{\mu} n^{\nu} n^{\sigma} - \frac{1}{2} R_{\mu\nu} \left(n^{\mu} n^{\nu} K + K^{\mu\nu} \right) + \frac{1}{6} K R, \qquad h^{\mu\rho} \hat{K}^{\nu\sigma} W_{\mu\nu\rho\sigma} = -K^{pq} W_{npnq}.$$

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Anomaly coefficients in free theories

Before calculating the A & B anomaly coefficients for the D3-D5 dCFT, let us go through some results for codimension-1:

- In $d = 2$ the relation of the anomaly coefficient a to the central charge is $c = 12a$... For free scalar & Dirac fields:

$$a^{s=0} = a^{s=1/2} = \frac{1}{12} \qquad \text{(see e.g. Cardy, 2004).}$$

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$$a^{s=0}\big|_{\mathrm{D}} = -\frac{1}{96}, \qquad a^{s=0}\big|_{\mathrm{R}} = \frac{1}{96}, \qquad a^{s=1/2} = 0, \qquad b^{s=0}\big|_{\mathrm{D/R}} = \frac{1}{64}, \qquad b^{s=1/2} = \frac{1}{32}.$$

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(see e.g. Birrell-Davies)... For the boundary charges of free fields, b_1 generally depends on the boundary conditions...

$$b_1^{s=0}\big|_{\mathrm{D}} = \frac{2}{35}, \qquad b_1^{s=0}\big|_{\mathrm{R}} = \frac{2}{45}, \qquad b_1^{s=1/2}\big|_{\mathrm{D/R}} = \frac{2}{7}, \qquad b_1^{s=1}\big|_{\mathrm{D/R}} = \frac{16}{35}$$

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Melmed (1988), Moss (1989)

whereas the (free field) boundary charge b_2 is independent of the BCs and proportional to the bulk central charge c:

$$b_2=8c.$$

Dowker-Schofield (1990)
Fursaev (2015), Solodukhin (2015)

Anomalies as observables (bulk)

All types (A, B, C) of anomaly coefficients show up in CFT and dCFT data... For the bulk charges,

- In $d = 2$, the central charge $c = 12a$ shows up in the two and three-point function of the (traceless) stress tensor:

$$\langle T(\mathfrak{z}_1)\, T(\mathfrak{z}_2)\rangle = \frac{c/2}{(\mathfrak{z}_1 - \mathfrak{z}_2)^4}, \qquad \langle T(\mathfrak{z}_1)\, T(\mathfrak{z}_2)\, T(\mathfrak{z}_3)\rangle = \frac{c}{(\mathfrak{z}_1 - \mathfrak{z}_2)^2 (\mathfrak{z}_2 - \mathfrak{z}_3)^2 (\mathfrak{z}_3 - \mathfrak{z}_1)^2},$$

  where $T \equiv T_{\mathfrak{z}\mathfrak{z}}$, and $\mathfrak{z} \equiv x_1 + i x_2$, $\bar{\mathfrak{z}} \equiv x_1 - i x_2$ are the holomorphic/anti-holomorphic coordinates.

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  On the other hand, the (type A & C) conformal anomaly coefficients become:

$$c = \frac{N_0 + 3N_{1/2} + 12N_1}{120} = \frac{\pi^4 C_T}{40}, \qquad a = \frac{2N_0 + 11N_{1/2} + 124N_1}{720}$$

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  so that in the case of $U(N_c)$, $\mathcal{N} = 4$ SYM, all three coefficients turn out to be equal:

$$a = c = \frac{N_c^2}{4} = \frac{\pi^4 C_T}{40}.$$

Anomalies as observables (boundary)

The boundary charges show up in two and three-point functions of the displacement operator $\mathcal{D}$. In d dimensions,

$$\langle \mathcal{D}(\mathbf{x}_1)\,\mathcal{D}(\mathbf{x}_2)\rangle = \frac{c_{nn}}{\mathbf{x}_{12}^{2d}}, \qquad \langle \mathcal{D}(\mathbf{x}_1)\,\mathcal{D}(\mathbf{x}_2)\,\mathcal{D}(\mathbf{x}_3)\rangle = \frac{c_{nnn}}{\mathbf{x}_{12}^{d}\mathbf{x}_{23}^{d}\mathbf{x}_{31}^{d}}.$$

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It can be shown that the single 3d B-type anomaly coefficient and the two 4d B-type anomaly coefficients are given by:

$$b = \frac{\pi^2}{8}c_{nn}, \qquad b_1 = \frac{2\pi^3}{35}c_{nnn}, \qquad b_2 = \frac{2\pi^4}{15}c_{nn},$$

whereas there is no known relation for the 3d A-type anomaly coefficient a... Interestingly, the displacement operator computations confirm the (old) heat kernel results...

The D3-D5 stress tensor

Let us now compute the anomaly coefficients for the (codimension-1) dCFT that is dual to the D3-D5 probe-brane system... Because we are in 4d, there are 4 of them: the bulk charges c & a and the boundary charges b_1 & b_2...

Start off from the Lagrangian of $\mathcal{N}=4$ SYM...

$$\mathcal{L}_{\mathcal{N}=4} = \frac{2}{g_{\rm YM}^2} \cdot \mathrm{tr}\Bigg\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(D_\mu \varphi_i\right)^2 + i\,\bar{\psi}_\alpha \not{D} \psi_\alpha + \frac{1}{4} \left[\varphi_i, \varphi_j\right]^2 + \\ + \sum_{i=1}^{3} G^i_{\alpha\beta} \bar{\psi}_\alpha \left[\varphi_i, \psi_\beta\right] + \sum_{i=4}^{6} G^i_{\alpha\beta} \bar{\psi}_\alpha \gamma_5 \left[\varphi_i, \psi_\beta\right] \Bigg\}.$$

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$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial^\mu A_\rho}\,\partial_\nu A_\rho + \frac{\partial\mathcal{L}}{\partial\partial^\mu \varphi_i}\,\partial_\nu \varphi_i + \frac{\partial\mathcal{L}}{\partial\partial^\mu \bar{\psi}_\alpha}\,\partial_\nu \bar{\psi}_\alpha + \frac{\partial\mathcal{L}}{\partial\partial^\mu \psi_\alpha}\,\partial_\nu \psi_\alpha - g_{\mu\nu}\mathcal{L}.$$

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$$\Theta_{\mu\nu} = \frac{2}{g_{\text{YM}}^2} \cdot \text{tr}\left\{ - F_{\mu}{}^{\varrho} F_{\nu\varrho} - \frac{2}{3} \left(D_\mu \varphi_i\right)\left(D_\nu \varphi_i\right) + \frac{1}{3}\, \varphi_i\, D_{(\mu} D_{\nu)} \varphi_i + \frac{i}{2}\, \bar{\psi}_\alpha \gamma_{(\mu} \overleftrightarrow{D}_{\nu)} \psi_\alpha \right\} - g_{\mu\nu} \Lambda$$

$$\Lambda \equiv \frac{2}{g_{\text{YM}}^2} \cdot \text{tr}\left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{6} \left(D_\mu \varphi_i\right)^2 - \frac{1}{12} \left[\varphi_i, \varphi_j\right]^2 \right\}, \qquad a_{(\mu\nu)} \equiv \frac{1}{2} \left(a_{\mu\nu} + a_{\nu\mu}\right).$$

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$$\left\langle\Theta_{\mu\nu}\left(\mathrm{x}_1\right)\Theta_{\rho\sigma}\left(\mathrm{x}_2\right)\right\rangle = \frac{640c}{\pi^4\mathrm{x}_{12}^8}\cdot I_{\mu\nu\rho\sigma}\left(\mathrm{x}_1-\mathrm{x}_2\right), \qquad c = \frac{N_c^2}{4},$$

which is found by Wick-contracting the perturbed fields with the $\mathcal{N}=4$ SYM Feynman rules (2 contractions for the LO)...

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To compute the defect anomaly coefficients, we will need only the scalar part of the (improved) stress tensor (since only scalars acquire vevs):

$$\Theta_{\mu\nu(\text{scalars})} = \frac{2}{g_{\text{YM}}^2} \cdot \text{tr}\Big\{ -\frac{2}{3} \left(\partial_\mu \varphi_i\right)\left(\partial_\nu \varphi_i\right) + \frac{1}{3}\, \varphi_i \left(\partial_\mu \partial_\nu \varphi_i\right) + \frac{1}{6}\, g_{\mu\nu} \left[\left(\partial_\varrho \varphi_i\right)^2 + \frac{1}{2} \left[\varphi_i, \varphi_j\right]^2 \right] \Big\}.$$

Stress tensor two-point function

Plugging the fuzzy funnel solution for the D3-D5 interface, we find that the stress tensor one-point function vanishes:

$$\langle \Theta_{\mu\nu} (\mathrm{x}) \rangle = 0,$$

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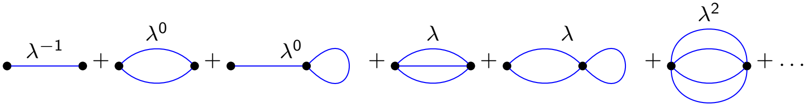
The LO contribution (order λ^{-1}) to the (connected) stress tensor two-point function consists of a single Wick contraction:

$\langle \Theta_{\mu\nu}(\mathrm{x}_1)\, \Theta_{\rho\sigma}(\mathrm{x}_2) \rangle =$ 

By expanding the $\mathcal{N} = 4$ fields around the fuzzy funnel solution of the D3-D5 interface we find:

$$\Theta^{(1)}_{\mu\nu}(x) = \frac{1}{g^2_{\mathrm{YM}}} \frac{4}{3z^2} \cdot \mathrm{tr}\left\{ \left(\frac{1}{z} (n_\mu n_\nu - g_{\mu\nu})\, \tilde{\varphi}_i + n_\mu \partial_\nu \tilde{\varphi}_i + n_\nu \partial_\mu \tilde{\varphi}_i - \frac{g_{\mu\nu}}{2} \partial_3 \tilde{\varphi}_i + \frac{z}{2} \partial_\mu \partial_\nu \tilde{\varphi}_i \right) t_i \right\}.$$

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contracting with the propagator of the D3-D5 dCFT (Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm, 2016)...

$$X_\mu \equiv \mathrm{z}_1\cdot\frac{\upsilon}{\xi}\frac{\partial\xi}{\partial\mathrm{x}_1^\mu} = \upsilon\left(\frac{2\mathrm{z}_1}{\mathrm{x}_{12}^2}(\mathrm{x}_{1\mu}-\mathrm{x}_{2\mu}) - n_\mu\right), \qquad X'_\rho \equiv \mathrm{z}_2\cdot\frac{\upsilon}{\xi}\frac{\partial\xi}{\partial\mathrm{x}_2^\rho} = -\upsilon\left(\frac{2\mathrm{z}_2}{\mathrm{x}_{12}^2}(\mathrm{x}_{1\rho}-\mathrm{x}_{2\rho}) + n_\rho\right).$$

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$$A(\upsilon) = 4\gamma\left(6\upsilon^6 + 3\upsilon^4 + \upsilon^2\right), \quad B(\upsilon) = -\gamma\left(3\upsilon^6 - \upsilon^4 - 2\upsilon^2\right), \quad C(\upsilon) = \gamma\upsilon^2\left(\upsilon^2 - 1\right)^2,$$

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which is valid for $k \geq 2$, while we have also defined,

$$\gamma \equiv \frac{32 c_k N_c}{9\pi^2\lambda}, \qquad c_k \equiv \frac{k\left(k^2-1\right)}{4}, \qquad \xi \equiv \frac{\mathrm{x}_{12}^2}{4\mathrm{z}_1\mathrm{z}_2}, \qquad \upsilon^2 \equiv \frac{\xi}{1+\xi}, \qquad \lambda \equiv g_{\mathrm{YM}}^2 N_c.$$

b_2 anomaly coefficient: D3-D5

As we have already mentioned, the b_2 coefficient can be read off the two-point function of the displacement operator $\mathcal{D}$:

$$\langle \mathcal{D}(\mathbf{x}_1)\, \mathcal{D}(\mathbf{x}_2) \rangle = \frac{c_{nn}}{\mathbf{x}_{12}^8}, \qquad c_{nn} = \frac{15 b_2}{2\pi^4}.$$

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Integrating over the transverse coordinate z from 0^- to 0^+ (and using the conformal invariance of the defect) we find:

$$\mathcal{D}(\mathbf{x}) = \lim_{\mathrm{z}\to 0+} \Theta_{33}(\mathrm{z},\mathbf{x}) - \lim_{\mathrm{z}\to 0-} \Theta_{33}(\mathrm{z},\mathbf{x})\,.$$

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$$b_2 = \frac{8\pi^2 k\,(k^2-1)N_c}{3\lambda} \neq 8c = 0.$$

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The two-point function of the displacement operator then becomes:

$$\left\langle \mathcal{D}^{(1)}(\mathbf{x}_1)\,\mathcal{D}^{(1)}(\mathbf{x}_2)\right\rangle = \lim_{\mathrm{z}_1,\mathrm{z}_2\to 0^+} \left\langle \Theta_{33}^{(1)}(\mathrm{z}_1,\mathbf{x}_1)\,\Theta_{33}^{(1)}(\mathrm{z}_2,\mathbf{x}_2)\right\rangle = \frac{c_{nn}}{\mathbf{x}_{12}^8}, \quad c_{nn} = \frac{20k\,(k^2-1)N_c}{\pi^2\lambda},$$

and the b_2 anomaly coefficient (one contraction) is given by

$$b_2 = \frac{8\pi^2 k\,(k^2-1)N_c}{3\lambda} \neq 8c = 0.$$

de Leeuw-Kristjansen-GL-Volk (2023)

Despite not verifying the free-theory relation $b_2 = 8c$ (at the level of one Wick contraction), the value of b_2 confirms

$$\{\alpha(0),\alpha(1)\} = \{C_T, c_{nn}\} \xrightarrow{d=4} \left\{\frac{640c}{\pi^4}, \frac{15b_2}{2\pi^4}\right\}, \quad \alpha(v) = \frac{d-1}{d^2}\cdot\left[(d-1)(A(v)+4B(v)) + dC(v)\right],$$

for $d = 4$ at the level of a single Wick contraction... These expressions appeared in Herzog-Huang (2017)...

Subsection 3

D3-D7 anomaly coefficients

b_2 anomaly coefficient: D3-D7

To compute the anomaly coefficients for the D3-D7 system (both $SO(5)$ and $SO(3) \times SO(3)$), we plug the corresponding fuzzy funnel solutions into the expression for the stress tensor... We find that the one-point function vanishes:

$$\langle \Theta_{\mu\nu}(\mathrm{x}) \rangle = 0,$$

work in progress

to lowest order in perturbation theory, as it should for a codimension-1 defect (McAvity-Osborn 1993 & 1995)...

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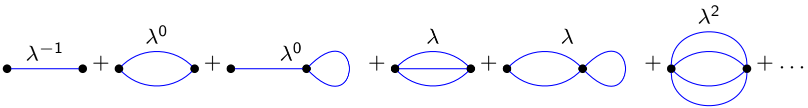
$$\langle \Theta_{\mu\nu}(\mathrm{x}_1)\, \Theta_{\rho\sigma}(\mathrm{x}_2) \rangle =$$



By expanding the $\mathcal{N} = 4$ fields around the fuzzy funnel solution of the D3-D7 interface we find:

$$\Theta^{(1)}_{\mu\nu}(x) = \frac{1}{g^2_{\mathrm{YM}}} \frac{4}{3z^2} \cdot \mathrm{tr}\left\{ \left(\frac{1}{z} (n_\mu n_\nu - g_{\mu\nu})\, \tilde{\varphi}_i + n_\mu \partial_\nu \tilde{\varphi}_i + n_\nu \partial_\mu \tilde{\varphi}_i - \frac{g_{\mu\nu}}{2} \partial_3 \tilde{\varphi}_i + \frac{z}{2} \partial_\mu \partial_\nu \tilde{\varphi}_i \right) \tau_i \right\}.$$

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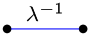

$$=\left\langle \Theta^{(1)}_{\mu\nu}(\mathrm{x}_1)\,\Theta^{(1)}_{\rho\sigma}(\mathrm{x}_2)\right\rangle = \frac{1}{\mathrm{x}_{12}^8}\cdot\Bigg\{\left(X_\mu X_\nu - \frac{g_{\mu\nu}}{4}\right)\left(X'_\rho X'_\sigma - \frac{g_{\rho\sigma}}{4}\right)A(\upsilon) + \Big(X_\mu X'_\rho I_{\nu\sigma} + X_\mu X'_\sigma I_{\nu\rho} +$$

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contracting with the propagator of the D3-D7 dCFT (Gimenez-Grau, Kristjansen, Volk, Wilhelm, 2019)...

$$X_\mu \equiv \mathrm{z}_1\cdot\frac{\upsilon}{\xi}\frac{\partial\xi}{\partial \mathrm{x}_1^\mu} = \upsilon\left(\frac{2\mathrm{z}_1}{\mathrm{x}_{12}^2}(\mathrm{x}_{1\mu}-\mathrm{x}_{2\mu}) - n_\mu\right), \qquad X'_\rho \equiv \mathrm{z}_2\cdot\frac{\upsilon}{\xi}\frac{\partial\xi}{\partial \mathrm{x}_2^\rho} = -\upsilon\left(\frac{2\mathrm{z}_2}{\mathrm{x}_{12}^2}(\mathrm{x}_{1\rho}-\mathrm{x}_{2\rho}) + n_\rho\right).$$

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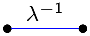

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$$A(\upsilon) = 4\gamma \left(6\upsilon^6 + 3\upsilon^4 + \upsilon^2 \right), \quad B(\upsilon) = -\gamma \left(3\upsilon^6 - \upsilon^4 - 2\upsilon^2 \right), \quad C(\upsilon) = \gamma \upsilon^2 \left(\upsilon^2 - 1 \right)^2,$$

$$\gamma \equiv \frac{32 c_k N_c}{9\pi^2 \lambda}, \quad c_k \equiv \begin{cases} n(n+1)(n+2)(n+3)(n+4)/48, & SO(5) \\ k_1 k_2 \left(k_1^2 + k_2^2 - 2 \right)/4, & SO(3) \times SO(3) \end{cases}, \qquad \xi \equiv \frac{\mathrm{x}_{12}^2}{4 \mathrm{z}_1 \mathrm{z}_2}, \quad \upsilon^2 \equiv \frac{\xi}{1+\xi}.$$

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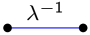

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The b_2 anomaly coefficient (at the level of a single Wick contraction) is found to be:

$$b_2 = \frac{32\pi^2 c_k N_c}{3\lambda} \neq 8c = 0.$$

work in progress

Summary & outlook

We can summarize our results for the (LO) anomaly coefficients of the D3-D5 and D3-D7 holographic defects as follows:

$$c = 0, \quad b_2 = \frac{32\pi^2 c_k N_c}{3\lambda} \neq 8c = 0, \quad c_k \equiv \begin{cases} k\left(k^2-1\right)/4, \quad k \geq 2 & \text{D3-D5} \\ n(n+1)(n+2)(n+3)(n+4)/48, \quad n \geq 1 & \text{D3-D7 } [SO(5)] \\ k_1 k_2\left(k_1^2+k_2^2-2\right)/4, \quad k_{1,2} \geq 2 & \text{D3-D7 } [SO(3)\times SO(3)]. \end{cases}$$

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More results are underway...

- b_1 anomaly coefficient related to the stress tensor/displacement operator 3-point function ($b_1 = 2\pi^3 c_{nnn}/35$)...
- Crosscheck the D3-D5 results (analytically continued to $k = 0$) from the 3d SCFT point of view...
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Ευχαριστώ!

Extra slides

4 The D3-D5 probe-brane system
- AdS_5/CFT_4 duality
- Probe D5-brane
- Gamma matrices
- One-point functions
- $\mathfrak{su}(2)_k$ representations

5 The D3-D7 defect
- The D3-D7 geometries
- Symmetrized direct products & fuzzy S^4 matrices
- One-point functions

6 The D2-D4 defect
- AdS_4/CFT_3 duality
- The D2-D4 geometries
- T and R-matrices

7 Correlation functions in CFTs and dCFTs
- Conformal field theories
- Defect conformal field theories
- Boundary conformal bootstrap
- Conformal anomalies

8 Codimension-1 determinant formulas
- D3-D5 domain wall
- D3-D7 domain wall
- D2-D4 domain wall

9 Chiral primary operators
- $SO(3) \times SO(3)$ spherical harmonics
- $SO(4)$ spherical harmonics

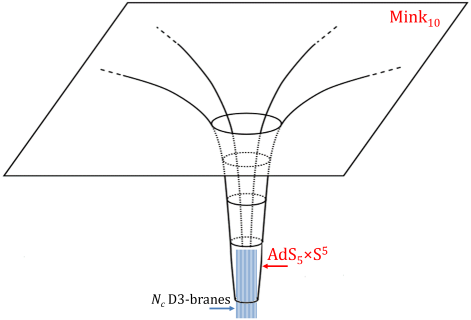
Section 4

The D3-D5 probe-brane system

The AdS$_5$/CFT$_4$ correspondence

Let us briefly revisit Maldacena's argument leading to the AdS/CFT correspondence.

- We consider 2 different descriptions of a system of N_c coincident D3-branes...



