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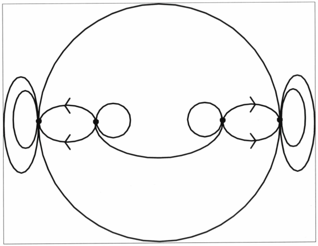
On braided Majorana qubits and Volichenko algebras

Talk given at Corfu 2024 (Sep. 17 - Sep. 24)

Workshop on Noncommutative and Generalized Geometry in String theory, Gauge theory and Related Modeld



(logo of the CNPq group: **Algebraic Structures in Field Theory**)

Multiparticle quantization of braided Majorana qubits:

- **F. T.,** *First quantization of braided Majorana fermions*, **Nucl. Phys. B 980 (2022), 115834; arXiv:2203.01776**

- **F. T.,** *The parastatistics of braided Majorana fermions*, **SciPost Phys. Proc. 14, 046 (2023); arXiv:2312.06693**

Fundamental ingredients:

The mathematical framework for braids appeared in:

- L. Kauffman and H. Saleur, *Free fermions and the Alexander-Conway polynomial*, **Comm. Math. Phys. 141, 293 (1991).**

Parastatistics recovered from graded Hopf algebras endowed with a braided tensor product:

- S. Majid, *Foundations of Quantum Group Theory*, **Cambridge University Press, Cambridge (1995).**

Recent results:

- **F.T.**, *Volichenko-type metasymmetry of braided Majorana qubits*, arXiv:2406.00876

1 - Quantum group interpretation of the roots of unity truncations recovered from a (superselected) set of reps of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$;

2 - Reconstruction, via suitable intertwining operators, of the braided tensor products as ordinary tensor products;

3 - Introduction of mixed-brackets for the braided creation/annihilation operators which define generalized Heisenberg-Lie algebras;

4 - Braided creation/annihilation operators as (*meta*)symmetries of ordinary differential equations given by matrix Schrödinger equations in $0+1$ dimension;

5 - Special case of a third root of unity truncation, a nonminimal realization of the intertwining operators defines the system as a ternary algebra.

Symmetries wider than supersymmetry:

Volume 252, number 1 PHYSICS LETTERS B 6 December 1990

Metasymmetry and Volichenko algebras

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Received 12 April 1990; revised manuscript received 14 September 1990

We continue the study of a generalization of supermanifolds (called here metamanifolds) on which "functions" form a metabelian algebra (one for which $[[x, y], z]=0$). The usual superspaces considered as metaspaces and some conventional lagrangeans have a symmetry wider than supersymmetry. Infinitesimal transformations of these metaspaces constitute Volichenko algebras. The Volichenko algebras are natural generalizations of Lie superalgebras. Here we classify simple finite-dimensional complex Volichenko algebras (under a technical hypothesis). Their list is as discrete as the list of simple Lie superalgebras. The results may be significant for applications to physics in connection with parastatistics.

What is a Majorana qubit:

A $\mathbb{Z}_2$-graded qubit describes the Hilbert space $\mathcal{H}^{(1)}$ of a single Majorana fermion. $|vac\rangle$ is the bosonic vacuum and $|\psi\rangle$ is the fermionic excited state:

$$|vac\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\psi\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The 2×2 matrix operators acting on the graded qubit are:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where α, δ are even (bosonic) and β, γ are odd (fermionic) matrices.
Their (anti)commutators define the $\mathfrak{gl}(1|1)$ superalgebra:

$$[\alpha, \beta] = \beta, \qquad [\alpha, \gamma] = -\gamma, \qquad [\alpha, \delta] = 0, \qquad [\delta, \beta] = -\beta, \qquad [\delta, \gamma] = \gamma,$$
$$\{\beta, \beta\} = \{\gamma, \gamma\} = 0, \qquad \{\beta, \gamma\} = \alpha + \delta.$$

The $\mathbb{Z}_2$-grading is given by

$$\mathfrak{gl}(1|1) = \mathfrak{gl}(1|1)_{[0]} \oplus \mathfrak{gl}(1|1)_{[1]}, \qquad \text{with} \quad \alpha, \delta \in \mathfrak{gl}(1|1)_{[0]} \quad \text{and} \quad \beta, \gamma \in \mathfrak{gl}(1|1)_{[1]}.$$