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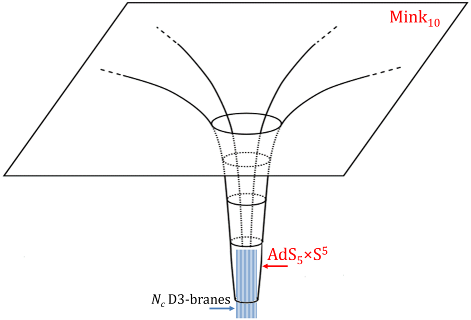

- The D3-branes are extended along the directions x_1, x_2, x_3...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	●	●	●	●						

The D3-brane system: open string description

In the *open string description* the system contains (1) open strings ending on the N_c D3-branes and (2) closed strings propagating in the bulk:

$$S = S_{\text{branes}} + S_{\text{bulk}} + S_{\text{interactions}}$$

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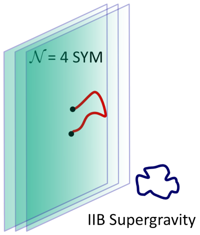

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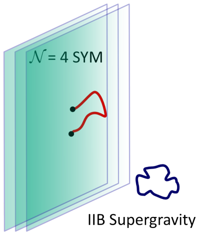


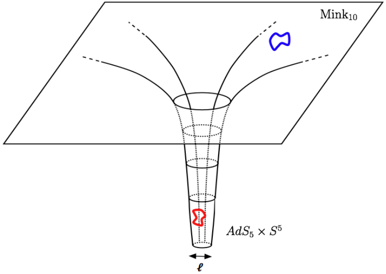
At low energies $S_{\text{interactions}}$ can be ignored and the system only contains free open & closed strings, or equivalently

$$\left\{ \begin{array}{c} \text{Open string description} \\ \text{low energy limit} \end{array} \right\} \Rightarrow \mathcal{N} = 4,\ \mathfrak{su}(N_c)\ \text{super Yang-Mills} \ + \ \text{Free type IIB supergravity}.$$

The D3-brane system: closed strings description

In the *closed strings description* the N_c D3-branes act as sources to the bulk fields:

$$ds^2 = H^{-1/2}\left(-dt^2 + d\mathbf{x}_3^2\right) + H^{1/2}\left(dz^2 + z^2 d\Omega_5^2\right), \quad H(z) \equiv 1 + \left(\frac{\ell}{z}\right)^4, \quad \ell^4 = 4\pi g_s N_c \ell_s^4.$$



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Far from the horizon ($z \to \infty$), the above metric describes 10-dimensional Minkowski spacetime. Close to the horizon ($z \to 0$) it reduces to the metric of $AdS_5 \times S^5$ in Poincaré coordinates:

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At low energies, the excitations that live far from the horizon decouple from the excitations that are close to the horizon and so again the system can be written as the sum of two non-interacting systems:

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The AdS/CFT correspondence

This leads us to the AdS$_5$/CFT$_4$ correspondence:

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On the lhs, $\mathcal{N}=4$, super Yang-Mills (SYM) theory is a 4-dimensional superconformal gauge theory:

$$\mathcal{L}_{\mathcal{N}=4} = \frac{2}{g_{\text{YM}}^2} \cdot \text{tr}\Big\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (D_\mu \varphi_i)^2 + i\,\bar{\psi}_\alpha \not{D} \psi_\alpha + \frac{1}{4} [\varphi_i, \varphi_j]^2 + \\ + \sum_{i=1}^{3} G^i_{\alpha\beta} \bar{\psi}_\alpha [\varphi_i, \psi_\beta] + \sum_{i=4}^{6} G^i_{\alpha\beta} \bar{\psi}_\alpha \gamma_5 [\varphi_i, \psi_\beta] \Big\}.$$

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The AdS/CFT correspondence

This leads us to the AdS_5/CFT_4 correspondence:

$\mathcal{N}=4$, $\mathfrak{su}(N_c)$ super Yang-Mills theory in 4d $\Leftrightarrow$ Type IIB superstring theory on $AdS_5 \times S^5$

Maldacena (1997)

On the lhs, $\mathcal{N}=4$, super Yang-Mills (SYM) theory is a 4-dimensional superconformal gauge theory:

$$\mathcal{L}_{\mathcal{N}=4} = \frac{2}{g_{\text{YM}}^2}\cdot \text{tr}\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(D_\mu\varphi_i\right)^2 + i\,\bar{\psi}_\alpha \not{D}\psi_\alpha + \frac{1}{4}\left[\varphi_i,\varphi_j\right]^2 + \\ + \sum_{i=1}^{3} G^i_{\alpha\beta}\bar{\psi}_\alpha\left[\varphi_i,\psi_\beta\right] + \sum_{i=4}^{6} G^i_{\alpha\beta}\bar{\psi}_\alpha\gamma_5\left[\varphi_i,\psi_\beta\right]\Big\}.$$

- Beta function vanishes, $\beta_{(\mathcal{N}=4)} = 0$... exact superconformal symmetry $PSU(2,2|4)$...
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- Spectral problem solved (Gromov-Kazakov-Leurent-Volin, 2013)... solution of full planar theory by computing all observables (correlators, scattering amplitudes, Wilson loops, etc) underway...
- Half-BPS boundary conditions in $\mathcal{N}=4$ SYM were studied by Gaiotto-Witten (2008)...

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Type IIB superstring theory on $AdS_5 \times S^5$ is described by a nonlinear σ-model on a supercoset:

$$AdS_5 \times S^5 = \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} \subseteq \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}.$$

Green-Schwarz superstring action on $AdS_5 \times S^5$ is a WZW sigma model (Metsaev-Tseytlin, 1998):

$$S = -\frac{T_2}{2}\int \ell^2 \text{str}\left[J^{(2)} \wedge \star J^{(2)} + J^{(1)} \wedge J^{(3)}\right], \qquad J \equiv \mathfrak{g}^{-1} d\mathfrak{g}, \qquad T_2 \equiv \frac{1}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi\ell^2}.$$

The $AdS_5 \times S^5$ supercoset is a semi-symmetric space, i.e. its elements afford a $\mathbb{Z}_4$ decomposition:

$$J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}, \qquad \Omega\left[J^{(n)}\right] = i^n J^{(n)}, \qquad \Omega\left(M\right) = -\mathcal{K} M^{st} \mathcal{K}^{-1}, \quad \mathcal{K} = \begin{bmatrix} \gamma_{13} & 0 \\ 0 & \gamma_{13} \end{bmatrix}.$$

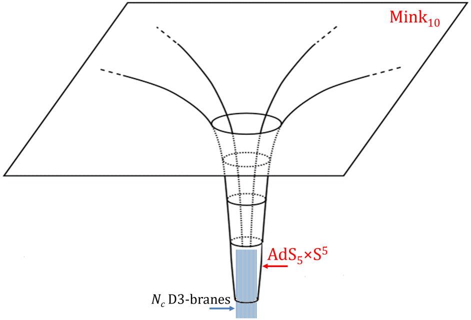
Nonlinear sigma models on semi-symmetric spaces are classically integrable (Bena-Polchinski-Roiban, 2003)...

Subsection 2

Probe D5-brane

The D3-D5 system: bulk geometry

Type IIB string theory on $AdS_5 \times S^5$ is encountered very close to a system of N_c coincident D3-branes:



The D3-branes extend along x_1, x_2, x_3...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	●	●	●	●						

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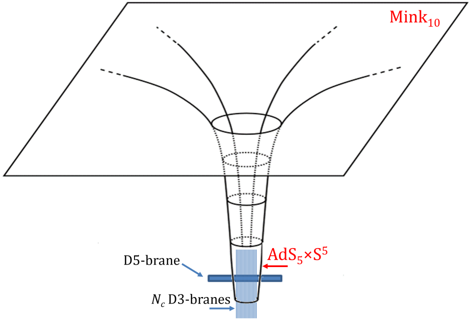

Now insert a single (probe) D5-brane at $x_3 = x_7 = x_8 = x_9 = 0$...

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
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D5	•	•	•		•	•	•			

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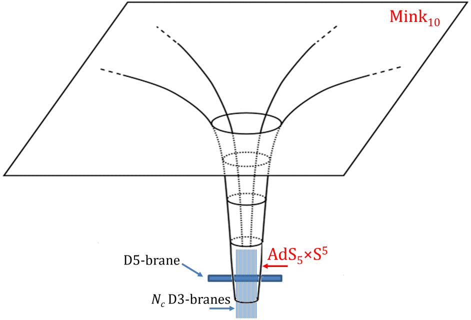

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D5	•	•	•		•	•	•			

... its geometry will be $AdS_4 \times S^2$ (Karch-Randall, 2001b)...

The D3-D5 system: bulk geometry (zero flux)

Here's a quick way to figure out the geometry of the D3-brane. Write the $AdS_5 \times S^5$ metric as follows:

$$ds^2 = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right) = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\sum_{i=4}^{9} dx_i^2 ,$$

where $r^2 \equiv x_4^2 + \ldots + x_9^2$ and

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The line element of $AdS_5 \times S^5$ takes the following form:

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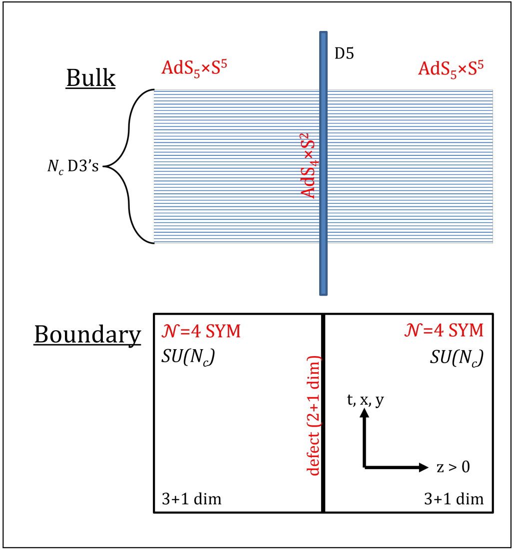
and its geometry is $AdS_4 \times S^2$ (Karch-Randall, 2001b)... result confirmed from the DBI analysis...

The D3-D5 system: description



- The defect reduces the total bosonic symmetry of the system from $SO(4,2)\times SO(6)$ to $SO(3,2)\times SO(3)\times SO(3)$. The corresponding superalgebra $\mathfrak{psu}(2,2|4)$ becomes $\mathfrak{osp}(4|4)$. Supersymmetry studied by Domokos-Royston (2022)...
- The D3-D5 system describes IIB string theory on $\mathrm{AdS}_5\times \mathrm{S}^5$ bisected by a D5 brane with worldvolume geometry $\mathrm{AdS}_4\times \mathrm{S}^2$.
- The D5-brane is stable... the tachyonic instability in the fluctuations of ψ does not violate the BF bound (Karch-Randall, 2001b)...
- The probe D5-brane is classically integrable... i.e. infinite conserved charges for open strings with D5-brane BCs (Dekel-Oz, 2011)...
- The dual field theory is still $SU(N_c)$, $\mathcal{N}=4$ SYM in $3+1$ dimensions, that interacts with a CFT living on the $2+1$ dimensional defect: $S=S_{\mathcal{N}=4}+S_{2+1}$ (DeWolfe-Freedman-Ooguri, 2001).
- $\mathcal{N}=4$ spin chain not modified by the presence of the defect... open spin chain ending on defect fields remains integrable (DeWolfe-Mann, 2004)...

The D3-D5 defect action

The action of the $SU(2)$ symmetric D3-D5 dCFT consists of a 4d bulk theory coupled to a 3d boundary theory:

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$$\mathcal{L}_{\text{pot}} = -\frac{1}{g_{\text{YM}}^2} \cdot \left\{ q_m^\dagger X_V^A X_V^A q_m + i\epsilon_{ABC} q_m^\dagger \sigma_{mn}^A X_H^B X_H^C q_n + q_m^\dagger \sigma_{mn}^A (D_z X_H^A) q_n \right\}, \quad \mathcal{L}_{\text{delta}} = -\frac{\delta(0)}{2g_{\text{YM}}^2} \cdot \left\{ \left(q_m^\dagger \sigma_{mn}^A q_n \right)^2 \right\},$$

for $\{\dot{\mu} = 0,1,2\}$, $\{m,n,i,j = 1,2\}$, and $\{A,B,C = 1,2,3\}$. Moreover, σ_A denote the Pauli matrices and

$$\mathfrak{D}_{\dot{\mu}} f \equiv \partial_{\dot{\mu}} f - i A_{\dot{\mu}} f, \qquad \bar{\lambda}_i \equiv \lambda_i^\dagger \rho^0, \qquad \not{\mathfrak{D}} \equiv \rho^{\dot{\mu}} \mathfrak{D}_{\dot{\mu}}, \qquad P_\pm \equiv (1 \pm \gamma_5 \gamma^3)/2.$$

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The bulk fields split into a vector multiplet $\{A_{\dot{\mu}}, P_+\psi_\alpha, X_V^A, D_z X_H^A\}$ and a hypermultiplet $\{A_z, P_-\psi_\alpha, X_H^A, D_z X_V^A\}$, with $X_H = \{\varphi_1, \varphi_2, \varphi_3\}$ and $X_V = \{\varphi_4, \varphi_5, \varphi_6\}$. The 4d bulk spinors ψ_α are split into two pairs of 3d spinors by using the projectors $P_\pm \psi_\alpha$. Their indices $\alpha = 1, \ldots, 4$ have been rearranged as follows:

$$\psi_{im} \equiv \psi_4 \delta_{im} - i\psi_\alpha \sigma_{im}^\alpha, \qquad \bar{\psi}_{mi} \equiv \bar{\psi}_4 \delta_{mi} + i\bar{\psi}_\alpha \sigma_{mi}^\alpha, \qquad i,m = 1,2, \quad \alpha = 1,2,3.$$

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- Because of the Yukawa terms in the defect action, the bulk 4d fermions ψ_α of $\mathcal{N} = 4$ SYM (4-component spinors) couple directly to defect 3d fermions λ_i (2-component spinors)...

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which affords a unique solution

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- Accordingly, the 3d Dirac matrices can be encoded into three 4×4 matrices $\rho_{\dot{\mu}}$ which are defined as:

$$\rho^{\dot{\mu}} \equiv \gamma^{\dot{\mu}}\gamma_5\gamma^3.$$

They satisfy the Clifford algebra (for $\dot{\mu}, \dot{\nu} = 0, 1, 2$),

$$\rho_{\dot{\mu}}\rho_{\dot{\nu}} + \rho_{\dot{\nu}}\rho_{\dot{\mu}} = -2g_{\dot{\mu}\dot{\nu}} = 2 \times \text{diag}\,(1, -1, -1)\,.$$

The D3-D5 defect action

- Because of the Yukawa terms in the defect action, the bulk 4d fermions ψ_α of $\mathcal{N}=4$ SYM (4-component spinors) couple directly to defect 3d fermions λ_i (2-component spinors)...
- We either express the bulk 4d fermions in terms of (two-component) 3d spinors (DeWolfe-Freedman-Ooguri, 2001)... or express the 3d defect fermions in terms of (four-component) 4d spinors (DeWolfe-Mann, 2004)...
- Here we adopt the latter approach... using the projectors $P_\pm$, the (4-component) defect fermions λ_i should satisfy:

$$P_+\lambda = \lambda, \qquad P_-\lambda = 0,$$

which affords a unique solution

$$\lambda^t = (\lambda_1, \lambda_2, -\lambda_1, \lambda_2)\,.$$

- Accordingly, the 3d Dirac matrices can be encoded into three 4×4 matrices $\rho_{\dot{\mu}}$ which are defined as:

$$\rho^{\dot{\mu}} \equiv \gamma^{\dot{\mu}}\gamma_5\gamma^3.$$

They satisfy the Clifford algebra (for $\dot{\mu},\dot{\nu} = 0,1,2$),

$$\rho_{\dot{\mu}}\rho_{\dot{\nu}} + \rho_{\dot{\nu}}\rho_{\dot{\mu}} = -2g_{\dot{\mu}\dot{\nu}} = 2\times \text{diag}\,(1,-1,-1)\,.$$

- We also note that bulk fields carry adjoint $\mathfrak{u}(N_c)$ color indices, and defect fields q_m, λ_i carry fundamental $\mathfrak{u}(N_c)$ color indices. For simplicity we have also omitted the traces over the color degrees of freedom from the defect Lagrangian...

The $(D3\text{-}D5)_k$ system: bulk geometry (nonzero flux)

Despite stability, we can still add $k \neq 0$ units of background magnetic flux over the S^2 part of the D5-brane... The D5-brane geometry should be determined from the equations of motion of the DBI+WZ action:

$$S_{\mathrm{D5}} = -\frac{T_5}{g_s}\int\left[d^6\zeta\sqrt{\det\left(G_{ab} + 2\pi\alpha' F_{ab}\right)} + 2\pi\alpha' F\wedge C\right], \quad T_5 \equiv \frac{1}{(2\pi)^5\,\alpha'^3}, \quad g_s = \frac{g_{\mathrm{YM}}^2}{4\pi}.$$

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- G_{ab} is the metric of $AdS_5 \times S^5$ (in the conformal Poincaré frame):

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2\right) + \ell^2 d\Omega_5^2, \qquad z \equiv \frac{1}{r},$$

  where the line element of the unit 5-sphere has been written as:

$$d\Omega_5^2 = d\psi^2 + \cos^2\psi\, d\Omega_2^2 + \sin^2\psi\, d\tilde{\Omega}_2^2, \qquad d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2.$$

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- There are also N_c units of self-dual 5-form RR flux through AdS_5 and S^5... the 4-form potential is

$$\hat{C} = \ell^4\left[-\frac{1}{z^4}\left(dt\wedge dx_1\wedge dx_2\wedge dx_3\right) + \frac{1}{8}\left(4\psi - \sin 4\psi\right) d\cos\theta\wedge d\varphi\wedge d\cos\vartheta\wedge d\chi\right],$$

while the components of the corresponding 5-form field strength $\hat{f} \equiv d\hat{C}$ are

$$\hat{f}_{mnpqr} = \epsilon_{mnpqr}, \qquad \hat{f}_{\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau},$$

where Latin and Greek indices, (m, n, p, q, r) and $(\mu, \nu, \rho, \sigma, \tau)$, refer to AdS_5 and S^5 respectively.

D5-brane embedding

There are also k units of magnetic flux through the S^2... forcing k out of N_c D3-branes to end on the D5-brane...

$$F = dA = \frac{k}{2} \cdot d\cos\theta \wedge d\varphi, \qquad A = \frac{k}{2} \cos\theta \cdot d\varphi, \qquad \int_{S^2} \frac{F}{2\pi} = k \quad \text{(first Chern class)}.$$

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$$F = \frac{k}{4} \cdot \sum_{a,b,c=4}^{6} \varepsilon_{abc}\, x_a\, dx_b \wedge dx_c, \qquad \{F_{ab}\} = -\frac{k}{2} \begin{pmatrix} 0 & x_6 & -x_5 \\ -x_6 & 0 & x_4 \\ x_5 & -x_4 & 0 \end{pmatrix}, \qquad \int_{S^2} \frac{F}{2\pi} = k \quad \text{(first Chern class)},$$

where $a, b = 4, 5, 6$...

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where $a, b = 4, 5, 6$... The geometry of the D5-brane in $AdS_5 \times S^5$ is still $AdS_4 \times S^2$... its embedding is described by:

$$x_3 = \kappa \cdot z, \qquad \kappa \equiv \frac{\pi k}{\sqrt{\lambda}} \equiv \tan\alpha. \qquad \psi = 0.$$

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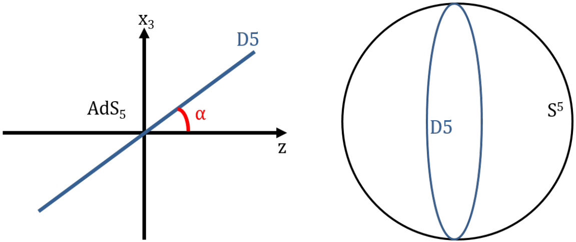
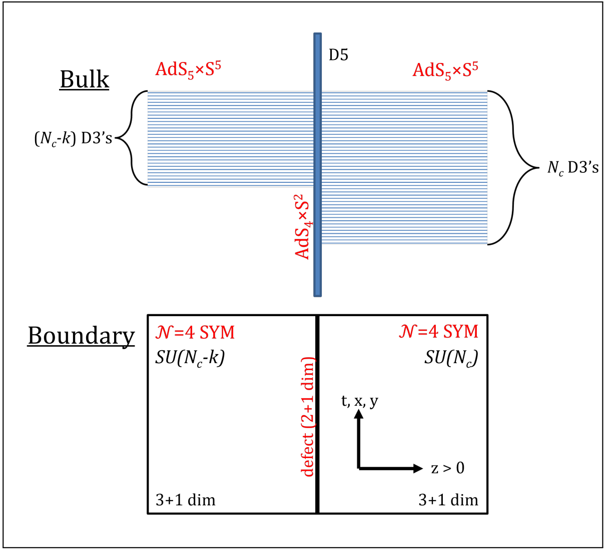


The (D3-D5)$_k$ dSCFT



- D5-brane with flux preserves classical integrability of open strings (Zarembo-GL, 2021)...
- The SCFT gauge group $SU(N_c) \times SU(N_c)$ breaks to $SU(N_c - k) \times SU(N_c)$...
- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs (Karch-Randall, 2001b)... dCFT correlators = Higgs condensates of gauge-invariant operators of $\mathcal{N} = 4$ SYM (Nagasaki-Yamaguchi, 2012)...
- Matrix product states... overlaps with Bethe states... Scalar one-point functions (de Leeuw, Kristjansen, Zarembo, 2015)... closed-form det formulas... integrable quench criteria satisfied (Piroli, Pozsgay, Vernier, 2017; de Leeuw-Kristjansen-GL, 2018)...
- Two-point functions of (spin-2) stress tensor, displacement operator, anomaly coefficients (de Leeuw-Kristjansen-GL-Volk 2023)...
- Strong-coupling computations were recently set up (Georgiou-GL-Zoakos, 2023)...
- Before going through the weak-coupling results, we revisit CFT and dCFT correlation functions...

Subsection 3

Gamma matrices

Gamma matrices in $3+1$ dimensions

In the Weyl (chiral) representation, the 4×4 gamma matrices γ^μ (in 4-dimensional Minkowski spacetime) are given by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

where $i = 1, 2, 3$ and the Pauli matrices σ_μ are as usual defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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The gamma matrices obey the following Clifford algebra:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu} = 2 \times \text{diag}\,(1, -1, -1, -1)\,, \qquad \gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 2\,\delta_5^\mu.$$

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We also define the gamma matrix commutators,

$$\gamma^{\mu\nu} \equiv \gamma^{[\mu\nu]} = \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right].$$

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The charge conjugation matrix C is defined as:

$$C \equiv i\,\sigma_3 \otimes \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = i\gamma^{02}.$$

It obeys among others the following properties

$$C^t = C^{-1} = -C, \qquad \gamma_\mu^t = -C\gamma_\mu C^{-1}, \qquad \gamma_5^t = C\gamma_5 C^{-1}.$$

The G-matrices of $\mathcal{N} = 4$ SYM

The 4×4 matrices G^i that show up in the Lagrangian density of $\mathcal{N} = 4$ SYM are given by:

$$G^1 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}, \qquad G^2 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \qquad G^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

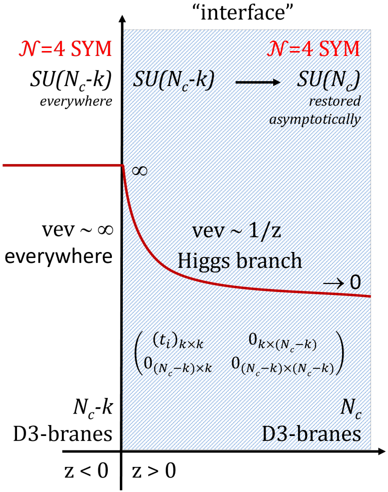
$$G^4 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad G^5 = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \qquad G^6 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}.$$

These matrices are all antisymmetric. The first three are Hermitian, while the other three anti-Hermitian. One can work out explicit expressions for the commutators and anticommutators of the $\mathcal{N} = 4$ SYM G-matrices (see e.g. Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm, 2016)...

Subsection 4

One-point functions

The D3-D5 interface



- An interface is a wall between two (different/same) QFTs...
- It can be described by means of classical solutions that are known as "fuzzy-funnel" solutions (Constable-Myers-Tafjord, 1999 & 2001)...

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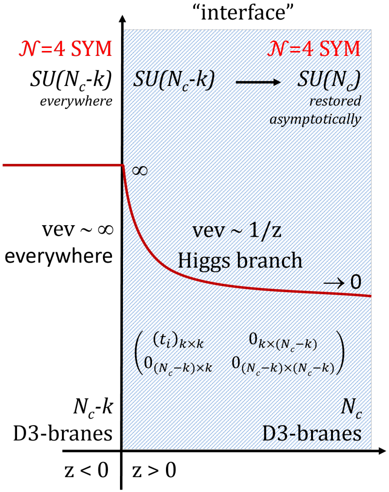

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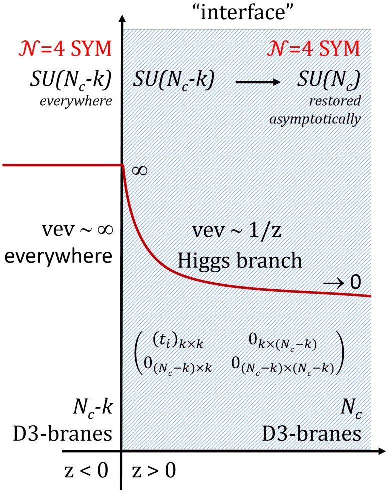

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- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = [\varphi_j, [\varphi_j, \varphi_i]], \quad i, j = 1, \ldots, 6.$$

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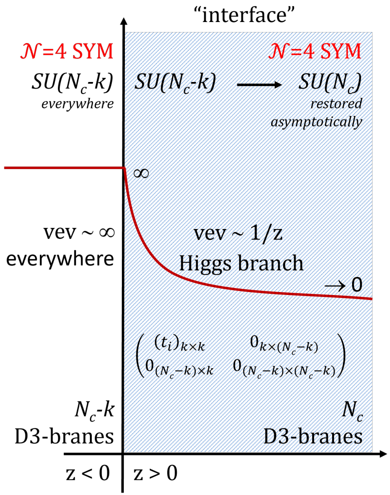

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- A manifestly $SO(3) \simeq SU(2)$ symmetric solution is given by ($z > 0$):

$$\varphi_{2i-1}(z) = \frac{1}{z}\begin{bmatrix} (t_i)_{k\times k} & 0_{k\times(N_c-k)} \\ 0_{(N_c-k)\times k} & 0_{(N_c-k)\times(N_c-k)} \end{bmatrix} \quad \& \quad \varphi_{2i} = 0,$$

Diaconescu (1996), Giveon-Kutasov (1998)

where the matrices t_i furnish a k-dimensional representation of $\mathfrak{su}(2)$:

$$[t_i, t_j] = i\epsilon_{ijk} t_k.$$

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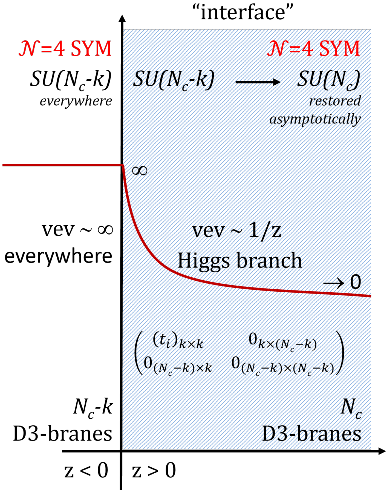

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Diaconescu (1996), Giveon-Kutasov (1998)

- The solution also satisfies the Nahm equations:

$$\frac{d\varphi_i}{dz} = \frac{i}{2}\epsilon_{ijk}\left[\varphi_j, \varphi_k\right],$$

as expected for a half-BPS interface (Gaiotto-Witten, 2008)...

One-point functions

Following Nagasaki & Yamaguchi (2012), the one-point functions of local gauge-invariant scalar operators,

$$\langle \mathcal{O}(z, \mathbf{x}) \rangle = \frac{\mathcal{C}}{z^{\Delta}}, \qquad z > 0,$$

can be calculated within the D3-D5 defect CFT from the corresponding fuzzy-funnel solution, for example:

$$\boxed{\mathcal{O}(z, \mathbf{x}) = \Psi^{\mu_1 \dots \mu_L} \text{tr} \left[\varphi_{2\mu_1 - 1} \dots \varphi_{2\mu_L - 1} \right] \xrightarrow[\text{interface}]{SU(2)} \frac{1}{z^L} \cdot \Psi^{\mu_1 \dots \mu_L} \text{tr} \left[t_{\mu_1} \dots t_{\mu_L} \right]}$$

where $\Psi^{\mu_1 \cdots \mu_L}$ is an $SO(6)$ symmetric tensor and the constant $\mathcal{C}$ is given by (MPS="*matrix product state*"),

$$\mathcal{C} = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda} \right)^{L/2} \cdot \frac{\langle \text{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \qquad \left\{ \begin{array}{ll} \langle \text{MPS} | \Psi \rangle \equiv \Psi^{\mu_1 \dots \mu_L} \text{tr} \left[t_{\mu_1} \dots t_{\mu_L} \right] & (\text{“overlap”}) \\ \langle \Psi | \Psi \rangle \equiv \Psi^{\mu_1 \dots \mu_L} \Psi_{\mu_1 \dots \mu_L} & \end{array} \right\},$$

which ensures that the 2-point function will be normalized to unity ($\mathcal{O} \rightarrow (2\pi)^L \left(L\lambda^L \right)^{-1/2} \cdot \mathcal{O}$):

$$\langle \mathcal{O}(x_1) \, \mathcal{O}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}},$$

within $SU(N_c)$, $\mathcal{N} = 4$ SYM (i.e. without the defect). Once more, we set $x_i \equiv (z_i, \mathbf{x}_i)$, where $\mathbf{x}_i \equiv \{x_i^{(0,1,2)}\}$.

Chiral primary operators

The one-point functions of $SO(3) \times SO(3) \subseteq SO(6)$ invariant chiral primary operators (CPOs),

$$\mathcal{O}_{\text{CPO}}(x) = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot K^{\mu_1 \cdots \mu_L} \text{tr}\left[\varphi_{\mu_1}(x) \ldots \varphi_{\mu_L}(x)\right],$$

where $K^{\mu_1 \cdots \mu_L}$ are symmetric & traceless $SO(3) \times SO(3) \subseteq SO(6)$ tensors satisfying,

$$K^{\mu_1 \cdots \mu_L} K^{\mu_1 \cdots \mu_L} = 1 \qquad \& \qquad Y_L = K^{\mu_1 \cdots \mu_L} x_{\mu_1} \ldots x_{\mu_L}, \qquad \sum_{\mu=4}^{6} x_\mu^2 = \cos^2 \psi, \qquad \sum_{\mu=7}^{9} x_\mu^2 = \sin^2 \psi,$$

and $Y_L(\psi)$ is the $SO(3) \times SO(3) \subseteq SO(6)$ spherical harmonic, have been calculated at weak coupling:

$$\langle \mathcal{O}_{\text{CPO}}(\mathrm{x}) \rangle = \frac{1}{\sqrt{L}} \left(\frac{2\pi^2}{\lambda}\right)^{L/2} k \left(k^2 - 1\right)^{L/2} \frac{Y_L(0)}{\mathrm{z}^L}, \qquad k \ll N_c \to \infty,$$

Nagasaki-Yamaguchi (2012)

where $L = 2j$, $j = 0, 1, \ldots$ The large-k limit agrees with the supergravity calculation (details in Part III):

$$\langle \mathcal{O}_{\text{CPO}}(\mathrm{x}) \rangle = \frac{k^{L+1}}{\sqrt{L}} \left(\frac{2\pi^2}{\lambda}\right)^{L/2} \frac{Y_L(0)}{\mathrm{z}^L} \cdot \left[1 + \frac{\lambda \, \mathrm{I}_1}{\pi^2 k^2} + \ldots\right], \qquad \mathrm{I}_1 \equiv \frac{3}{2} + \frac{(L-2)(L-3)}{4(L-1)}.$$

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$$\mathcal{O}_{\text{CPO}}(x) = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot K^{\mu_1 \cdots \mu_L} \text{tr}\left[\varphi_{\mu_1}(x) \ldots \varphi_{\mu_L}(x)\right],$$

where $K^{\mu_1 \cdots \mu_L}$ are symmetric & traceless $SO(3) \times SO(3) \subseteq SO(6)$ tensors satisfying,

$$K^{\mu_1 \cdots \mu_L} K^{\mu_1 \cdots \mu_L} = 1 \qquad \& \qquad Y_L = K^{\mu_1 \cdots \mu_L} x_{\mu_1} \ldots x_{\mu_L}, \qquad \sum_{\mu=4}^{6} x_\mu^2 = \cos^2 \psi, \qquad \sum_{\mu=7}^{9} x_\mu^2 = \sin^2 \psi,$$

and $Y_L(\psi)$ is the $SO(3) \times SO(3) \subseteq SO(6)$ spherical harmonic, have been calculated at weak coupling:

$$\langle \mathcal{O}_{\text{CPO}}(\mathrm{x}) \rangle = \frac{1}{\sqrt{L}} \left(\frac{2\pi^2}{\lambda}\right)^{L/2} k \left(k^2 - 1\right)^{L/2} \frac{Y_L(0)}{\mathrm{z}^L}, \qquad k \ll N_c \to \infty,$$

Nagasaki-Yamaguchi (2012)

where $L = 2j, \ j = 0, 1, \ldots$ The large-k limit agrees with the supergravity calculation (details in Part III):

$$\langle \mathcal{O}_{\text{CPO}}(\mathrm{x}) \rangle = \frac{k^{L+1}}{\sqrt{L}} \left(\frac{2\pi^2}{\lambda}\right)^{L/2} \frac{Y_L(0)}{\mathrm{z}^L} \cdot \left[1 + \frac{\lambda\, \mathrm{I}_1}{\pi^2 k^2} + \ldots\right], \qquad \mathrm{I}_1 \equiv \frac{3}{2} + \frac{(L-2)(L-3)}{4(L-1)}.$$

We can go beyond (bulk) CPOs... by computing the one-point functions of (scalar) gauge invariant operators of $\mathcal{N} = 4$ SYM with definite scaling dimensions...

Dilatation operator

The mixing of single-trace operators $\mathcal{O}(x)$ is generally described by the integrable $\mathfrak{so}(6)$ spin chain:

$$\mathbb{D} = L \cdot \mathbb{I} + \frac{\lambda}{8\pi^2} \cdot \mathbb{H} + \sum_{n=2}^{\infty} \lambda^n \cdot \mathbb{D}_n, \qquad \mathbb{H} = \sum_{j=1}^{L} \left(\mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1} + \frac{1}{2} \mathbb{K}_{j,j+1} \right), \qquad \lambda = g_{\text{YM}}^2 N,$$

Minahan-Zarembo (2002)
Beisert-Kristjansen-Staudacher (2003)
Beisert (2003)

up to one loop in $\mathcal{N} = 4$ SYM, where

$$\mathbb{I} \cdot |\ldots \varphi_a \varphi_b \ldots\rangle = |\ldots \varphi_a \varphi_b \ldots\rangle$$

$$\mathbb{P} \cdot |\ldots \varphi_a \varphi_b \ldots\rangle = |\ldots \varphi_b \varphi_a \ldots\rangle$$

$$\mathbb{K} \cdot |\ldots \varphi_a \varphi_b \ldots\rangle = \delta_{ab} \sum_{c=1}^{6} |\ldots \varphi_c \varphi_c \ldots\rangle \, .$$

The above result is unaffected by the presence of a defect (DeWolfe-Mann, 2004; Ipsen-Vardinghus, 2019)...

Bethe eigenstates

- In the following we will examine eigenstates of the $\mathfrak{so}(6)$ spin chain which can be written as:

$$|\Psi\rangle \equiv \sum_{x_i} \psi_i(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \cdot |\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\downarrow} \bullet \ldots \bullet \underset{x_3}{\Uparrow} \bullet \ldots \bullet \underset{x_4}{\Downarrow} \bullet \ldots\rangle,$$

where $\mathbf{u}_{1,2,3}$ are the rapidities of the excitations at x_i. The corresponding single-trace operator is

$$|\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\downarrow} \bullet \ldots \bullet \underset{x_3}{\Uparrow} \bullet \ldots \bullet \underset{x_4}{\Downarrow} \ldots\rangle \sim \text{tr}\left[\mathcal{Z}^{x_1-1}\mathcal{W}\mathcal{Z}^{x_2-x_1-1}\mathcal{Y}\mathcal{Z}^{x_3-x_2-1}\overline{\mathcal{W}}\mathcal{Z}^{x_4-x_3-1}\overline{\mathcal{Y}}\ldots\right],$$

where $\mathcal{Z}$ (ground state field), $\mathcal{W}$, $\mathcal{Y}$ (excitations) are the following three complex scalars:

$$\mathcal{W} = \varphi_1 + i\varphi_2 \sim \uparrow \qquad \mathcal{Y} = \varphi_3 + i\varphi_4 \sim \downarrow \qquad \mathcal{Z} = \varphi_5 + i\varphi_6 \sim \bullet$$

$$\overline{\mathcal{W}} = \varphi_1 - i\varphi_2 \sim \Uparrow \qquad \overline{\mathcal{Y}} = \varphi_3 - i\varphi_4 \sim \Downarrow \qquad \overline{\mathcal{Z}} = \varphi_5 - i\varphi_6 \sim \circ$$

- The wavefunction $\psi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ can be constructed with the (nested) coordinate Bethe ansatz (details can be found in Basso-Coronado-Komatsu-Lam-Vieira-Zhong, 2017)...

Nesting

- Let us first construct the kets $|\bullet\ldots\bullet\underset{x_1}{\uparrow}\bullet\ldots\bullet\underset{x_2}{\downarrow}\bullet\ldots\bullet\underset{x_3}{\Uparrow}\bullet\ldots\bullet\underset{x_4}{\Downarrow}\bullet\ldots\rangle\ldots$

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- Because the excitations can have 5 different polarizations, we apply a procedure called "nesting"...

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- Start from a closed $\mathfrak{so}(6)$ spin chain of length L:

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- Because the excitations can have 5 different polarizations, we apply a procedure called "nesting"...
- Start from a closed $\mathfrak{so}(6)$ spin chain of length L. Excite exactly N_1 sites of the chain:

Nesting

- Let us first construct the kets $|\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\downarrow} \bullet \ldots \bullet \underset{x_3}{\Uparrow} \bullet \ldots \bullet \underset{x_4}{\Downarrow} \bullet \ldots\rangle \ldots$
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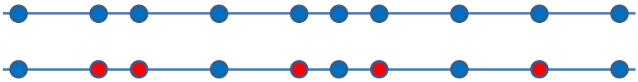
- Start from a closed $\mathfrak{so}(6)$ spin chain of length L. Excite exactly N_1 sites of the chain:

Now take the N_1 excitations to be the ground state.

Nesting

- Let us first construct the kets $|\bullet\ldots\bullet\underset{x_1}{\uparrow}\bullet\ldots\bullet\underset{x_2}{\downarrow}\bullet\ldots\bullet\underset{x_3}{\Uparrow}\bullet\ldots\bullet\underset{x_4}{\Downarrow}\bullet\ldots\rangle\ldots$
- Because the excitations can have 5 different polarizations, we apply a procedure called "nesting"...
- Start from a closed $\mathfrak{so}(6)$ spin chain of length L. Excite exactly N_1 sites of the chain:

Now take the N_1 excitations to be the ground state. Excite N_2 sites of the new chain...



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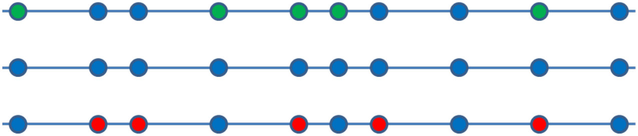
- Start from a closed $\mathfrak{so}(6)$ spin chain of length L. Excite exactly N_1 sites of the chain:

Now take the N_1 excitations to be the ground state. Excite N_2 sites of the new chain... or N_3 sites:



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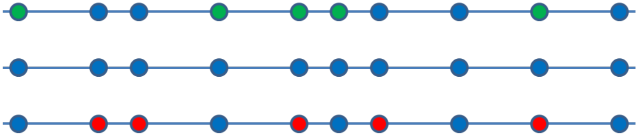

- We end up with three sets/levels of rapidities, one rapidity for each excitation:

$$\mathbf{u}_1 = \{u_{1,j}\}_{j=1}^{N_1}, \qquad \mathbf{u}_2 = \{u_{2,j}\}_{j=1}^{N_2}, \qquad \mathbf{u}_3 = \{u_{3,j}\}_{j=1}^{N_3},$$

each set corresponds to a simple root $\alpha_{1,2,3}$ of $\mathfrak{so}(6)$...

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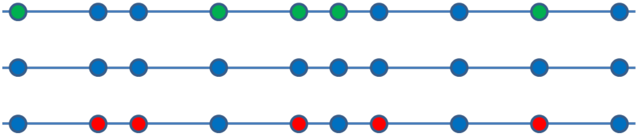

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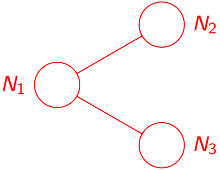
each set corresponds to a simple root $\alpha_{1,2,3}$ of $\mathfrak{so}(6)$...
- To construct the kets, we must map the sets of rapidities to the available complex scalar fields...

Rapidities & fields

- As we've just seen, each set of rapidities can be associated to a node of the $\mathfrak{so}(6)$ Dynkin diagram:



$$(0 \leq N_1 \leq L,\ 0 \leq N_2 \leq N_1/2,\ 0 \leq N_3 \leq N_2)\,.$$

- Setting $\mathbf{q} \equiv (1,0,0)$ as the highest weight of $\mathfrak{so}(6)$, the total weight of the representation is given by:

$$\mathbf{w} = L\mathbf{q} - N_1\boldsymbol{\alpha}_1 - N_2\boldsymbol{\alpha}_2 - N_3\boldsymbol{\alpha}_3,$$

where $\boldsymbol{\alpha}_1 \equiv (1,-1,0)$, $\boldsymbol{\alpha}_2 \equiv (0,1,-1)$, $\boldsymbol{\alpha}_3 \equiv (0,1,1)$ are the simple roots of $\mathfrak{so}(6)$.

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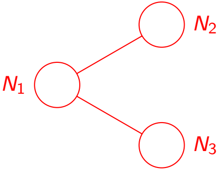

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- The corresponding Cartan charges are given by:

$$\mathbf{w} = (J_1, J_2, J_3) = (L - N_1, N_1 - N_2 - N_3, N_2 - N_3)\,, \qquad J_1 \geq J_2 \geq J_3 \geq 0.$$

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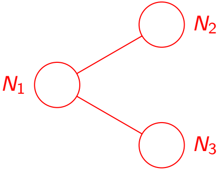

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- Here are the corresponding Dynkin indices:

$$[\mathbf{w}\cdot\boldsymbol{\alpha}_2, \mathbf{w}\cdot\boldsymbol{\alpha}_1, \mathbf{w}\cdot\boldsymbol{\alpha}_3] = [J_2 - J_3, J_1 - J_2, J_2 + J_3] = [N_1 - 2N_2, L - 2N_1 + N_2 + N_3, N_1 - 2N_3]\,.$$

Rapidities & fields

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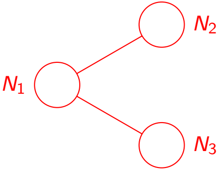

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- The $\mathfrak{so}(6)$ Cartan matrix is

$$M_{ab} = \frac{2\alpha_a \cdot \alpha_b}{\alpha_b^2} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

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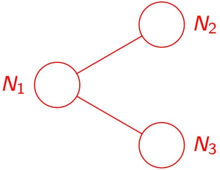

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- Each complex scalar field is associated to the following set of weights:

$$\mathcal{Z} \sim \mathbf{q} \qquad \mathcal{W} \sim \mathbf{q} - \boldsymbol{\alpha}_1 \qquad \mathcal{Y} \sim \mathbf{q} - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2$$
$$\overline{\mathcal{Z}} \sim \mathbf{q} - 2\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 \qquad \overline{\mathcal{W}} \sim \mathbf{q} - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 \qquad \overline{\mathcal{Y}} \sim \mathbf{q} - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_3.$$

Coordinate Nested Bethe Ansatz

Here's the nested $\mathfrak{so}(6)$ wavefunction (in a somewhat simplified form):

$$\psi_i(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \sum_{P_1} A_1(P_1) \prod_{j=1}^{N_1} \frac{1}{u_{1,P_{1,j}} - i/2} \left(\frac{u_{1,P_{1,j}} + i/2}{u_{1,P_{1,j}} - i/2} \right)^{n_{1,j}-1} \cdot \psi_{(2,i)}(\mathbf{u}_1, \mathbf{u}_2) \cdot \psi_{(3,i)}(\mathbf{u}_1, \mathbf{u}_3),$$

where

$$\psi_{(a,i)}(\mathbf{u}_1, \mathbf{u}_a) = \sum_{P_a} A_a(P_a) \prod_{j=1}^{N_a} \frac{1}{u_{a,P_{a,j}} - u_{1,P_{1,n_{a,j}}} - i/2} \prod_{k=1}^{n_{a,j}-1} \frac{u_{a,P_{a,j}} - u_{1,P_{1,k}} + i/2}{u_{a,P_{a,j}} - u_{1,P_{1,k}} - i/2}, \qquad a = 2, 3,$$

and

$$A_a(\ldots, k, j, \ldots) = A_a(\ldots, j, k, \ldots)\, S_a(u_{a,k}, u_{a,j}), \quad S_a(u_{a,k}, u_{a,j}) \equiv \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i}.$$

Bethe equations

- The periodicity of the Bethe wavefunction ψ (at each nesting level) leads to the Bethe equations:

$$\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^L = \prod_{j\neq i}^{N_1} \frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i} \prod_{k=1}^{N_2} \frac{u_{1,i}-u_{2,k}-i/2}{u_{1,i}-u_{2,k}+i/2} \prod_{l=1}^{N_3} \frac{u_{1,i}-u_{3,l}-i/2}{u_{1,i}-u_{3,l}+i/2}, \quad i=1,\ldots,N_1 \equiv M$$

$$1 = \prod_{l\neq i}^{N_2} \frac{u_{2,i}-u_{2,l}+i}{u_{2,i}-u_{2,l}-i} \prod_{k=1}^{N_1} \frac{u_{2,i}-u_{1,k}-i/2}{u_{2,i}-u_{1,k}+i/2}, \quad i=1,\ldots,N_2 \equiv N_+$$

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which must be satisfied by the rapidities of the excitations/Bethe roots.

- Because of the cyclicity of the trace, the momentum carrying roots obey the following relation:

$$\prod_{i=1}^{N_1} \frac{u_{1,i}+i/2}{u_{1,i}-i/2} = 1 \Leftrightarrow \sum_{i=1}^{N_1} p_{1,i} = 0 \qquad \text{(momentum conservation)},$$

where the relation of the rapidities to momenta is $u_{a,i} \equiv 1/2 \cot(p_{a,i}/2)$...

- Solving the Bethe system fast and efficiently is a hot topic... best method we will also use: *fast Bethe solver* (Marboe-Volin, 2014 & 2017; Marboe, 2017), based on the QQ system (requiring the solutions to be polynomials)...

Bethe state overlaps

- The matrix product state projects the 3 complex scalars on the $SU(2)$ fuzzy funnel solution:

$$\langle \text{MPS}|\Psi\rangle = z^L \cdot \sum_{1\leq x_k \leq L} \psi(x_k) \cdot \text{tr}\left[\mathcal{Z}^{x_1-1}\mathcal{W}\mathcal{Z}^{x_2-x_1-1}\mathcal{Y}\mathcal{Z}^{x_3-x_2-1}\overline{\mathcal{W}}\mathcal{Z}^{x_4-x_3-1}\overline{\mathcal{Y}}\ldots\right],$$

where the complex scalar fields $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{Y}$ are expressed in terms of the $\mathfrak{su}(2)$ matrices as follows:

$$\mathcal{W} = \overline{\mathcal{W}} = \frac{t_1}{z}, \qquad \mathcal{Y} = \overline{\mathcal{Y}} = \frac{t_2}{z}, \qquad \mathcal{Z} = \overline{\mathcal{Z}} = \frac{t_3}{z}.$$

- The corresponding matrix product state (MPS) is given by:

$$|\text{MPS}\rangle = \text{tr}_a\left[\prod_{\mathrm{l}=1}^{L} |\mathcal{Z}\rangle_{\mathrm{l}} \otimes t_3 + |\mathcal{W}\rangle_{\mathrm{l}} \otimes t_1 + |\mathcal{Y}\rangle_{\mathrm{l}} \otimes t_2 + \text{c.c.}\right].$$

The $\mathfrak{su}(2)$ subsector

For example, let us first consider the subsector that contains only two complex scalars:

$$\mathcal{W} = \varphi_1 + i\varphi_2 \quad \longleftrightarrow \quad |\uparrow\rangle \ \sim t_1$$

$$\mathcal{Z} = \varphi_5 + i\varphi_6 \quad \longleftrightarrow \quad |\bullet\rangle \ \sim t_3.$$

This is also known as the $\mathfrak{su}(2)$ subsector of the dCFT. In the $\mathfrak{su}(2)$ subsector, the trace operator $\mathbb{K}_{j,j+1}$ does not contribute to the mixing matrix $\mathbb{D}$:

$$\mathbb{H}_{\mathfrak{su}(2)} = \sum_{j=1}^{L} \left(\mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1}\right).$$

This is just the Hamiltonian of the Heisenberg $\text{XXX}_{1/2}$ spin chain. The MPS can be written as follows:

$$|\text{MPS}\rangle = \text{tr}_a \left[\prod_{j=1}^{L} \Big(|\uparrow_j\rangle \otimes t_1 + |\bullet_j\rangle \otimes t_3 \Big)\right],$$

and it corresponds to the above choice of fields.

$\mathfrak{su}(2)$ Bethe states

In the $\mathfrak{su}(2)$ subsector, $|\Psi\rangle$ is just the coordinate Bethe state $|\mathbf{p}\rangle$:

$$|\mathbf{p}\rangle = \mathfrak{N} \cdot \sum_{\sigma \in S_M} \sum_{1 \leq n_1 \leq \ldots \leq n_M \leq L} \exp\left[i \sum_k p_{\sigma(k)} n_k + \frac{i}{2} \sum_{j<k} \theta_{\sigma(j)\sigma(k)}\right] |\mathbf{x}\rangle, \qquad |\mathbf{p}\rangle \equiv |p_1, p_2, \ldots, p_M\rangle \, .$$

where

$$|\mathbf{x}\rangle \equiv |x_1, x_2, \ldots, x_M\rangle \equiv |\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\uparrow} \bullet \ldots \bullet \underset{x_M}{\uparrow} \bullet \ldots \bullet\rangle = S^-_{n_1} \ldots S^-_{n_M} |0\rangle \, ,$$

and the vacuum state $|0\rangle$ and the raising and lowering operators $S^{\pm}$ have been defined as

$$|0\rangle = \bigotimes_{i=1}^{L} |\bullet\rangle \, , \qquad S^+ |\uparrow\rangle = |\bullet\rangle \quad \& \quad S^- |\bullet\rangle = |\uparrow\rangle \, .$$

The matrix θ_{jk} and the normalization constant $\mathfrak{N}$ are given by:

$$e^{i\theta_{jk}} = \frac{u_j - u_k + i}{u_j - u_k - i} \equiv S_{jk} \, , \qquad u_j \equiv \frac{1}{2} \cot \frac{p_j}{2} \, , \qquad \mathfrak{N} \equiv \exp\left[-\frac{i}{2} \sum_{j<k} \theta_{jk}\right] .$$

The $\mathfrak{su}(3)$ and $\mathfrak{so}(6)$ subsectors

- In the $\mathfrak{su}(3)$ subsector all the three real complex scalars contribute:

$$\mathcal{W} = \varphi_1 + i\varphi_2 \sim t_1, \qquad \mathcal{Y} = \varphi_3 + i\varphi_4 \sim t_2, \qquad \mathcal{Z} = \varphi_5 + i\varphi_6 \sim t_3.$$

The corresponding wavefunction is constructed by means of the nested coordinate Bethe ansatz:

$$\psi = \sum_{P_1, P_2} A_1(P_1) A_2(P_2) \prod_{j=1}^{N_1} \prod_{j=1}^{N_2} \left(\frac{u_{1,P_{1,j}} + i/2}{u_{1,P_{1,j}} - i/2}\right)^{n_{1,j}} \prod_{k=1}^{n_{2,j}} \frac{\left(u_{2,P_{2,j}} - u_{1,P_{1,k}} + i/2\right)^{\delta_{k \neq n_{2,j}}}}{u_{2,P_{2,j}} - u_{1,P_{1,k}} - i/2}$$

$$A_a(\ldots, k, j, \ldots) = A_a(\ldots, j, k, \ldots) S_a(u_{a,k}, u_{a,j}), \qquad S_a(u_{a,k}, u_{a,j}) \equiv \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i}.$$

- In the $\mathfrak{so}(6)$ subsector all the three real complex scalars contribute:

$$\mathcal{W} = \overline{\mathcal{W}} = \varphi_1 + i\varphi_2 \sim t_1, \qquad \mathcal{Y} = \overline{\mathcal{Y}} = \varphi_3 + i\varphi_4 \sim t_2, \qquad \mathcal{Z} = \overline{\mathcal{Z}} = \varphi_5 + i\varphi_6 \sim t_3,$$

and similarly the $\mathfrak{so}(6)$ wavefunction can be constructed by the nested coordinate Bethe ansatz.

Subsection 5

$\mathfrak{su}(2)_k$ representations

k-dimensional Representation of $\mathfrak{su}(2)$

We use the following $k \times k$ dimensional representation of $\mathfrak{su}(2)$:

$$t_+ = \sum_{i=1}^{k-1} c_{k,i} E^i_{i+1}, \qquad t_- = \sum_{i=1}^{k-1} c_{k,i} E^{i+1}_i, \qquad t_3 = \sum_{i=1}^{k} d_{k,i} E^i_i$$

$$t_1 = \frac{t_+ + t_-}{2}, \qquad t_2 = \frac{t_+ - t_-}{2i}$$

$$c_{k,i} = \sqrt{i(k-i)}, \qquad d_{k,i} = \frac{1}{2}(k - 2i + 1),$$

where E^i_j are the standard matrix unities that are zero everywhere except (i,j) where they're 1.

Section 5

The D3-D7 defect

S^4 geometry

Start again from the near-horizon geometry ($r \to 0$) of a system of N_c coincident D3-branes,

$$ds^2 = H^{-1/2}\left(-dt^2 + d\mathbf{x}_3^2\right) + H^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \quad H(r) \equiv 1 + \left(\frac{\ell}{r}\right)^4, \quad \ell^4 = 4\pi g_s N_c \ell_s^4,$$

that is $AdS_5 \times S^5$ in the so-called Poincaré coordinates:

$$ds^2 = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right) = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\sum_{i=4}^{9} dx_i^2,$$

where $\mathbf{x}_3^2 = x_1^2 + x_2^2 + x_3^2$ and $r^2 = x_4^2 + \ldots + x_9^2$.

S^4 geometry

Start again from the near-horizon geometry ($r \to 0$) of a system of N_c coincident D3-branes,

$$ds^2 = H^{-1/2}\left(-dt^2 + d\mathbf{x}_3^2\right) + H^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \quad H(r) \equiv 1 + \left(\frac{\ell}{r}\right)^4, \quad \ell^4 = 4\pi g_s N_c \ell_s^4,$$

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where $\mathbf{x}_3^2 = x_1^2 + x_2^2 + x_3^2$ and $r^2 = x_4^2 + \ldots + x_9^2$. If we set $r^2 \equiv \rho^2 + x_9^2$, the metric becomes:

$$ds^2 = \frac{r^2}{\ell^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{\ell^2}{r^2}\left(d\rho^2 + \rho^2 d\Omega_4^2 + dx_9^2\right).$$

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Now insert a single D7-brane at $x_3 = x_9 = 0$:

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D7	•	•	•		•	•	•	•	•	

S^4 geometry

Start again from the near-horizon geometry ($r \to 0$) of a system of N_c coincident D3-branes,

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Now insert a single D7-brane at $x_3 = x_9 = 0$. The geometry it sees is $AdS_4 \times S^4$.

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D7	•	•	•		•	•	•	•	•	

S^4 geometry

Start again from the near-horizon geometry ($r \to 0$) of a system of N_c coincident D3-branes,

$$ds^2 = H^{-1/2}\left(-dt^2 + d\mathbf{x}_3^2\right) + H^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \quad H(r) \equiv 1 + \left(\frac{\ell}{r}\right)^4, \quad \ell^4 = 4\pi g_s N_c \ell_s^4,$$

that is $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the so-called Poincaré coordinates:

$$ds^2 = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right) = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\sum_{i=4}^{9} dx_i^2,$$

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$$ds^2 = \left\{\frac{r^2}{\ell^2}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \frac{\ell^2}{r^2}\,dr^2\right\} + \ell^2 d\Omega_4^2.$$

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D3	•	•	•	•						
D7	•	•	•		•	•	•	•	•	

The same result is of course obtained from the DBI analysis (Davis-Kraus-Shah, 2008; Myers-Wapler, 2008)...

$S^2 \times S^2$ geometry

Start from the metric of $AdS_5 \times S^5$:

$$ds^2 = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right) = \frac{r^2}{\ell^2}\left(-dt^2 + d\mathbf{x}_3^2\right) + \frac{\ell^2}{r^2}\sum_{i=4}^{9} dx_i^2,$$

where $r^2 = x_4^2 + \ldots + x_9^2$ and

$$x_4 = r\cos\psi\sin\theta\cos\varphi, \quad x_5 = r\cos\psi\sin\theta\sin\varphi, \quad x_6 = r\cos\psi\cos\theta,$$
$$x_7 = r\sin\psi\sin\vartheta\cos\chi, \quad x_8 = r\sin\psi\sin\vartheta\sin\chi, \quad x_9 = r\sin\psi\cos\vartheta.$$

Then the metric of $AdS_5 \times S^5$ is written as:

$$ds^2 = \left\{\frac{r^2}{\ell^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{\ell^2}{r^2}\,dr^2\right\} + \ell^2\left(d\psi^2 + \cos^2\psi d\Omega_2^2 + \sin^2\psi d\tilde{\Omega}_2^2\right).$$

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Now insert a single D7 brane at $x_3 = 0$, $\psi = \pi/4$...

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$S^2 \times S^2$ geometry

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Now insert a single D7 brane at $x_3 = 0$, $\psi = \pi/4$... The D7-brane geometry is $AdS_4 \times S^2 \times S^2$... Same result follows from the DBI analysis (Bergman-Jokela-Lifschytz-Lippert, 2010)...

- The two S^2's have equal sizes and sit on the equator of S^5...
- The configuration is again unstable towards slipping off each side of the equator...

$S^2 \times S^2$ geometry

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- The two S^2's have equal sizes and sit on the equator of S^5...
- The configuration is again unstable towards slipping off each side of the equator...
- The D7-brane can be stabilized by adding k units of abelian flux on each S^2...

The D3-D7 system

The probe D7-brane geometry is either $AdS_4 \times S^2 \times S^2$ or $AdS_4 \times S^4$. The brane sits at $x_3 = x_9 = 0$:

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•	•						
D7	•	•	•		•	•	•	•	•	

Again there's a tachyonic instability... this time it violates the BF bound and the brane is unstable (Davis-Kraus-Shah, 2008; Myers-Wapler, 2008; Bergman-Jokela-Lifschytz-Lippert, 2010)...

The D3-D7 system

The probe D7-brane geometry is either $AdS_4 \times S^2 \times S^2$ or $AdS_4 \times S^4$. The brane sits at $x_3 = x_9 = 0$:

	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	●	●	●	●						
D7	●	●	●		●	●	●	●	●	

Again there's a tachyonic instability... this time it violates the BF bound and the brane is unstable (Davis-Kraus-Shah, 2008; Myers-Wapler, 2008; Bergman-Jokela-Lifschytz-Lippert, 2010)... To stabilize it we add:

- An instanton bundle on the S^4 component of the $AdS_4 \times S^4$ probe D7-brane, with instanton number d_G:

$$d_G = \frac{1}{6}(n+1)(n+2)(n+3)\,.$$

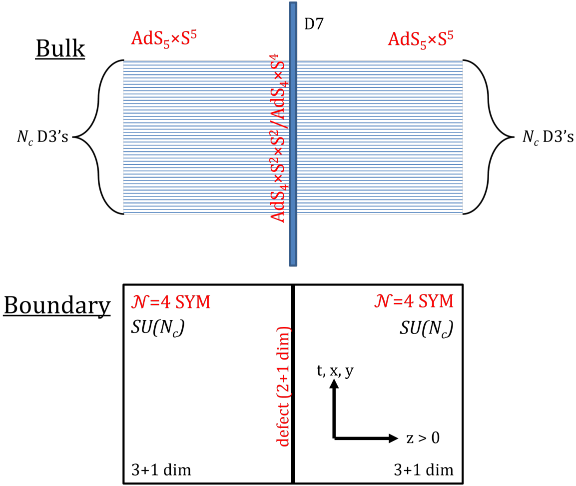
Myers-Wapler (2008)

- $k_{1,2}$ units of $U(1)$ flux on each of the S^2 components of the $AdS_4 \times S^2 \times S^2$ probe D7-brane...

Bergman-Jokela-Lifschytz-Lippert (2010)

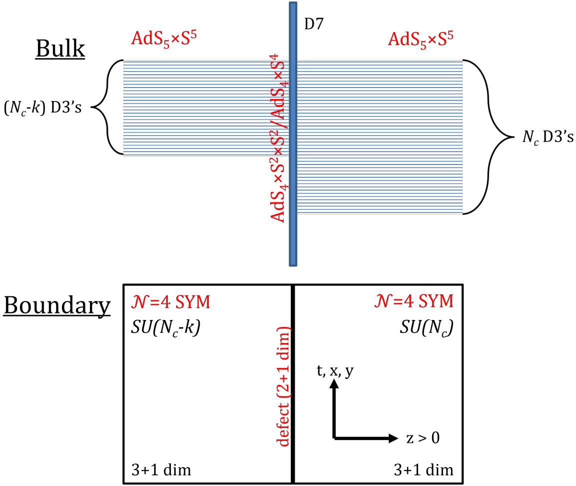
The $(D3\text{-}D7)_k$ system

Same picture as before: begin with $SU(N_c) \times SU(N_c)$, $\mathcal{N} = 4$ SYM,



The (D3-D7)$_k$ system

End up with $SU(N_c - k) \times SU(N_c)$, $k = k_1 \cdot k_2$ or $k = d_G = (n+1)(n+2)(n+3)/6$ $(k \ll N_c \to \infty)$:



$(\text{D3-D7})_k$ solutions

For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N}=4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = [\varphi_j, [\varphi_j, \varphi_i]], \quad i, j = 1, \ldots, 6.$$

We find two solutions (Kristjansen-Semenoff-Young, 2012):

$$SU(2)\times SU(2): \quad \varphi_i = \begin{cases} -\frac{1}{z}\left[(t_i)_{k_1} \otimes \mathbb{1}_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 1,2,3 \\ -\frac{1}{z}\left[\mathbb{1}_{k_1} \otimes (t_i)_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 4,5,6 \end{cases}$$

$$SO(5): \quad \varphi_i = \frac{G_i}{\sqrt{8}z}, \quad i = 1, \ldots, 5, \qquad \varphi_6 = 0,$$

where the matrices t_i furnish a k_i-dimensional $(i = 1, 2)$ representation of $\mathfrak{su}(2)$...

$$[t_i, t_j] = i\epsilon_{ijk} t_k,$$

and the five $d_G \times d_G$ matrices G_i are known as "fuzzy" S^4 matrices...

Subsection 2

Symmetrized direct products & fuzzy S^4 matrices

Symmetrized direct products

Consider the following symmetrized matrix direct product

$$\left[A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes \ldots \otimes A^{(n)}\right]_{\text{sym}},$$

where the A's are $k \times k$ matrices. In Dirac's notation this can be written as follows:

$$_{\text{sym}}\left\langle i_1, i_2, \ldots, i_n \right| A^{(1)} \otimes A^{(2)} \otimes \ldots \otimes A^{(n)} \left| j_1, j_2, \ldots, j_n \right\rangle_{\text{sym}}, \quad i_1, \ldots, i_n,\ j_1, \ldots, j_n = 1, 2, \ldots, k,$$

with

$$\left| j_1, j_2, \ldots, j_n \right\rangle_{\text{sym}} = \frac{1}{\sqrt{\|\sigma(j)\|}} \sum_{\sigma} \left| \sigma(j_1), \sigma(j_2), \ldots, \sigma(j_n) \right\rangle,$$

where $\|\sigma(j)\|$ gives the number of permutations of $(j_1, j_2, \ldots, j_n)$. For $n = k = 2$,

$$\begin{pmatrix} A^{(1)}_{11} A^{(2)}_{11} & \frac{A^{(1)}_{12} A^{(2)}_{11} + A^{(1)}_{11} A^{(2)}_{12}}{\sqrt{2}} & A^{(1)}_{12} A^{(2)}_{12} \\ \frac{A^{(1)}_{21} A^{(2)}_{11} + A^{(1)}_{11} A^{(2)}_{21}}{\sqrt{2}} & \frac{1}{2}\left(A^{(1)}_{22} A^{(2)}_{11} + A^{(1)}_{21} A^{(2)}_{12} + A^{(1)}_{12} A^{(2)}_{21} + A^{(1)}_{11} A^{(2)}_{22}\right) & \frac{A^{(1)}_{22} A^{(2)}_{12} + A^{(1)}_{12} A^{(2)}_{22}}{\sqrt{2}} \\ A^{(1)}_{21} A^{(2)}_{21} & \frac{A^{(1)}_{22} A^{(2)}_{21} + A^{(1)}_{21} A^{(2)}_{22}}{\sqrt{2}} & A^{(1)}_{22} A^{(2)}_{22} \end{pmatrix}.$$

Symmetrized direct products

The dimension equals the # of different arrangements of n stars and $k-1$ bars (k multichoose n):

$$\left(\!\!\binom{k}{n}\!\!\right) = \binom{n+k-1}{n} \xrightarrow{k=4} \left(\!\!\binom{4}{n}\!\!\right) = \binom{n+4-1}{n} = \frac{1}{6}(n+1)(n+2)(n+3) = d_G .$$

Here's the dimensionality of the symmetrized matrix product for various k's and n's:

k/n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8	9	10	11
3	3	6	10	15	21	28	36	45	55	66
4	4	10	20	35	56	84	120	165	220	286
5	5	15	35	70	126	210	330	495	715	1001
6	6	21	56	126	252	462	792	1287	2002	3003
7	7	28	84	210	462	924	1716	3003	5005	8008
8	8	36	120	330	792	1716	3432	6435	11440	19448
9	9	45	165	495	1287	3003	6435	12870	24310	43758
10	10	55	220	715	2002	5005	11440	24310	48620	92378

The fuzzy S^4 matrices: construction

The five fuzzy $d_G \times d_G$ dimensional S^4 matrices $G_i = nX_i/r$ obey the following properties:

Castelino-Lee-Taylor (1997)

- Spherical locus: $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = r\,\mathbb{I}$.
- Longitudinal 5-brane charge: $\epsilon_{ijklm} X_i X_j X_k X_l = \alpha X_m$.
- Local flatness ...
- Rotational invariance: $R_{ij} X_j = U(R) \cdot X_i \cdot U\left(R^{-1}\right)$,
  where R_{ij} is an element of $SO(5)$ and $U(R)$ is a d_G dimensional unitary representation of $SO(5)$.
- Spectrum ...

The fuzzy S^4 G-matrices

Here's the definition of the five $d_G \times d_G$ fuzzy S^4 matrices (G-matrices) G_i:

$$G_i \equiv \Big[\underbrace{\overbrace{\gamma_i \otimes \mathbb{1}_4 \otimes \ldots \otimes \mathbb{1}_4}^{n \text{ factors}} + \mathbb{1}_4 \otimes \gamma_i \otimes \ldots \otimes \mathbb{1}_4 + \ldots + \mathbb{1}_4 \otimes \ldots \otimes \mathbb{1}_4 \otimes \gamma_i}_{n \text{ terms}}\Big]_{\text{sym}} \qquad (i = 1, \ldots, 5),$$

Castelino-Lee-Taylor (1997)

where γ_i are the five 4×4 Euclidean Dirac matrices:

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

and σ_i are the three 2×2 Pauli matrices. The ten commutators of the five G-matrices,

$$G_{ij} \equiv \frac{1}{2}\left[G_i, G_j\right],$$

furnish a d_G-dimensional (anti-hermitian) irreducible representation of $\mathfrak{so}\,(5) \simeq \mathfrak{sp}\,(4)$:

$$\left[G_{ij}, G_{kl}\right] = 2\left(\delta_{jk}\,G_{il} + \delta_{il}\,G_{jk} - \delta_{ik}\,G_{jl} - \delta_{jl}\,G_{ik}\right).$$

The fuzzy S^4 G-matrices

The dimension of the G-matrices is equal to the instanton number $d_G = (n+1)(n+2)(n+3)/6$:

n	1	2	3	4	5	6	7	8	9	10	...
d_G	4	10	20	35	56	84	120	165	220	286	...

E.g., for $n = 2$, here are the 10×10 G-matrices:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

The real diagonal matrices $G_{5(n)}$

The elements the diagonal matrices $G_{5(n)}$ are:

n	d_G	$G_{5(n)}$
1	4	$\{-1,-1,1,1\}$
2	10	$\{-2,-2,-2,0,0,0,0,2,2,2,2\}$
3	20	$\{-3,-3,-3,-3,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,3,3,3,3\}$
4	35	$\{-4,-4,-4,-4,-4,-2,-2,-2,-2,-2,-2,-2,-2,0,0,0,0,0,0,0,0,0,2,2,2,2,2,2,2,2,4,4,4,4,4\}$
		$\vdots$

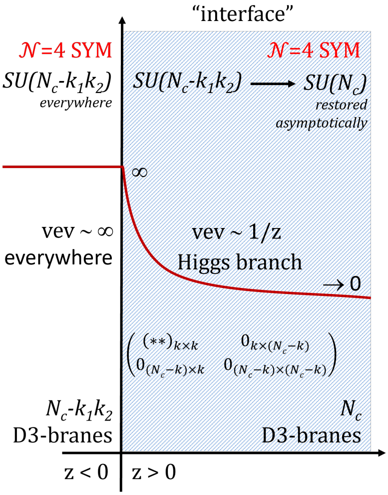
Therefore the general form of the matrices $G_{5(n)}$ is the following (for $j = 1, 2, \ldots, n+1$):

$$G_{5(n)} = 2\Big\{\underbrace{\Big\{-\frac{n}{2},\ldots\Big\}}_{(n+1)\ \text{terms}},\ \underbrace{\Big\{-\frac{n}{2}+1,\ldots\Big\}}_{2n\ \text{terms}},\ \ldots,\ \underbrace{\Big\{-\frac{n}{2}+j-1,\ldots\Big\}}_{j\cdot(n-j+2)\ \text{terms}},\ \ldots,\ \underbrace{\Big\{\frac{n}{2}-1,\ldots\Big\}}_{2n\ \text{terms}},\ \underbrace{\Big\{\frac{n}{2},\ldots\Big\}}_{(n+1)\ \text{terms}}\Big\}.$$

Subsection 3

One-point functions

The D3-D7 interface: $SU(2) \times SU(2)$ symmetry



- To compute correlation functions in the dCFT that is dual to the $SU(2) \times SU(2)$ symmetric D3-D7 system, we set up the corresponding interface...
- The interface (placed at $z = 0$) separates the $SU(N_c)$ and $SU(N_c - k_1 k_2)$ regions of the $(\text{D3-D7})_{k_1 k_2}$ dCFT... It will be described by a fuzzy funnel solution...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = \left[\varphi_j, \left[\varphi_j, \varphi_i\right]\right], \quad i, j = 1, \ldots, 6.$$

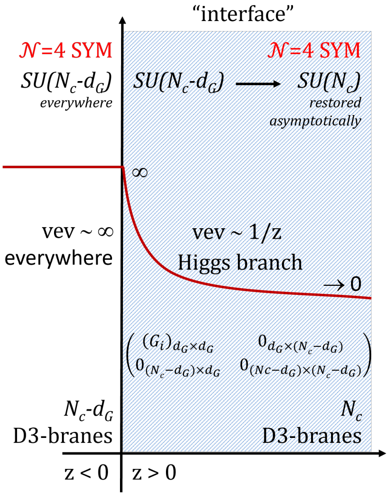
- The wanted $SU(2) \times SU(2) \subset SU(3,2) \times SU(2) \times SU(2)$ solution is:

$$\varphi_i(z) = -\frac{1}{z} \times \begin{cases} \left[(t_i)_{k_1} \otimes \mathbb{1}_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 1, 2, 3 \\ \left[\mathbb{1}_{k_1} \otimes (t_i)_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 4, 5, 6. \end{cases}$$

Kristjansen-Semenoff-Young (2012)

- The defect CFT is not supersymmetric so that the interface does not satisfy the Nahm equations...

The D3-D7 interface: $SO(5)$ symmetry



- The interface for the dCFT that is dual to the $SO(5)$ symmetric D3-D7 system (placed at $z=0$) separates the $SU(N_c)$ and $SU(N_c-d_G)$ regions of the $(\text{D3-D7})_{d_G}$ dCFT... It will be described by a fuzzy funnel solution...
- For no vectors/fermions, we solve the equations of motion for the scalar fields of $\mathcal{N}=4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\varphi_i}{dz^2} = \left[\varphi_j, \left[\varphi_j, \varphi_i\right]\right], \quad i,j = 1,\ldots,6.$$

- A manifestly $SO(5) \subset SO(3,2) \times SO(5)$ symmetric solution is given by:

$$\varphi_i(z) = \frac{G_i \oplus 0_{(N_c-d_G)\times(N_c-d_G)}}{\sqrt{8}\,z}, \quad i = 1,\ldots,5, \qquad \varphi_6 = 0\,.$$

Kristjansen-Semenoff-Young (2012)

- Once more, the defect CFT is not supersymmetric so that the interface does not satisfy the Nahm equations...
- The five $d_G \times d_G$ matrices G_i are known as the "fuzzy" S^4 matrices...

One-point functions

One-point functions of local gauge-invariant scalar operators,

$$\langle \mathcal{O}(\mathrm{z}, \mathbf{x}) \rangle = \frac{\mathcal{C}}{\mathrm{z}^{\Delta}}, \qquad \mathrm{z} > 0,$$

can again be calculated within the D3-D7 defect CFT from the corresponding fuzzy funnel solution...

$$\boxed{\mathcal{O}(\mathrm{z}, \mathbf{x}) = \Psi^{i_1 \ldots i_L} \mathrm{tr}\left[\varphi_{i_1} \ldots \varphi_{i_L}\right] \xrightarrow[\text{interface}]{SO(5),\ SO(3)\times SO(3)} \frac{1}{z^L} \cdot \Psi^{i_1 \ldots i_L} \mathrm{tr}\left[\tau_{i_1} \ldots \tau_{i_L}\right]},$$

where the matrices τ_i are defined in terms of the corresponding fuzzy funnel solution:

$$\tau_i = \begin{cases} \left.\begin{array}{ll} G_i/\sqrt{8}, & i = 1, \ldots, 5 \\ 0, & i = 6 \end{array}\right\}, & SO(5) \text{ symmetric interface} \\ \left.\begin{array}{ll} \left[(t_i)_{k_1} \otimes \mathbb{1}_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 1, 2, 3 \\ \left[\mathbb{1}_{k_1} \otimes (t_i)_{k_2}\right] \oplus 0_{(N_c - k_1 k_2)}, & i = 4, 5, 6 \end{array}\right\}, & SO(3) \times SO(3) \text{ symmetric interface.} \end{cases}$$

Again, $\Psi^{i_1 \cdots i_L}$ is an $\mathfrak{so}(6)$-symmetric tensor and the constant $\mathcal{C}$ is given by (MPS="*matrix product state*"),

$$\mathcal{C} = \frac{1}{\sqrt{L}} \left(\frac{\pi^2}{\lambda}\right)^{L/2} \cdot \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \qquad \left\{ \begin{array}{ll} \langle \mathrm{MPS} | \Psi \rangle \equiv \Psi^{i_1 \cdots i_L} \mathrm{tr}\left[G_{i_1} \ldots G_{i_L}\right] & (\text{"overlap"}) \\ \langle \Psi | \Psi \rangle \equiv \Psi^{i_1 \cdots i_L} \Psi_{i_1 \ldots i_L} & \end{array} \right\}.$$

Chiral primary operators

The one-point functions of $SO(5) \subseteq SO(6)$ invariant chiral primary operators (CPOs),

$$\mathcal{O}_{\mathrm{CPO}}(x) = \frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot K^{\mu_1\cdots\mu_L}\mathrm{tr}\left[\varphi_{\mu_1}(x)\ldots\varphi_{\mu_L}(x)\right],$$

where $K^{\mu_1\cdots\mu_L}$ are symmetric & traceless $SO(5) \subseteq SO(6)$ tensors satisfying,

$$K^{\mu_1\cdots\mu_L}K^{\mu_1\cdots\mu_L} = 1 \qquad \& \qquad Y_L = K^{\mu_1\cdots\mu_L}x_{\mu_1}\ldots x_{\mu_L}, \qquad \sum_{\mu=4}^{9} x_\mu^2 = 1,$$

and $Y_L(\psi)$ is the $SO(5) \subseteq SO(6)$ spherical harmonic, have been calculated at weak coupling:

$$\langle\mathcal{O}_{\mathrm{CPO}}(\mathrm{x})\rangle = \frac{d_G}{\sqrt{L}}\left(\frac{\pi^2 c_{\mathrm{G}}}{\lambda}\right)^{L/2}\frac{Y_L(0)}{\mathrm{z}^L}, \quad c_{\mathrm{G}} \equiv n(n+4), \quad d_G \equiv \frac{1}{6}\cdot(n+1)(n+2)(n+3) \ll N_c \to \infty,$$

Kristjansen-Semenoff-Young (2012)

where $n = 1, 2, \ldots, L = 2j, \ j = 0, 1, \ldots$ The large-n limit agrees with the supergravity calculation:

$$\langle\mathcal{O}_{\mathrm{CPO}}(\mathrm{x})\rangle \xrightarrow{n\to\infty} \frac{Y_L(0)}{\sqrt{L}}\left(\frac{\pi^2 n^2}{\lambda}\right)^{L/2}\frac{n^3}{\mathrm{z}^L}.$$

Once more, we can go beyond CPOs (de Leeuw-Kristjansen-GL, 2016)...

Chiral primary operators

The one-point functions of $SU(2) \times SU(2) \subseteq SO(6)$ invariant chiral primary operators (CPOs),

$$\mathcal{O}_{\text{CPO}}(x) = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot K^{\mu_1 \cdots \mu_L} \text{tr}\left[\varphi_{\mu_1}(x) \ldots \varphi_{\mu_L}(x)\right],$$

where $K^{\mu_1 \cdots \mu_L}$ are symmetric & traceless $SO(3) \times SO(3) \subseteq SO(6)$ tensors satisfying,

$$K^{\mu_1 \cdots \mu_L} K^{\mu_1 \cdots \mu_L} = 1 \qquad \& \qquad Y_L = K^{\mu_1 \cdots \mu_L} x_{\mu_1} \ldots x_{\mu_L}, \qquad \sum_{\mu=4}^{6} x_\mu^2 = \cos^2\psi, \qquad \sum_{\mu=7}^{9} x_\mu^2 = \sin^2\psi,$$

and $Y_L(\psi)$ is the $SO(3) \times SO(3) \subseteq SO(6)$ spherical harmonic, have been calculated at weak coupling:

$$\langle \mathcal{O}_{\text{CPO}}(\mathrm{x}) \rangle = \frac{k_1 k_2}{\sqrt{L}} \left(\frac{2\pi^2\left(k_1^2 + k_2^2\right)}{\lambda}\right)^{L/2} \frac{Y_L\left(\arctan\left(k_2/k_1\right)\right)}{\mathrm{z}^L}, \qquad k \equiv k_1 k_2 \ll N_c \to \infty,$$

Kristjansen-Semenoff-Young (2012)

where $L = 2j$, $j = 0, 1, \ldots$ The large-n limit completely agrees with the supergravity calculation...

Section 6

The D2-D4 defect

The AdS/CFT correspondence

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The prototype of holographic dualities is the AdS$_5$/CFT$_4$ correspondence:

$$\mathcal{N} = 4,\ \mathfrak{su}(N_c) \text{ super Yang-Mills theory in 4d} \ \Leftrightarrow\ \text{Type IIB superstring theory on AdS}_5 \times \text{S}^5$$

Maldacena (1997)

the spectrum of which is quantum integrable in the planar ('t Hooft/large-N_c) limit $N_c \to \infty$, $\lambda \equiv g_{\text{YM}}^2 N_c = \text{const.}$

Minahan-Zarembo (2002), Beisert-Kristjansen-Staudacher (2003), Beisert (2003)

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Minahan-Zarembo (2002), Beisert-Kristjansen-Staudacher (2003), Beisert (2003)

There also exists an $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality... reading, for $k^5\gg N_c$:

$\mathcal{N}=6,\ U(N_c)_k\times \hat{U}(N_c)_{-k}$ super Chern-Simons theory in 3d with Chern-Simons levels $\pm k\in\mathbb{Z}$ $\Leftrightarrow$ Type IIA string theory on $\mathrm{AdS}_4\times\mathbb{CP}^3$ with N_c units of flux in AdS_4 and k units in $\mathbb{CP}^3$

Aharony-Bergman-Jafferis-Maldacena (2008)

the spectrum of IIA/ABJM is also quantum integrable in the planar limit $k, N_c\to\infty$, $\lambda\equiv g_{\mathrm{CS}}^2 N_c=\text{const.}$ ($g_{\mathrm{CS}}^2\equiv 1/k$).

Minahan-Zarembo (2008), Gaiotto-Giombi-Yin (2008), Bak-Rey (2008)

M/ABJM correspondence

In its full version, the AdS$_4$/CFT$_3$ duality takes the form of the M/ABJM correspondence:

$\mathcal{N} = 6$, $U(N_c)_k \times \hat{U}(N_c)_{-k}$ super Chern-Simons theory in 3d with Chern-Simons levels $\pm k \in \mathbb{Z}$ $\Leftrightarrow$ M-theory on AdS$_4 \times S^7/\mathbb{Z}_k$ with N_c units of flux in AdS$_4$

Aharony-Bergman-Jafferis-Maldacena (2008)

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- The duality emerges in the low-energy limit of N_c coincident M2-branes...

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Aharony-Bergman-Jafferis-Maldacena (2008)

- The duality emerges in the low-energy limit of N_c coincident M2-branes... the M2-branes live in an 8d transverse toric hyper-Kähler manifold with an $\mathbb{R}^8/\mathbb{Z}_k = \mathbb{C}^4/\mathbb{Z}_k$ singularity...

Gauntlett-Gibbons-Papadopoulos-Townsend (1997)

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$$\begin{array}{c}\mathcal{N}=6,\ U(N_c)_k \times \hat{U}(N_c)_{-k} \text{ super Chern-Simons} \\ \text{theory in 3d with Chern-Simons levels } \pm k \in \mathbb{Z}\end{array} \Leftrightarrow \begin{array}{c}\text{M-theory on AdS}_4 \times S^7/\mathbb{Z}_k \\ \text{with } N_c \text{ units of flux in AdS}_4\end{array}$$

Aharony-Bergman-Jafferis-Maldacena (2008)

- The duality emerges in the low-energy limit of N_c coincident M2-branes... the M2-branes live in an 8d transverse toric hyper-Kähler manifold with an $\mathbb{R}^8/\mathbb{Z}_k = \mathbb{C}^4/\mathbb{Z}_k$ singularity...

  Gauntlett-Gibbons-Papadopoulos-Townsend (1997)

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- For $k=2$, the dual gauge theory becomes the so-called BLG theory:

  $\mathcal{N}=8$, $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Bagger-Lambert-Gustavsson theory $\Leftrightarrow$ M-theory on $AdS_4 \times S^7/\mathbb{Z}_2$

  Bagger-Lambert (2007) & Gustavsson (2007)

D2-branes

- String theory limit: for $k^5 \gg N_c$, M-theory on the rhs of the duality becomes weakly coupled...

$$g_s \equiv \left(\frac{N_c}{k^5}\right)^{1/4} \to 0$$

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- Close to the horizon the spacetime becomes $AdS_4 \times \mathbb{CP}^3$, the metric of which is given by:

$$ds^2 = \frac{\ell^2}{z^2}\left(-dx_0^2 + dx_1^2 + dx_2^2 + dz^2\right) + 4\ell^2\Bigg[d\xi^2 + \cos^2\xi\,\sin^2\xi\left(d\psi + \frac{1}{2}\cos\theta_1\,d\phi_1 - \frac{1}{2}\cos\theta_2\,d\phi_2\right)^2 + \\ + \frac{1}{4}\cos^2\xi\left(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2\right) + \frac{1}{4}\sin^2\xi\left(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2\right)\Bigg],$$

where $\xi \in [0, \pi/2)$, $\theta_{1,2} \in [0, \pi]$, $\phi_{1,2} \in [0, 2\pi)$ and $\psi \in [-2\pi, 2\pi]$.

ABJM theory

Consider the IIA/ABJM correspondence we have just mentioned in its integrable limit:

$\mathcal{N}=6$, $U(N_c)_k \times \hat{U}(N_c)_{-k}$ super Chern-Simons theory with $k, N_c \to \infty$ & $\lambda \equiv N_c/k = \text{const.}$	$\Leftrightarrow$	Type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ with N_c units of flux in AdS_4 and k units in $\mathbb{CP}^3$

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On the one side of the duality lies a 3-dimensional superconformal gauge theory:

$$\mathcal{L}_{\text{ABJM}} = \frac{k}{4\pi}\cdot\Bigg[\epsilon^{\mu\nu\rho}\text{tr}\Big\{A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho - \hat{A}_\mu\partial_\nu\hat{A}_\rho - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\Big\} - \text{tr}\Big\{(D_\mu Y_B)^\dagger D^\mu Y_B + $$
$$+ i\psi_B^\dagger \not{D}\psi_B\Big\} - V_{\text{ferm}} - V_{\text{bos}}\Bigg], \text{ where } B = 1,\ldots,4,\ D_\mu Y \equiv \partial_\mu Y + iA_\mu Y - iY\hat{A}_\mu,$$

and the potential contains mixed quartic and sextic bosonic terms which read

$$V_{\text{ferm}} = \frac{i}{2}\text{tr}\Big\{Y_A^\dagger Y_A\psi_B^\dagger\psi_B - Y_A Y_A^\dagger\psi_B\psi_B^\dagger + 2Y_A Y_B^\dagger\psi_A\psi_B^\dagger - 2Y_A^\dagger Y_B\psi_A^\dagger\psi_B - \epsilon^{ABCD}Y_A^\dagger\psi_B Y_C^\dagger\psi_D + \epsilon^{ABCD}Y_A\psi_B^\dagger Y_C\psi_D^\dagger\Big\}$$

$$V_{\text{bos}} = -\frac{1}{12}\text{tr}\Big\{Y_A Y_A^\dagger Y_B Y_B^\dagger Y_C Y_C^\dagger + Y_A^\dagger Y_A Y_B^\dagger Y_B Y_C^\dagger Y_C + 4Y_A Y_B^\dagger Y_C Y_A^\dagger Y_B Y_C^\dagger - 6Y_A Y_B^\dagger Y_B Y_A^\dagger Y_C Y_C^\dagger\Big\}.$$

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- Global symmetry group is $OSP(2,2|6)$ for $k > 2$, enhanced to $OSP(2,2|8)$ for $k = 1, 2$.

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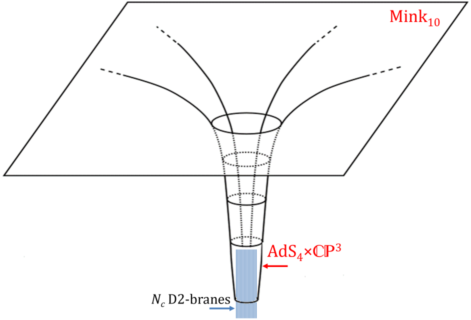
- Deformed ABJM: CS levels k and $\hat{k}$ do not sum to zero... less supersymmetry... no integrability...

Subsection 2

The D2-D4 geometries

The D2-D4 probe-brane system

Type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ is encountered very close to a system of N_c coincident D2-branes:



The D2-branes extend along x_1, x_2...

	t	x_1	x_2	z	ξ	θ_1	ϕ_1	θ_2	ϕ_2	ψ
D2	•	•	•							

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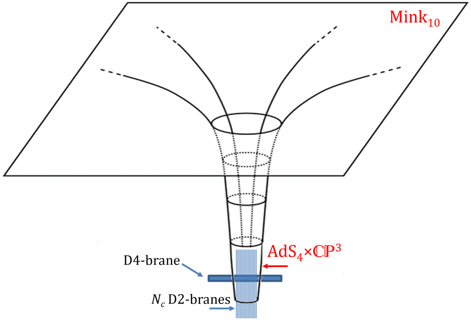

Now insert a single D4-brane at $x_1 = \xi = \theta_2 = \phi_2 = \psi = 0$...

	t	x_1	x_2	z	ξ	θ_1	ϕ_1	θ_2	ϕ_2	ψ
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D4	•		•	•		•	•			

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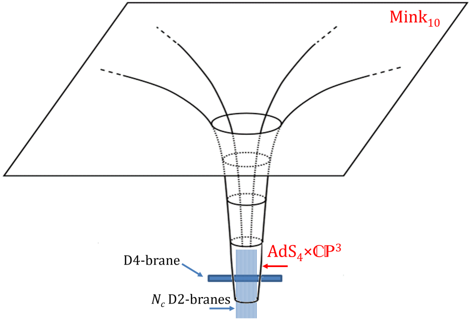

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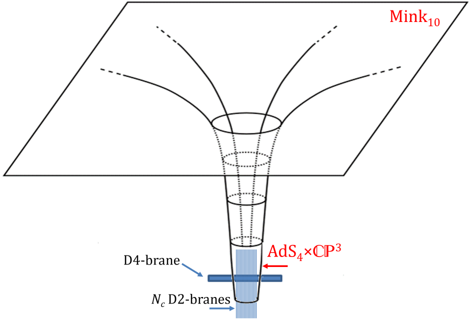

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where $\xi \in [0, \pi/2)\,,\ \theta_{1,2} \in [0, \pi]\,,\ \phi_{1,2} \in [0, 2\pi)\,,\ \psi \in [-2\pi, 2\pi]$.

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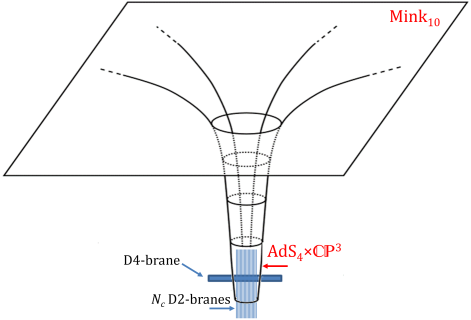

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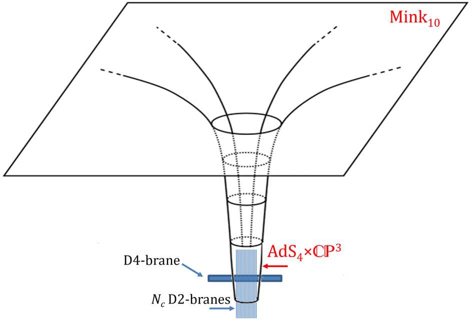

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Note that $\mathbb{CP}^1$ is just a 2-sphere: $ds^2_{\mathbb{CP}^1} = \ell^2\left(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2\right) = \sum_{i=4}^6 dx_i\, dx_i, \quad \sum_{i=4}^6 x_i\, x_i = \ell^2.$

D4-brane embedding

The brane geometry is also supported by Q units of magnetic flux through $\mathbb{CP}^1$...

$$F = \ell^2 \, Q \, d\cos\theta_1 \wedge d\phi_1 = -\ell^2 \, Q \, \sin\theta_1 \, d\theta_1 \, d\phi_1 = dA.$$

The flux forces exactly $q \equiv \sqrt{2\lambda}\, Q$ of the D2-branes to terminate on one side of the D4-brane...

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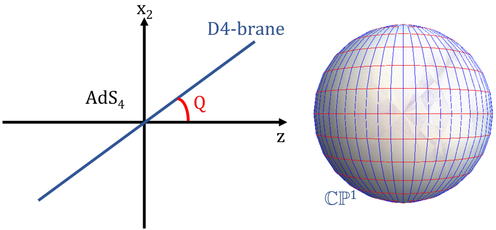
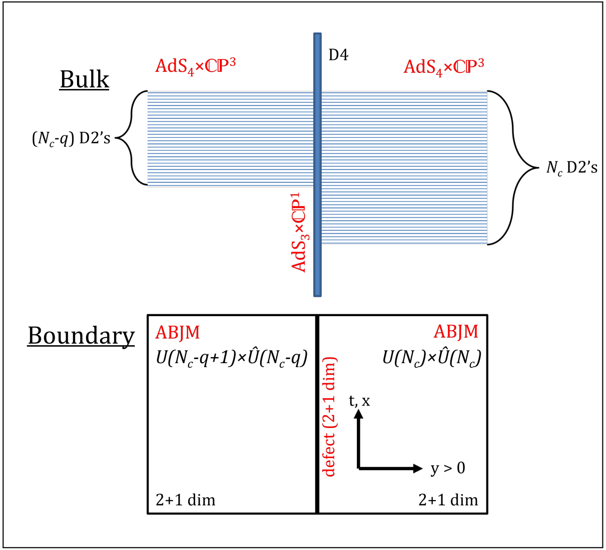


The (D2-D4)$_q$ dSCFT



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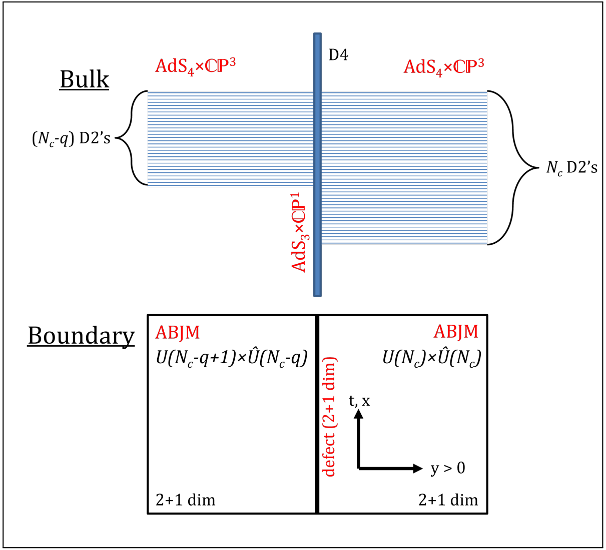

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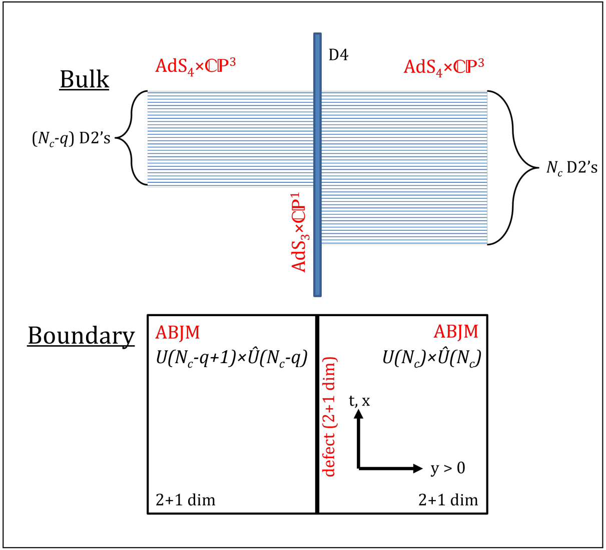

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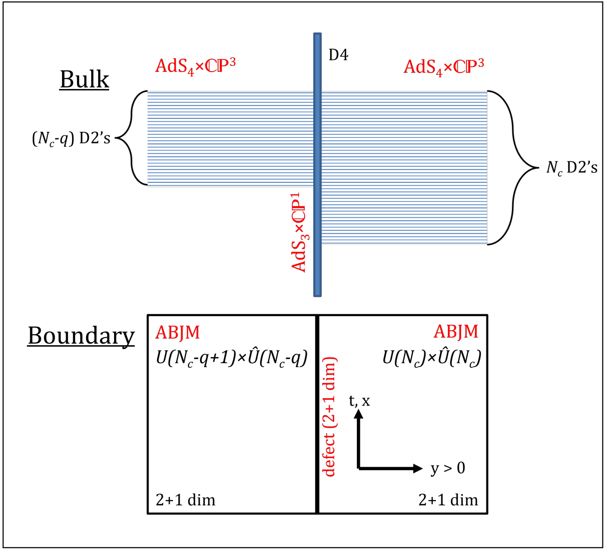
- The D4-brane is classically integrable... i.e. infinite conserved charges for open strings with D4-brane BCs (Dekel-Oz, 2011; GL, 2022)...

The $(\text{D2-D4})_q$ dSCFT



- The defect reduces the total bosonic symmetry of the system from $SO(3,2) \times SO(6)$ to $SO(2,2) \times SU(2) \times SU(2) \times U(1)$...
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- The D4-brane is classically integrable... i.e. infinite conserved charges for open strings with D4-brane BCs (Dekel-Oz, 2011; GL, 2022)...
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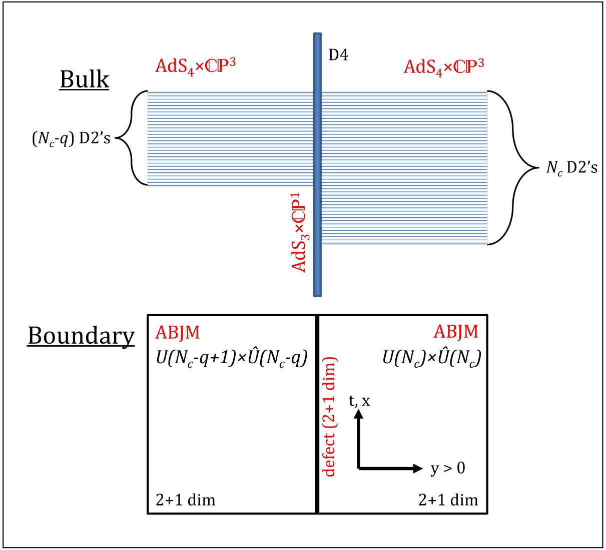

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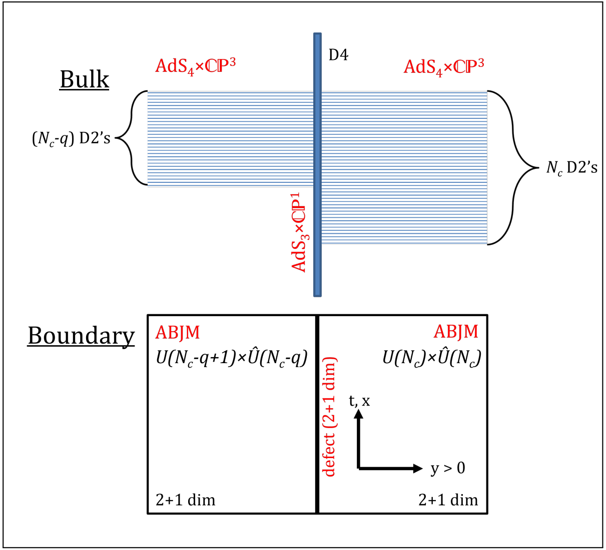

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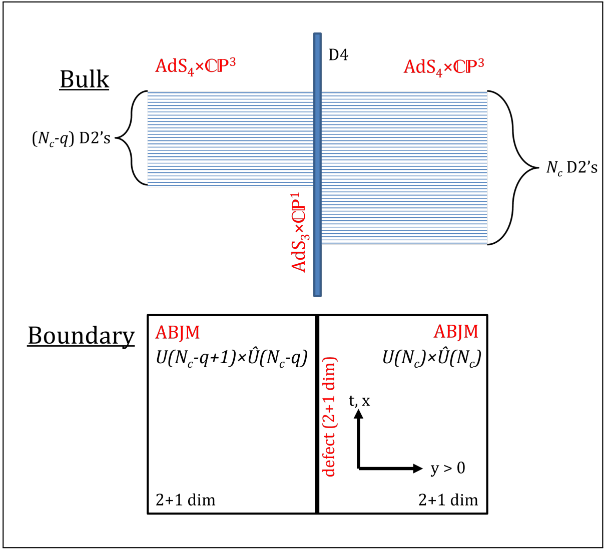

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- Strong-coupling computations were recently set up (Georgiou-GL-Zoakos, 2023)...

Subsection 3

T and R-matrices

T and R-matrices

The Lie algebra of $\mathfrak{so}(6)$ is generated by 15 matrices M_{ij},

$$[M_{ij}, M_{kl}] = \delta_{il} M_{jk} + \delta_{jk} M_{il} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik}, \qquad i, j = 1, \ldots 6.$$

The $\mathfrak{u}(3)$ subalgebra of $\mathfrak{so}(6)$ is generated by the 9 antisymmetric R-matrices (graded-0 generators):

$$R_1 = \frac{1}{2}(M_{13} + M_{24}), \quad R_2 = \frac{1}{2}(M_{23} - M_{14}), \quad R_3 = \frac{1}{2}(M_{15} + M_{26}), \quad R_4 = \frac{1}{2}(M_{25} - M_{16})$$
$$R_5 = \frac{1}{2}(M_{35} + M_{46}), \quad R_6 = \frac{1}{2}(M_{45} - M_{36}), \quad R_7 = M_{12}, \quad R_8 = M_{34}, \quad R_9 = M_{56}.$$

The graded-2 generators belong to the orthogonal space of $\mathfrak{u}(3)$ inside $\mathfrak{so}(6)$:

$$T_1 = \frac{1}{2}(M_{13} - M_{24}), \quad T_2 = \frac{1}{2}(M_{14} + M_{23}), \quad T_3 = \frac{1}{2}(M_{15} - M_{26})$$
$$T_4 = \frac{1}{2}(M_{16} + M_{25}), \quad T_5 = \frac{1}{2}(M_{35} - M_{46}), \quad T_6 = \frac{1}{2}(M_{36} + M_{45}).$$

The T-matrices anticommute, while the R-matrices commute with K_6.

Section 7

Correlation functions in CFTs and dCFTs

Conformal field theory: scalars

- A well-known result in CFT is that the form of 2 and 3-point functions of scalar operators is completely determined by conformal symmetry, while 1-point functions are generally zero (Polyakov, 1970):

$$\langle \mathcal{O}_1(x_1)\rangle = 0 \qquad (\text{except } \langle c\rangle = c)$$

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{x_{12}^{\Delta_1+\Delta_2}}, \qquad x_{ij} \equiv |x_i - x_j|$$

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle = \frac{\mathcal{C}_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{23}^{\Delta_2+\Delta_3-\Delta_1}\,x_{31}^{\Delta_3+\Delta_1-\Delta_2}}.$$

- If we have more than 3 points we may construct conformally invariant cross/anharmonic ratios, as e.g. in the case of 4 points:

$$\frac{x_{12}x_{34}}{x_{13}x_{24}} \quad \& \quad \frac{x_{12}x_{34}}{x_{14}x_{23}}.$$

- The corresponding n-point function ($n \geq 4$) has an arbitrary dependence on them, e.g. for $n = 4$:

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\,\mathcal{O}_4(x_4)\rangle = f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{14}x_{23}}\right)\cdot\prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j}, \qquad \Delta \equiv \sum_{i=1}^{4}\Delta_i.$$

Conformal field theory: fields with spin

- For fields with spin, such as conserved currents V_μ and the (improved!) stress (aka energy-momentum) tensor $T_{\mu\nu}$, similar results apply. These fields generally obey,

$$\partial^\mu V_\mu = 0, \qquad \partial^\mu T_{\mu\nu} = 0, \qquad T_{\mu\nu} = T_{\nu\mu}, \qquad g^{\mu\nu} T_{\mu\nu} = 0.$$

- In d dimensions the corresponding two-point functions take the following forms (case $d = 2$ is included):

$$\langle V_\mu(\mathrm{x}_1)\, V_\nu(\mathrm{x}_2)\rangle = \frac{C_V}{\mathrm{x}_{12}^{2(d-1)}} \cdot I_{\mu\nu}(\mathrm{x}_1 - \mathrm{x}_2)$$

$$\langle T_{\mu\nu}(\mathrm{x}_1)\, T_{\rho\sigma}(\mathrm{x}_2)\rangle = \frac{C_T}{\mathrm{x}_{12}^{2d}} \cdot I_{\mu\nu\rho\sigma}(\mathrm{x}_1 - \mathrm{x}_2).$$

Osborn-Petkou (1993)

Sometimes (e.g. in the case of free theories) the structure constants C_T can be related to the anomaly coefficients (or central charges) of CFTs... The inversion tensors $I_{\mu\nu}$, $I_{\mu\nu\rho\sigma}$ are defined as:

$$I_{\mu\nu}(\mathrm{x}) \equiv g_{\mu\nu} - \frac{2\,\mathrm{x}_\mu \mathrm{x}_\nu}{\mathrm{x}^2}$$

$$I_{\mu\nu\rho\sigma}(\mathrm{x}) \equiv \frac{1}{2}\left(I_{\mu\rho}(\mathrm{x})\, I_{\nu\sigma}(\mathrm{x}) + I_{\mu\sigma}(\mathrm{x})\, I_{\nu\rho}(\mathrm{x})\right) - \frac{1}{d}\, g_{\mu\nu} g_{\rho\sigma}.$$

Operator product expansion (OPE)

- Generally, we don't need a Lagrangian to define a QFT. As shown by Wightman (1956), any QFT can be reconstructed (or solved) from its local operators and their n-point correlation functions:

$$\{\mathcal{O}_i(x)\} \qquad \langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle\,.$$

- In CFTs, the latter can always be determined by means of a convergent operator product expansion (OPE) (Ferrara-Grillo-Gatto, 1973; Polyakov, 1974). E.g. for scalars:

$$\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2) = \frac{\delta_{12}}{x_{12}^{\Delta_1+\Delta_2}} + \sum_j \frac{\mathcal{C}_{12}^{j}}{x_{12}^{\Delta_1+\Delta_2-\Delta_j}}\cdot\mathcal{P}_j(x_{12},\partial_2)\,\mathcal{O}_j(x_2)\,,$$

where the sum is over all the primary operators of the CFT (normalizing $\mathcal{P}_j = 1+\mathcal{O}(x_{12})$).

- In general, the $(n+2)$-point function can be computed recursively:

$$\left\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\prod_{i=3}^{n}\mathcal{O}_i(x_i)\right\rangle = \sum_j \mathcal{C}_{12}^{j}\cdot\tilde{\mathcal{P}}_j(x_{12},\partial_2)\left\langle \mathcal{O}_j(x_2)\prod_{i=3}^{n}\mathcal{O}_i(x_i)\right\rangle.$$

- CFTs are fully specified by the CFT data: $\{\Delta_i, \ell_i, f_i, \mathcal{C}_{ij}=1, \mathcal{C}_{ijl}\}$... Conformal bootstrap program...

Subsection 2

Defect conformal field theories

Defect conformal field theory

Now consider a CFT_d and introduce a boundary at $z = 0$, where $x_\mu = (z, \mathbf{x})$... (Cardy, 1984)

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The subgroup of the d-dimensional (Euclidean) conformal group $SO(d+1, 1)$ that leaves the plane $z = 0$ invariant contains:

- $(d-1)$ dimensional translations: $\mathbf{x}' = \mathbf{x} + \mathbf{a}$
- $(d-1)$ dimensional rotations $SO(d-1)$
- d dimensional rescalings $x'_\mu = \alpha\, x_\mu$ & inversions $x'_\mu = x_\mu / x^2$

This is just the conformal group in $d-1$ dimensions, $SO(d, 1)$...

The resulting setup that contains a CFT_d and a codimension-1 boundary/interface/domain wall/defect upon which a CFT_{d-1} lives, is a defect Conformal Field Theory (dCFT).

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Boundaries of higher dimensionalities p and codimensionalities q (with $p + q = d$) are of course possible... In what follows, we will just focus on codimension-1 dCFTs for which $q = 1$...

dCFT correlators: bulk scalars

Due to the presence of the $z = 0$ boundary we may form invariant ratios from only 2 bulk points:

$$\xi = \frac{x_{12}^2}{4\,|z_1|\,|z_2|} \qquad \& \qquad \upsilon^2 = \frac{\xi}{\xi+1} = \frac{x_{12}^2}{x_{12}^2 + 4\,|z_1|\,|z_2|}, \qquad x_i \equiv (z_i, \mathbf{x}_i)\,, \quad i = 1, 2.$$

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This means that 1-point bulk functions are nonzero and the only ones fully determined by symmetry:

$$\langle \mathcal{O}_1\,(z_1, \mathbf{x}_1)\rangle = \frac{\mathcal{C}_1}{|z_1|^{\Delta_1}}.$$

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$$\langle \mathcal{O}_1\,(z_1, \mathbf{x}_1)\,\mathcal{O}_2\,(z_2, \mathbf{x}_2)\rangle = \frac{f_{12}\,(\xi)}{|z_1|^{\Delta_1}\,|z_2|^{\Delta_2}},$$

McAvity-Osborn (1995)

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- 1-point functions are the fundamental building blocks of dCFTs (along with bulk/boundary CFT data)...
- Boundary conformal bootstrap program (Liendo-Rastelli-van Rees, 2012)...

dCFT correlators: bulk fields with spin

One-point functions of fields with spin generally vanish (McAvity-Osborn 1993 & 1995),

$$\langle V_\mu(\mathrm{x}_1)\rangle = \langle T_{\mu\nu}(\mathrm{x}_1)\rangle = 0, \qquad \mathrm{x}_\mathrm{i} \equiv (\mathrm{z}_\mathrm{i}, \mathbf{x}_\mathrm{i})$$

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whereas two-point functions are given by:

$$\langle V_\mu(\mathrm{x}_1)\, V_\nu(\mathrm{x}_2)\rangle = \frac{1}{\mathrm{x}_{12}^{2(d-1)}}\Big[I_{\mu\nu} C(v) - X_\mu X'_\nu D(v)\Big]$$

$$\langle T_{\mu\nu}(\mathrm{x}_1)\, T_{\rho\sigma}(\mathrm{x}_2)\rangle = \frac{1}{\mathrm{x}_{12}^{2d}}\cdot\Bigg\{\left(X_\mu X_\nu - \frac{g_{\mu\nu}}{d}\right)\left(X'_\rho X'_\sigma - \frac{g_{\rho\sigma}}{d}\right)A(v) + \Big(X_\mu X'_\rho I_{\nu\sigma} + X_\mu X'_\sigma I_{\nu\rho} +$$

$$+ X_\nu X'_\sigma I_{\mu\rho} + X_\nu X'_\rho I_{\mu\sigma} - \frac{4}{d} g_{\mu\nu} X'_\rho X'_\sigma - \frac{4}{d} g_{\rho\sigma} X_\mu X_\nu + \frac{4}{d^2} g_{\mu\nu} g_{\rho\sigma}\Big) B(v) + I_{\mu\nu\rho\sigma} C(v)\Bigg\},$$

where $A(v)$, $B(v)$, $C(v)$ are functions of the dCFT invariant v. We have set,

$$X_\mu \equiv \mathrm{z}_1 \cdot \frac{v}{\xi}\frac{\partial \xi}{\partial \mathrm{x}_1^\mu} = v\left(\frac{2\mathrm{z}_1}{\mathrm{x}_{12}^2}(\mathrm{x}_{1\mu} - \mathrm{x}_{2\mu}) - n_\mu\right), \quad X'_\rho \equiv \mathrm{z}_2 \cdot \frac{v}{\xi}\frac{\partial \xi}{\partial \mathrm{x}_2^\rho} = -v\left(\frac{2\mathrm{z}_2}{\mathrm{x}_{12}^2}(\mathrm{x}_{1\rho} - \mathrm{x}_{2\rho}) + n_\rho\right),$$

where $n \equiv (1, \mathbf{0})$ is the unit normal to the $z = 0$ boundary. X, X' obey

$$X_\mu X_\mu = X'_\rho X'_\rho = 1, \qquad X'_\rho = I_{\rho\mu} X_\mu.$$

dCFT correlators: boundary scalars

Now suppose that we insert a boundary scalar operator $\widehat{\mathcal{O}}(\mathbf{x})$. We find:

$$\left\langle \mathcal{O}_1(\mathrm{z}_1, \mathbf{x}_1)\, \widehat{\mathcal{O}}_2(\mathbf{x}_2) \right\rangle = \frac{\mathcal{B}_{12}}{|\mathrm{z}_1|^{\Delta_1-\Delta_2}\, \mathrm{x}_{12}^{2\Delta_2}}, \qquad \mathrm{x}_{12}^2 = \mathrm{z}_1^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2 .$$

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Since the conformal symmetry is intact on the $z = 0$ defect, the n-point correlators of boundary operators satisfy the usual relations of $\mathrm{CFT}_{(d-1)}$:

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while all the higher correlators will have an explicit dependence on the boundary CFT_{d-1} cross ratios...

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There is also a boundary operator expansion (BOE) which reads (normalizing $\widehat{\mathcal{P}}_j = 1 + \mathcal{O}(z^2)$):

$$\mathcal{O}_1(x_1) = \frac{\mathcal{C}_1}{|z_1|^{\Delta_1}} + \sum_j \frac{\mathcal{B}_{1j}}{|z_1|^{\Delta_1-\Delta_j}} \cdot \widehat{\mathcal{P}}_j(z_1, \partial_{\mathbf{x}_1})\, \widehat{\mathcal{O}}_j(\mathbf{x}_1) .$$

Subsection 3

Boundary conformal bootstrap

Defect two-point functions (bulk channel)

Let us now compute the bulk-bulk two-point function from the CFT+dCFT data and the bulk OPE,

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{\Delta_1+\Delta_2}} + \sum_j \frac{\mathcal{C}_{12}^{j}}{x_{12}^{\Delta_1+\Delta_2-\Delta_j}} \cdot \mathcal{P}_j(x_{12},\partial_2)\,\langle \mathcal{O}_j(x_2)\rangle\,,$$

which is valid independently of the presence of defects.

Defect two-point functions (bulk channel)

Let us now compute the bulk-bulk two-point function from the CFT+dCFT data and the bulk OPE,

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\rangle = \frac{\delta_{12}}{x_{12}^{\Delta_1+\Delta_2}} + \sum_j \frac{\mathcal{C}_{12}^{j}}{x_{12}^{\Delta_1+\Delta_2-\Delta_j}} \cdot \mathcal{P}_j(x_{12},\partial_2)\,\langle \mathcal{O}_j(x_2)\rangle\,,$$

which is valid independently of the presence of defects. Plugging the one and two-point functions

$$\langle \mathcal{O}_1(z_1,\mathbf{x}_1)\,\mathcal{O}_2(z_2,\mathbf{x}_2)\rangle = \frac{f_{12}(\xi)}{|z_1|^{\Delta_1}\,|z_2|^{\Delta_2}}, \qquad \langle \mathcal{O}_j(z_2,\mathbf{x}_2)\rangle = \frac{\mathcal{C}_j}{|z_2|^{\Delta_j}},$$

into the OPE we obtain (the factor 2^{Δ_k} accounts for having $|z_i|$ instead of $2\,|z_i|$ in the denominators):

$$f_{12}(\xi) = (4\xi)^{-\frac{\Delta_1+\Delta_2}{2}} \left[\delta_{12} + \sum_j 2^{\Delta_j}\,\mathcal{C}_{12}^{j}\,\mathcal{C}_j \cdot F_{\text{bulk}}(\Delta_j,\delta\Delta,\xi)\right], \qquad \delta\Delta \equiv \Delta_1 - \Delta_2.$$

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$$\langle \mathcal{O}_1 (z_1, \mathbf{x}_1) \mathcal{O}_2 (z_2, \mathbf{x}_2) \rangle = \frac{f_{12}(\xi)}{|z_1|^{\Delta_1} |z_2|^{\Delta_2}}, \qquad \langle \mathcal{O}_j (z_2, \mathbf{x}_2) \rangle = \frac{\mathcal{C}_j}{|z_2|^{\Delta_j}},$$

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The bulk conformal blocks F_{bulk} can be determined from the expression $\mathcal{P}_j (x_{12}, \partial_2) |z_2|^{-\Delta_j}$:

$$F_{\text{bulk}} (\Delta_j, \delta\Delta, \xi) = \xi^{\frac{\Delta_j}{2}} \, {}_2F_1 \left(\frac{\Delta_j + \delta\Delta}{2}, \frac{\Delta_j + \delta\Delta}{2}, \Delta_j - 1; -\xi \right).$$

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We now compute the bulk-bulk two-point function from the boundary operator expansion (BOE),

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$$\langle \mathcal{O}_1(z_1,\mathbf{x}_1)\,\mathcal{O}_2(z_2,\mathbf{x}_2)\rangle = \frac{f_{12}(\xi)}{|z_1|^{\Delta_1}\,|z_2|^{\Delta_2}}, \qquad \langle\widehat{\mathcal{O}}_i(\mathbf{x}_1)\,\widehat{\mathcal{O}}_j(\mathbf{x}_2)\rangle = \frac{\widehat{\mathcal{B}}_{ij}}{\mathbf{x}_{12}^{\Delta_i+\Delta_j}}$$

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$$\langle \mathcal{O}_1(z_1,\mathbf{x}_1)\,\mathcal{O}_2(z_2,\mathbf{x}_2)\rangle = \frac{f_{12}(\xi)}{|z_1|^{\Delta_1}\,|z_2|^{\Delta_2}}, \qquad \big\langle\widehat{\mathcal{O}}_i(\mathbf{x}_1)\,\widehat{\mathcal{O}}_j(\mathbf{x}_2)\big\rangle = \frac{\widehat{\mathcal{B}}_{ij}}{\mathbf{x}_{12}^{\Delta_i+\Delta_j}},$$

and contracting the indices i, j inside the sum by $\widehat{\mathcal{B}}_{ij}$ we find

$$f_{12}(\xi) = \mathcal{C}_1\mathcal{C}_2 + \sum_j \mathcal{B}_{1j}\mathcal{B}_2^{j}\cdot F_{\text{boundary}}(\Delta_j,\xi)\,.$$

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$$f_{12}(\xi) = \mathcal{C}_1\mathcal{C}_2 + \sum_j \mathcal{B}_{1j}\mathcal{B}_2^j \cdot F_{\text{boundary}}(\Delta_j,\xi).$$

The boundary conformal blocks F_{boundary} are determined from $\widehat{\mathcal{P}}_i(z_1,\partial_{\mathbf{x}_1})\widehat{\mathcal{P}}_j(z_2,\partial_{\mathbf{x}_2})\mathbf{x}_{12}^{-(\Delta_i+\Delta_j)}$:

$$F_{\text{boundary}}(\Delta_j,\xi) = \xi^{-\Delta_j}\,{}_2F_1\Big(\Delta_j, \Delta_j - 1, 2\Delta_j - 2; -\xi^{-1}\Big).$$

Boundary conformal bootstrap program

Equating the two expressions for $f_{12}(\xi)$ we have found in the bulk and the boundary channel,

$$f_{12}(\xi) = (4\xi)^{-\frac{\Delta_1+\Delta_2}{2}} \left[\delta_{12} + \sum_j 2^{\Delta_j} \mathcal{C}_{12}^j \mathcal{C}_j \cdot \xi^{\frac{\Delta_j}{2}} \, {}_2F_1\left(\frac{\Delta_j + \delta\Delta}{2}, \frac{\Delta_j + \delta\Delta}{2}, \Delta_j - 1; -\xi\right)\right] =$$

$$= \mathcal{C}_1\mathcal{C}_2 + \sum_j \mathcal{B}_{1j}\mathcal{B}_2^j \cdot \xi^{-\Delta} \, {}_2F_1\left(\Delta_j, \Delta_j - 1, 2\Delta_j - 2; -\xi^{-1}\right)$$

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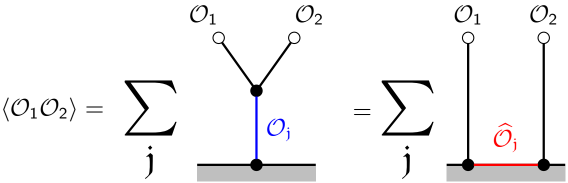
we may extract a set of defect bootstrap equations for the conformal data (Liendo-Rastelli-van Rees, 2012; Gliozzi-Liendo-Meineri-Rago, 2015; Billò-Gonçalves-Lauria-Meineri, 2016; Liendo-Meneghelli, 2016; Hogervorst, 2017)...



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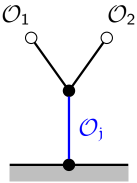
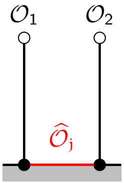
$$f_{12}(\xi) = (4\xi)^{-\frac{\Delta_1+\Delta_2}{2}}\left[\delta_{12} + \sum_j 2^{\Delta_j} C_{12}^j C_j \cdot \xi^{\frac{\Delta_j}{2}}\, {}_2F_1\left(\frac{\Delta_j+\delta\Delta}{2}, \frac{\Delta_j+\delta\Delta}{2}, \Delta_j - 1; -\xi\right)\right] =$$

$$= C_1 C_2 + \sum_j \mathcal{B}_{1j}\mathcal{B}_2^j \cdot \xi^{-\Delta}\, {}_2F_1\left(\Delta_j, \Delta_j - 1, 2\Delta_j - 2; -\xi^{-1}\right).$$

For the dCFT that is dual to the D3-D5 intersection, de Leeuw-Ipsen-Kristjansen-Vardinghus-Wilhelm (2017) have used its domain wall description to compute various bulk-bulk two-point functions at weak 't Hooft coupling, then used the bootstrap equations to mine for (unknown) conformal data...

$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \sum_j$  $= \sum_j$ 

Bulk-bulk two-point functions in the bulk channel

Let us form the ratio of the (bulk-bulk) dCFT two-point function over the CFT two-point function

$$\langle \mathcal{O}_1(\mathrm{x}_1)\,\mathcal{O}_2(\mathrm{x}_2)\rangle_{\mathrm{dCFT}} = \frac{f_{12}(\xi)}{|\mathrm{z}_1|^{\Delta_1}\,|\mathrm{z}_2|^{\Delta_2}}, \qquad \langle \mathcal{O}_1(\mathrm{x}_1)\,\mathcal{O}_2(\mathrm{x}_2)\rangle_{\mathrm{CFT}} = \frac{\delta_{12}}{\mathrm{x}_{12}^{\Delta_1+\Delta_2}}, \qquad \xi \equiv \frac{\mathrm{x}_{12}^2}{4\,|\mathrm{z}_1|\,|\mathrm{z}_2|},$$

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getting, in the general case...

$$\frac{\langle \mathcal{O}_1(\mathrm{x}_1)\, \mathcal{O}_2(\mathrm{x}_2)\rangle_{\mathsf{dCFT}}}{\langle \mathcal{O}_1(\mathrm{x}_1)\, \mathcal{O}_2(\mathrm{x}_2)\rangle_{\mathsf{CFT}}} = (4\xi)^{\frac{\Delta_1+\Delta_2}{2}} \cdot \frac{f_{12}(\xi)}{\delta_{12}}.$$

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in the bulk channel and in the case of a single scalar operator $\mathcal{O}_I$ of dimension $\Delta_I = L = 2j$:

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Expanding around $\xi = 0$, we obtain

$$\frac{\langle \mathcal{O}_1(\mathrm{x}_1)\,\mathcal{O}_2(\mathrm{x}_2)\rangle_{\text{dCFT}}}{\langle \mathcal{O}_1(\mathrm{x}_1)\,\mathcal{O}_2(\mathrm{x}_2)\rangle_{\text{CFT}}} = 1 + 2^L \mathcal{C}_{12}^l\,\mathcal{C}_l\,\xi^j \cdot \left\{1 - \frac{j^2}{2j-1}\cdot\xi + \frac{j(j+1)^2}{4(2j-1)}\cdot\xi^2 + \ldots\right\}.$$

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- For the dCFT that is dual to the D3-D5 intersection, we will now verify that this relation holds at strong 't Hooft coupling, in the case of two heavy BMN operators (Georgiou-GL-Zoakos, 2023)...

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- Working in the holographic description of the the dCFT, we need to set up the computation of generic dCFT correlation functions with semiclassical strings...

Subsection 4

Conformal anomalies

Defect anomalies

Even dimensional CFTs (in curved spacetimes) are afflicted by conformal/Weyl anomalies: the trace of the energy-momentum/stress tensor acquires non-vanishing expectation value that is given by (scheme-independent terms only)...

$$\langle T^{\mu}_{\mu} \rangle^{d=2n} = \frac{4}{d!\,\mathrm{Vol}[\mathrm{S}^d]} \times \left[\sum_i c_i\, I_i - (-1)^{d/2} a_d\, E_d \right], \quad n = 1, 2, \ldots$$

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Odd dimensional (compact) spacetimes have no conformal/Weyl (trace) anomalies...

$$\langle T_\mu^\mu \rangle^{d=2n+1} = 0, \quad n = 1, 2, \ldots$$

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$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2n} = \frac{4}{d!\,\mathrm{Vol}[\mathrm{S}^d]} \times \left[\sum_i c_i\, I_i + \delta\left(z\right)\sum_j b_j\, \mathrm{I}_j - (-1)^{d/2} a_d\, \left(E_d + \delta\left(z\right) E^{(\mathrm{bry})}\right)\right], \quad n = 1, 2, \ldots$$

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$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2n+1} = \frac{2\delta\left(z\right)}{(d-1)!\,\mathrm{Vol}[\mathrm{S}^{d-1}]} \times \left[\sum_j b_j\, \mathrm{I}_j + (-1)^{(d-1)/2} a_d\, \mathring{E}_{d-1}\right], \quad n = 1, 2, \ldots$$

The presence of (codimension-1) boundaries gives rise to extra A & B anomaly coefficients (localized on the boundary)... and extra central charges which can classify defect CFTs (much like central charges classify pure CFTs)...

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$$\left\langle T_{\mu}^{\mu}\right\rangle^{d=2} = \frac{a}{2\pi}\left(R + 2\delta(z)\, K\right)$$

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$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2} = \frac{a}{2\pi}\left(R + 2\delta(z)\, K\right), \qquad \left\langle T^{\mu}_{\mu}\right\rangle^{d=3} = \frac{\delta(z)}{4\pi}\left(a\,\mathring{R} + b\,\mathrm{tr}\hat{K}^2\right)$$

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$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2n} = \frac{4}{d!\,\mathrm{Vol}[S^d]} \times \left[\sum_i c_i\, I_i + \delta(z)\sum_j b_j\, \mathrm{I}_j - (-1)^{d/2} a_d \left(E_d + \delta(z)\, E^{(\mathrm{bry})}\right)\right], \quad n = 1, 2, \ldots$$

Odd dimensional (compact) spacetimes have no conformal/Weyl (trace) anomalies...

$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2n+1} = \frac{2\delta(z)}{(d-1)!\,\mathrm{Vol}[S^{d-1}]} \times \left[\sum_j b_j\, \mathrm{I}_j + (-1)^{(d-1)/2} a_d\, \mathring{E}_{d-1}\right], \quad n = 1, 2, \ldots$$

The presence of (codimension-1) boundaries gives rise to extra A & B anomaly coefficients (localized on the boundary)... and extra central charges which can classify defect CFTs (much like central charges classify pure CFTs)... Examples:

$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=2} = \frac{a}{2\pi}\left(R + 2\delta(z)\, K\right), \qquad \left\langle T^{\mu}_{\mu}\right\rangle^{d=3} = \frac{\delta(z)}{4\pi}\left(a\,\mathring{R} + b\,\mathrm{tr}\hat{K}^2\right)$$

$$\left\langle T^{\mu}_{\mu}\right\rangle^{d=4} = \frac{1}{16\pi^2}\left(c\, W^2_{\mu\nu\rho\sigma} - a\, E_4\right) + \frac{\delta(z)}{16\pi^2}\left(a\, E^{(\mathrm{bry})}_4 - b_1\,\mathrm{tr}\hat{K}^3 - b_2\, h^{pq}\hat{K}^{rs} W_{pqrs}\right),$$

where E_d, $\mathring{E}_{d-1}$ are the bulk/boundary Euler densities, and $E^{(\mathrm{bry})}$ the boundary term of the Euler characteristic... K_{pq} is the boundary extrinsic curvature, and h_{pq} is the induced metric on the boundary... dimensionalities $d = 5, 6$ not fully classified as of now (no nontrivial CFTs in $d > 6$)...

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$$\hat{K}_{pq} \equiv K_{pq} - \frac{h_{pq}}{d-1}K, \qquad \mathrm{tr}\hat{K}^2 \equiv \mathrm{tr}K^2 - \frac{1}{2}K^2, \qquad \mathrm{tr}\hat{K}^3 \equiv \mathrm{tr}K^3 - K\mathrm{tr}K^2 + \frac{2}{9}K^3$$

$$E_4 = \frac{1}{4}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma}, \qquad E_4^{(\mathrm{bry})} = -4\delta^{stw}_{pqr} K^p_s \left(\frac{1}{2}R^{qr}_{tw} + \frac{2}{3}K^q_t K^r_w\right)$$

$$h^{\mu\nu}\hat{K}^{\rho\sigma} W_{\mu\nu\rho\sigma} = R^{\nu\rho\sigma}_{\mu} K^{\rho}_{\mu} n^\nu n^\sigma - \frac{1}{2}R_{\mu\nu}\left(n^\mu n^\nu K + K^{\mu\nu}\right) + \frac{1}{6}KR, \qquad h^{\mu\rho}\hat{K}^{\nu\sigma} W_{\mu\nu\rho\sigma} = -K^{pq} W_{npnq}.$$

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a theorems

Type-A anomaly coefficients have been shown (in $d = 2, 3, 4$) to have the following monotonicity property (a-theorem):

$$a_{\text{UV}} > a_{\text{IR}},$$

under the renormalization group flow. Here are the main monotonicity properties:

- $d = 2$: the c (or $a = c/12$) theorem was shown by Zamolodchikov (1986)...
- $d = 3$: the a theorem for the (codimension-1) boundary anomaly coefficient was shown by Jensen-O'Bannon (2015)...
- $d = 4$: the a theorem conjectured by Cardy (1988) and proven by Komargodski-Schwimmer (2001)...

Proofs of the above a-theorems with entanglement entropy have been given by Casini-Huerta (2004), Casini-Landea-Torroba (2018) and Casini-Teste-Torroba (2017) respectively. The 2d central charge $c = 12a$ also shows up in:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad m, n \in \mathbb{Z}, \quad T = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad \text{(Virasoro algebra)}$$

$$\langle T(\mathfrak{z}_1)\, T(\mathfrak{z}_2)\rangle = \frac{c/2}{(\mathfrak{z}_1-\mathfrak{z}_2)^4}, \quad \langle T(\mathfrak{z}_1)\, T(\mathfrak{z}_2)\, T(\mathfrak{z}_3)\rangle = \frac{c}{(\mathfrak{z}_1-\mathfrak{z}_2)^2(\mathfrak{z}_2-\mathfrak{z}_3)^2(\mathfrak{z}_3-\mathfrak{z}_1)^2}, \qquad T \equiv T_{\mathfrak{z}\mathfrak{z}}$$

$$S_{\text{thermo}} = \frac{\pi}{3}c\,L\,T + \ldots \qquad \text{(Cardy, 1986)}$$

$$S_{\text{EE}} = \frac{c}{3}\ln\frac{\ell}{\epsilon} + \ldots \qquad \text{(Holzhey-Larsen-Wilczek, 1994 \& Calabrese-Cardy, 2004)}$$

where L is the system size, T the temperature, ℓ is the EE interval and ϵ the short-distance cutoff...

Anomaly coefficients in free theories

Before calculating the A & B anomaly coefficients for the D3-D5 dCFT, let us go through some results for codimension-1:

- In $d = 2$ the relation of the anomaly coefficient a to the central charge is $c = 12a$... For free scalar & Dirac fields:
$$a^{s=0} = a^{s=1/2} = \frac{1}{12} \qquad \text{(see e.g. Cardy, 2004).}$$
- In $d = 3$ there are two new central charges... for free scalars their value depends on the type of boundary conditions Dirichlet (D) or Robin (R) (Neumann (N) boundary conditions are not consistent with the residual symmetries)...
$$a^{s=0}\big|_{\mathrm{D}} = -\frac{1}{96}, \qquad a^{s=0}\big|_{\mathrm{R}} = \frac{1}{96}, \qquad a^{s=1/2} = 0, \qquad b^{s=0}\big|_{\mathrm{D/R}} = \frac{1}{64}, \qquad b^{s=1/2} = \frac{1}{32}.$$
Nozaki-Takayanagi-Ugajin (2012), Jensen-O'Bannon (2015)
- In $d = 4$ there are three new central charges... for free fields, bulk charges are independent of boundary conditions...
$$a^{s=0} = \frac{1}{360}, \qquad a^{s=1/2} = \frac{11}{360}, \qquad a^{s=1} = \frac{31}{180}, \qquad c^{s=0} = \frac{1}{120}, \qquad c^{s=1/2} = \frac{1}{120}, \qquad c^{s=1} = \frac{1}{10},$$
(see e.g. Birrell-Davies)... For the boundary charges of free fields, b_1 generally depends on the boundary conditions...
$$b_1^{s=0}\big|_{\mathrm{D}} = \frac{2}{35}, \qquad b_1^{s=0}\big|_{\mathrm{R}} = \frac{2}{45}, \qquad b_1^{s=1/2}\big|_{\mathrm{D/R}} = \frac{2}{7}, \qquad b_1^{s=1}\big|_{\mathrm{D/R}} = \frac{16}{35},$$
Melmed (1988), Moss (1989)
whereas the (free field) boundary charge b_2 is independent of the BCs and proportional to the bulk central charge c:
$$b_2 = 8c.$$
Dowker-Schofield (1990)
Fursaev (2015), Solodukhin (2015)

Anomalies as observables (bulk)

All types (A, B, C) of anomaly coefficients show up in CFT and dCFT data... For the bulk charges,

- In $d = 2$, the central charge $c = 12a$ shows up in the two and three-point function of the (traceless) stress tensor:

$$\langle T(\mathfrak{z}_1)\, T(\mathfrak{z}_2)\rangle = \frac{c/2}{(\mathfrak{z}_1 - \mathfrak{z}_2)^4}, \qquad \langle T(\mathfrak{z}_1)\, T(\mathfrak{z}_2)\, T(\mathfrak{z}_3)\rangle = \frac{c}{(\mathfrak{z}_1 - \mathfrak{z}_2)^2 (\mathfrak{z}_2 - \mathfrak{z}_3)^2 (\mathfrak{z}_3 - \mathfrak{z}_1)^2},$$

  where $T \equiv T_{\mathfrak{z}\mathfrak{z}}$, and $\mathfrak{z} \equiv x_1 + i x_2$, $\bar{\mathfrak{z}} \equiv x_1 - i x_2$ are the holomorphic/anti-holomorphic coordinates.

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$$\langle T_{\mu\nu}(\mathrm{x}_1)\, T_{\rho\sigma}(\mathrm{x}_2)\rangle = \frac{C_T}{\mathrm{x}_{12}^8} \cdot I_{\mu\nu\rho\sigma}(\mathrm{x}_1 - \mathrm{x}_2).$$

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  E.g. for free (scalar, Majorana-Weyl, and vector) fields and $\mathcal{N} = 4$ SYM, the 2-point function coefficient is given by

$$C_T = \frac{N_0 + 3N_{1/2} + 12N_1}{3\pi^4}.$$

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  On the other hand, the (type A & C) conformal anomaly coefficients become:

$$c=\frac{N_0+3N_{1/2}+12N_1}{120}=\frac{\pi^4 C_T}{40}, \qquad a=\frac{2N_0+11N_{1/2}+124N_1}{720}$$

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  so that in the case of $U(N_c)$, $\mathcal{N}=4$ SYM, all three coefficients turn out to be equal:

$$a=c=\frac{N_c^2}{4}=\frac{\pi^4 C_T}{40}.$$

Anomalies as observables (boundary)

The boundary charges show up in two and three-point functions of the displacement operator $\mathcal{D}$. In d dimensions,

$$\langle \mathcal{D}(\mathbf{x}_1)\,\mathcal{D}(\mathbf{x}_2)\rangle = \frac{c_{nn}}{\mathbf{x}_{12}^{2d}}, \qquad \langle \mathcal{D}(\mathbf{x}_1)\,\mathcal{D}(\mathbf{x}_2)\,\mathcal{D}(\mathbf{x}_3)\rangle = \frac{c_{nnn}}{\mathbf{x}_{12}^{d}\mathbf{x}_{23}^{d}\mathbf{x}_{31}^{d}}.$$

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It can be shown that the single 3d B-type anomaly coefficient and the two 4d B-type anomaly coefficients are given by:

$$b = \frac{\pi^2}{8}c_{nn}, \qquad b_1 = \frac{2\pi^3}{35}c_{nnn}, \qquad b_2 = \frac{2\pi^4}{15}c_{nn},$$

whereas there is no known relation for the 3d A-type anomaly coefficient a... Interestingly, the displacement operator computations confirm the (old) heat kernel results...

Section 8

Codimension-1 determinant formulas

D3-D5 domain wall

In the $\mathfrak{so}(6)$ sector of the $\mathcal{N}=4$ SYM domain wall (dCFT) that is dual to the D3-D5 probe-brane system,

$$\mathcal{C}_k(u;v;w) = \mathbb{T}_{k-1} \times Q_1(k/2) \times \sqrt{\frac{Q_1(0)\,Q_1(1/2)}{R_2(0)\,R_2(1/2)\,R_3(0)\,R_3(1/2)} \cdot \frac{\det G^+}{\det G^-}}$$

(modulo the overall factor $L^{-1/2}\left(8\pi^2/\lambda\right)^{L/2}$) for fully balanced excitations $u_i \equiv u_{1,i}$, $v_j \equiv u_{2,j}$, $w_k \equiv u_{3,k}$ and

$$\mathbb{T}_s \equiv \sum_{q=-s/2}^{s/2} q^L \cdot \frac{Q_2(q)\,Q_3(q)}{Q_1(q+1/2)\,Q_1(q-1/2)}.$$

de Leeuw-Kristjansen-GL (2018)

This formula has also been verified numerically. The $M/2 \times M/2$ matrices $G^{\pm}_{jk}$ and $K^{\pm}_{jk}$ are defined as:

$$G^{\pm}_{ab,jk} = \delta_{ab}\delta_{jk}\left[\frac{Lq_a^2}{u_{a,j}^2 + q_a^2/4} - \sum_{c=1}^{3}\sum_{l=1}^{\lceil N/2 \rceil} K^{+}_{ac,jl}\right] + K^{\pm}_{ab,jk}, \qquad K^{\pm}_{ab,jk} = \mathbb{K}^{-}_{ab,jk} \pm \mathbb{K}^{+}_{ab,jk}$$

$$\mathbb{K}^{\pm}_{ab,jk} \equiv \frac{M_{ab}}{(u_{a,j} \pm u_{b,k})^2 + \frac{1}{4}M_{ab}^2}.$$

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de Leeuw-Kristjansen-GL (2018)

Some more properties of one-point functions in $\mathfrak{so}(6)$ (easily reducible to $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$) are:

- One-point functions vanish (for all values of k) if M or $L+N_+ + N_-$ is odd.
- Because $\mathbb{Q}_3\,|\text{MPS}\rangle = 0$, all 1-point functions vanish (for all k) unless all the Bethe roots are fully balanced:

$$\left\{u_1,\ldots,u_{M/2},-u_1,\ldots,-u_{M/2}\right\}$$
$$\left\{v_1,\ldots,v_{N_+/2},-v_1,\ldots,-v_{N_+/2},(0)\right\}, \qquad \left\{w_1,\ldots,w_{N_-/2},-w_1,\ldots,-w_{N_-/2},(0)\right\}.$$

D3-D5 domain wall

Yet another definition of the norm matrix is the following:

$$G \equiv \partial_J \phi_I = \frac{\partial \phi_I}{\partial u_J} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} & B_1 & B_2 & D_1 \\ B_1^t & B_2^t & C_1 & C_2 & D_2 \\ B_2^t & B_1^t & C_2 & C_1 & D_2 \\ D_1^t & D_1^t & D_2^t & D_2^t & D_3 \end{bmatrix} & \begin{matrix} F_1 & F_2 & H_1 \\ F_2 & F_1 & H_1 \\ K_1 & K_2 & H_2 \\ K_2 & K_1 & H_2 \\ D_4^t & D_4^t & H_3 \end{matrix} \\ \begin{matrix} F_1^t & F_2^t & K_1^t & K_2^t & D_4 \\ F_2^t & F_1^t & K_2^t & K_1^t & D_4 \\ H_1^t & H_1^t & H_2^t & H_2^t & H_3^t \end{matrix} & \begin{matrix} L_1 & L_2 & H_4 \\ L_2 & L_1 & H_4 \\ H_4^t & H_4^t & H_5 \end{matrix} \end{bmatrix},$$

where the submatrices correspond to the norm matrices in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ subsectors, while

$$\phi_I \equiv \{\phi_{1,i}, \phi_{2,j}, \phi_{3,k}\}, \qquad i = 1, \ldots, N_1, \quad j = 1, \ldots, N_2, \quad k = 1, \ldots, N_3$$

$$u_J \equiv \{u_{1,i}, u_{2,j}, u_{3,k}\}, \qquad I, J = 1, \ldots, N_1 + N_2 + N_3,$$

and

$$\phi_{1,i} = -i \log \left[\left(\frac{u_{1,i} - i/2}{u_{1,i} + i/2} \right)^L \prod_{j \neq i}^{N_1} \frac{u_{1,i} - u_{1,j} + i}{u_{1,i} - u_{1,j} - i} \prod_{k=1}^{N_2} \frac{u_{1,i} - u_{2,k} - \frac{i}{2}}{u_{1,i} - u_{2,k} + \frac{i}{2}} \prod_{l=1}^{N_3} \frac{u_{1,i} - u_{3,l} - \frac{i}{2}}{u_{1,i} - u_{3,l} + \frac{i}{2}} \right]$$

$$\phi_{2,i} = -i \log \left[\prod_{l \neq i}^{N_2} \frac{u_{2,i} - u_{2,l} + i}{u_{2,i} - u_{2,l} - i} \prod_{k=1}^{N_1} \frac{u_{2,i} - u_{1,k} - \frac{i}{2}}{u_{2,i} - u_{1,k} + \frac{i}{2}} \right], \quad \phi_{3,i} = -i \log \left[\prod_{l \neq i}^{N_3} \frac{u_{3,i} - u_{3,l} + i}{u_{3,i} - u_{3,l} - i} \prod_{k=1}^{N_1} \frac{u_{3,i} - u_{1,k} - \frac{i}{2}}{u_{3,i} - u_{1,k} + \frac{i}{2}} \right].$$

D3-D5 domain wall

- It can be shown that the determinant of the norm matrix factorizes:

$$\det G = \det G^{+} \times \det G^{-},$$

with $A_{\pm} \equiv A_1 \pm A_2$ (and so on for $B_{\pm}$, $C_{\pm}$, $F_{\pm}$, $K_{\pm}$, $L_{\pm}$), while

$$G^{+} = \begin{pmatrix} A_+ & B_+ & D_1 & F_+ & H_1 \\ B_+^t & C_+ & D_2 & K_+ & H_2 \\ 2D_1^t & 2D_2^t & D_3 & 2D_4^t & H_3 \\ F_+^t & K_+^t & D_4 & L_+ & H_4 \\ 2H_1^t & 2H_2^t & 2H_3^t & 2H_4^t & H_5 \end{pmatrix} \qquad \& \qquad G^{-} = \begin{pmatrix} A_- & B_- & F_- \\ B_-^t & C_- & K_- \\ F_-^t & K_-^t & L_- \end{pmatrix}.$$

These forms are fully consistent with the $G^{\pm}$ matrices we've defined in $SU(2)$ and $SU(3)$... We have checked the equivalence of the two definitions of the matrices $G^{\pm}$ for a large number of states...

D3-D5 domain wall

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- Another unproven claim (Escobedo, 2012) is that the norm of any $\mathfrak{so}(6)$ Bethe eigenstate is given by the following expression which involves the determinant of the norm matrix:

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which obviously also shares the above factorization property of G... It is rather straightforward to extract the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ structure constants and selection rules from $\mathfrak{so}(6)$...

Subsection 2

D3-D7 domain wall

D3-D7 domain wall

In the $\mathfrak{so}(6)$ sector of the $\mathcal{N}=4$ SYM domain wall (dCFT), dual to the $SO(5)$ symmetric D3-D7 probe-brane system,

$$\boxed{\mathcal{C}_n(u;v;w) = \mathbb{T}_n \cdot \sqrt{\frac{Q_1(0)\,Q_1(1/2)}{R_2(0)\,R_2(1/2)\,R_3(0)\,R_3(1/2)} \cdot \frac{\det G^+}{\det G^-}}}$$

(modulo the overall factor $L^{-1/2}\left(8\pi^2/\lambda\right)^{L/2}$) for fully balanced excitations $u_i \equiv u_{1,i}$, $v_j \equiv u_{2,j}$, $w_k \equiv u_{3,k}$ and

$$\mathbb{T}_n = \sum_{q=-n/2}^{n/2} (2q)^L \left[\sum_{p=-n/2}^{q} \frac{Q_1\left(p-\frac{1}{2}\right)}{Q_1\left(q-\frac{1}{2}\right)} \frac{Q_3(q)\,Q_3\left(\frac{n}{2}+1\right)}{Q_3(p)\,Q_3(p-1)}\right]\left[\sum_{r=q}^{n/2} \frac{Q_1\left(r+\frac{1}{2}\right)}{Q_1\left(q+\frac{1}{2}\right)} \frac{Q_2(q)\,Q_2\left(\frac{n}{2}+1\right)}{Q_2(r)\,Q_2(r+1)}\right].$$

de Leeuw-Gombor-Kristjansen-GL-Pozsgay (2019)

This formula has also been verified numerically. The $M/2 \times M/2$ matrices $G^{\pm}_{jk}$ and $K^{\pm}_{jk}$ are defined as:

$$G^{\pm}_{ab,jk} = \delta_{ab}\delta_{jk}\left[\frac{Lq_a^2}{u_{a,j}^2 + q_a^2/4} - \sum_{c=1}^{3}\sum_{l=1}^{\lceil N/2\rceil} K^{+}_{ac,jl}\right] + K^{\pm}_{ab,jk}, \qquad K^{\pm}_{ab,jk} = \mathbb{K}^{-}_{ab,jk} \pm \mathbb{K}^{+}_{ab,jk}$$

$$\mathbb{K}^{\pm}_{ab,jk} \equiv \frac{M_{ab}}{\left(u_{a,j} \pm u_{b,k}\right)^2 + \frac{1}{4}M_{ab}^2}.$$

D3-D7 domain wall

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de Leeuw-Gombor-Kristjansen-GL-Pozsgay (2019)

Interesting special cases of the D3-D7 determinant formula are obtained for $n=1$,

$$\mathbb{T}_1=\left(1+(-1)^L\right)\cdot\frac{Q_1(1)}{Q_1(0)}+(-1)^{N_-}\cdot\frac{Q_3(3/2)}{Q_3(1/2)}+(-1)^{L+N_+}\cdot\frac{Q_2(3/2)}{Q_2(1/2)},$$

as well as for $n=2$,

$$\mathbb{T}_2=2^{L+1}\times\left\{\frac{\left(1+(-1)^L\right)}{2}\cdot\frac{Q_1(3/2)}{Q_1(1/2)}+\frac{Q_3(2)}{R_3(0)}\left[\frac{Q_1'(1/2)}{Q_1(1/2)}-\frac{Q_3'(1)}{Q_3(1)}\right]^{\delta_{M/2=\text{odd}}}+(-1)^L\cdot\frac{Q_2(2)}{R_2(0)}\left[\frac{Q_1'(1/2)}{Q_1(1/2)}-\frac{Q_2'(1)}{Q_2(1)}\right]^{\delta_{M/2=\text{odd}}}\right\}.$$

Subsection 3

D2-D4 domain wall

D2-D4 domain wall

In the $\mathfrak{su}(4)$ sector of the ABJM domain wall (dCFT) that is dual to the D2-D4 probe-brane system:

$$\mathcal{C}_q(u;v;w) = \mathbb{T}_q \cdot \frac{Q_1(1/2)\, Q_1(q-1/2)}{\sqrt{R_2(0)\, R_2(1/2)}} \cdot \sqrt{\frac{\det G^+}{\det G^-}},$$

Gombor-Kristjansen (2022)

where

$$\mathbb{T}_q \equiv \sum_{k=1}^{q-1} \left(\frac{k}{2}\right)^L \cdot \frac{Q_2(k)}{Q_1(k+1/2)\, Q_1(k-1/2)},$$

and the $G^{\pm}$ matrices have been defined above... The Baxter Q and R functions have been defined as:

$$Q_a(x) = \prod_{i=1}^{N_a} \left(i\, x - u_{a,i}\right), \qquad R_a(x) = \prod_{i=1}^{2\lfloor N_a/2 \rfloor} \left(i\, x - u_{a,i}\right), \qquad a = 1, 2, 3.$$

Section 9

Chiral primary operators

Definition of CPOs

The chiral primary operators (CPO's) of $SU(N_c)$, $\mathcal{N} = 4$ SYM theory are defined as:

$$\mathcal{O}_I^{\rm CPO}(\mathrm{x}) = \frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{\frac{L}{2}} \Psi_I^{\mu_1\cdots\mu_L}\mathrm{tr}\left[\varphi_{\mu_1}(\mathrm{x})\ldots\varphi_{\mu_L}(\mathrm{x})\right], \qquad \mathrm{x} \equiv \{\mathrm{x}^{(0,1,2,3)}\},$$

where $\Psi_I^{\mu_1\cdots\mu_L}$ are traceless symmetric tensors of $SO(6)$ defining the S^5 spherical harmonics

$$Y_I(x_\mu) \equiv \Psi_I^{\mu_1\cdots\mu_L} x_{\mu_1}\ldots x_{\mu_L}, \qquad \Psi_I^{\mu_1\cdots\mu_L}\Psi_J^{\mu_1\cdots\mu_L} = \delta_{IJ}, \qquad \sum_{\mu=4}^{9} x_\mu^2 = 1$$

and I, J the corresponding quantum numbers. The dual supergravity fields s_I have been identified...

$$S = \frac{4N_c^2}{(2\pi)^5}\int d^4x\, dz\sqrt{g}\left\{\sum_I \frac{A_I}{2}\left[-(\nabla s_I)^2 - L(L-4)s_I^2\right] + \sum_{I,J,K}\frac{1}{3}\mathfrak{G}^{I,J,K}s_I s_J s_K\right\}.$$

Lee-Minwalla-Rangamani-Seiberg (1998)

The overall factor in front of the CPO's ensures that their 2-point functions are normalized to unity:

$$\langle \mathcal{O}_I^{\rm CPO}(\mathrm{x}_1)\,\mathcal{O}_J^{\rm CPO}(\mathrm{x}_2)\rangle = \frac{\delta_{IJ}}{\mathrm{x}_{12}^{2L}}.$$

Differentiating the definition of Y_I we may also show

$$\Box Y_I = -L(L+4)\,Y_I.$$

Subsection 1

$SO(3) \times SO(3)$ spherical harmonics

$SO(3) \times SO(3)$ invariant spherical harmonics

The definition of the S^5 spherical harmonics was given above. Let us now determine the subset of S^5 spherical harmonics that is invariant under the $SO(3) \times SO(3)$ subgroup of $SO(6)$...

$SO(3) \times SO(3)$ invariant spherical harmonics

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The line element of the unit 5-sphere $d\Omega_5$ in a manifestly $SO(3) \times SO(3)$ invariant way reads:

$$d\Omega_5^2 = d\psi^2 + \cos^2\psi \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \sin^2\psi \left(d\vartheta^2 + \sin^2\vartheta d\chi^2\right),$$

where $\psi \in [0, \pi/2]$. The corresponding Cartesian coordinates $x_4, \ldots, x_9$ are

$$x_4 = \cos\psi \sin\theta \cos\varphi, \quad x_5 = \cos\psi \sin\theta \sin\varphi, \quad x_6 = \cos\psi \cos\theta,$$
$$x_7 = \sin\psi \sin\vartheta \cos\chi, \quad x_8 = \sin\psi \sin\vartheta \sin\chi, \quad x_9 = \sin\psi \cos\vartheta.$$

These obviously obey

$$\sum_{\mu=4}^{9} x_\mu^2 = 1, \qquad \sum_{\mu=4}^{6} x_\mu^2 = \cos^2\psi, \qquad \sum_{\mu=7}^{9} x_\mu^2 = \sin^2\psi,$$

so that the $SO(3) \times SO(3)$ invariant spherical harmonics on S^5 depend only on the angle ψ.

$SO(3)\times SO(3)$ invariant spherical harmonics

We can compute the spherical harmonics $Y(\psi)$ from the eigenfunctions of the Laplace operator on S^5:

$$\Box Y=\frac{1}{\sqrt{\hat{g}_s}}\,\partial_\mu\left[\sqrt{\hat{g}_s}\,\hat{g}^{\mu\nu}\partial_\nu Y\right]=\frac{1}{\cos^2\psi\sin^2\psi}\partial_\psi\left(\cos^2\psi\sin^2\psi\partial_\psi Y(\psi)\right).$$

Changing variables $z=\sin^2\psi$, the eigenvalue equation $\Box Y=-EY$ is brought to the following form

$$z(1-z)\,\partial_z^2 Y(z)+\left(\frac{3}{2}-3z\right)\partial_z Y(z)+\frac{E}{4}\,Y(z)=0.$$

which is just the hypergeometric equation with solution

$$E=2j(2j+4)\,,\quad Y_j(\psi)=\mathfrak{C}_j\cdot{}_2F_1\left(-j,j+2,\frac{3}{2};\sin^2\psi\right),\qquad j=0,1,\ldots,$$

where the normalization factor $\mathfrak{C}_j$ is determined from

$$\int_{S^5}\left|Y_j\right|^2=\frac{1}{2^{2j-1}(2j+1)(2j+2)}\int_{S^5}1.$$

We end up with the general formula,

$$Y_j(\psi)=\frac{(2j+2)!}{2^{j+\frac{1}{2}}\sqrt{(2j+1)(2j+2)}}\sum_{p=0}^{j}\frac{(-1)^p\cos^{2p}\psi\sin^{2j-2p}\psi}{(2p+1)!(2j-2p+1)!}\quad\Rightarrow\quad\mathfrak{C}_j=Y_j(0)=\left(-\frac{1}{2}\right)^j\sqrt{\frac{j+1}{2j+1}}.$$

Comparing the $SO(3)\times SO(3)$ eigenvalues with the above $SO(6)$ eigenvalues $L(L+4)$, we get $L=2j$.

Subsection 2

$SO(4)$ spherical harmonics

$SO(4)$ invariant spherical harmonics

Here we determine the subset of S^5 spherical harmonics that is invariant under the $SO(4)$ subgroup of $SO(6)$...

$SO(4)$ invariant spherical harmonics

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First we write the line element of the unit 5-sphere $d\Omega_5$ as:

$$ds^2 = d\theta^2 + \cos^2\theta \, d\Omega_4^2,$$

where $\theta \in [-\pi/2, \pi/2]$. The corresponding Cartesian coordinates $x_4, \ldots, x_9$ are

$$x_a = m_{(a-3)} \cos\theta, \qquad x_9 = \sin\theta, \qquad a = 4, \ldots, 8, \qquad \sum_{a=1}^{5} m_a^2 = 1,$$

where the variables m_a parametrize the unit 4-sphere, for instance

$$m_1 = c_1, \quad m_2 = s_1 c_2, \quad m_3 = s_1 s_2 c_3, \quad m_4 = s_1 s_2 s_3 c_4, \quad m_5 = s_1 s_2 s_3 s_4, \quad \sum_{a=1}^{5} m_a^2 = 1.$$

Obviously, the $SO(4)$ invariant spherical harmonics on S^5 will depend only on the angle θ...

$SO(4)$ invariant spherical harmonics

As before, we compute the spherical harmonics $Y(\psi)$ from the eigenfunctions of the Laplace operator on S^5:

$$\Box Y = \frac{1}{\sqrt{\hat{g}_s}}\,\partial_\mu\left[\sqrt{\hat{g}_s}\,\hat{g}^{\mu\nu}\partial_\nu Y\right] = \sec^4\theta\,\partial_\theta\left(\sec^4\theta\,\partial_\theta Y(\theta)\right) = -E\,Y(\theta)\,,$$

By changing variables $z = (1-\sin\theta)/2$, the eigenvalue equation $\Box Y = -EY$ can be brought to the following form

$$z(1-z)\,\partial_z^2 Y(z) + \left(\frac{5}{2} - 5z\right)\partial_z Y(z) + E\,Y(z) = 0,$$

which is again the hypergeometric equation with solution

$$E = 2j\,(2j+4)\,, \qquad Y_j(z) = \mathfrak{C}_j \cdot {}_2F_1\left(-2j, 2j+4, \frac{5}{2}; z\right), \qquad j = 0, 1, \ldots,$$

and the normalization factor $\mathfrak{C}_j$ is determined from

$$\int_{S^5} \left|Y_j\right|^2 = \frac{1}{2^{2j-1}\,(2j+1)\,(2j+2)} \int_{S^5} 1.$$

We end up with the general formula,

$$Y_j(\theta) = \frac{1}{2^j}\sqrt{\frac{(2j+2)(2j+3)}{6}} \cdot \sum_{p=0}^{2j} \frac{\Gamma(5/2)}{\Gamma(p+5/2)}\,\frac{(2j+p+3)!(2j)!}{(2j-p)!(2j+3)!p!}\left(\frac{\sin\theta - 1}{2}\right)^p \Rightarrow \mathfrak{C}_j = \frac{1}{2^j}\sqrt{\frac{(2j+2)(2j+3)}{6}}.$$

By comparing the $SO(4)$ eigenvalues with the above $SO(6)$ eigenvalues $L(L+4)$, we get again $L = 2j$.