The matrices γ, β are a pair of fermionic creation/annihilation operators:

$$\{\gamma,\gamma\} = \{\beta,\beta\} = 0, \quad \{\gamma,\beta\} = \mathbb{I}_2, \qquad \beta|vac\rangle = 0, \quad |\psi\rangle = \gamma|vac\rangle.$$

Since **bosons/fermions are superselected**, the linear superposition of states belonging to different graded sectors is not allowed. Therefore, the Hilbert space is graded:

$$\mathcal{H}^{(1)} = \mathcal{H}^{(1)}_{[0]} \oplus \mathcal{H}^{(1)}_{[1]} \quad \equiv \quad \mathbb{C}^{1|1}.$$

The elements of its even and odd sectors are

$$c_0|vac\rangle \in \mathcal{H}^{(1)}_{[0]}, \qquad c_1|\psi\rangle \in \mathcal{H}^{(1)}_{[1]}, \qquad \text{with} \qquad c_0, c_1 \in \mathbb{C}.$$

A physical state is recovered by taking into account the irrelevance of the phase of a normalized vector. The above system describes two inequivalent physical states which are just $|vac\rangle$ and $|\psi\rangle$.
They correspond to a classical 1 bit of information (off/on states).
Just like the physically inequivalent states of an ordinary qubit are specified by points of the $\mathbf{S}^2$ Bloch sphere, $\mathbf{Z}^2$ (which is equivalent to a classical bit) represents "the Bloch sphere of the graded qubit".
The single-particle quantum Hamiltonian can be taken to be

$$H \quad := \quad \gamma\beta = \delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then 0 is the vacuum energy and 1 the energy eigenvalue of the excited state.

The multi-particle Hilbert space

The $\mathbb{Z}_2$-graded N-particle Hilbert space $\mathcal{H}^{(N)}$ is a subset of the tensor product of N single-particle Hilbert spaces $\mathcal{H}^{(1)} = \mathbb{C}^{(1|1)}$:

$$\mathcal{H}^{(N)} \quad \subset \quad \mathcal{H}^{\otimes N}.$$

The N-particle vacuum $|vac\rangle_N$ is the tensor product of N single-particle vacua:

$$|vac\rangle_N \quad = \quad |vac\rangle \otimes \ldots \otimes |vac\rangle \qquad\qquad (N \text{ times}).$$

The construction of the multiparticle observables and excited states is made in terms of an operation, the coproduct, defined for a Hopf algebra.

In our case the Hopf algebra is a Universal Enveloping Algebra (denoted as $\mathcal{U} \equiv \mathcal{U}(\mathfrak{g})$) of a graded Lie algebra.

A Hopf algebra is characterized by compatible structures (unit and multiplication), costructures (counit and coproduct) and antipode.

The coproduct Δ is a map

$$\Delta \quad : \quad \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$$

which satisfies the coassociativity property

$$\begin{aligned} (\Delta \otimes id)\Delta(U) &= (id \otimes \Delta)\Delta(U) \qquad \text{for} \quad U \in \mathcal{U}, \\ \Delta^{(n+1)} &= (\Delta \otimes id)\Delta^{(n)} = (id \otimes \Delta)\Delta^{(n)}. \end{aligned}$$

For any $U_A, U_B \in \mathcal{U}$, the further property

$$\Delta(U_A U_B) \quad = \quad \Delta(U_A)\Delta(U_B)$$

implies that the action on any given $U \in \mathcal{U}(\mathfrak{g})$ is recovered from the action of the coproduct on the Hopf algebra unit $\mathbf{1}$ and the Lie algebra elements $g \in \mathfrak{g}$:

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \qquad \Delta(g) = \mathbf{1} \otimes g + g \otimes \mathbf{1}.$$

Let R be a representation of the Universal Enveloping Algebra $\mathcal{U}$ on a vector space V. The representation of the operators induced by the coproduct will be denoted with a hat:

$$\text{for} \quad R : \mathcal{U} \to V, \qquad \widehat{\Delta} := \Delta|_R \in End(V \otimes V), \qquad \text{with} \quad \widehat{\Delta(U)} \in V \otimes V.$$

The coassociativity implies

$$\widehat{\Delta^{(n)}(U)} \quad \in \quad V \otimes \ldots \otimes V \qquad (n+1 \quad \text{times}).$$

The N-particle Hamiltonians $H_{(N)}$ are obtained by applying the N-particle coproducts $\Delta^{(N-1)}$ to the single-particle Hamiltonian $H = \delta$;
an N-particle excited state is created by applying $\Delta^{(N-1)}$ to the creation operator γ:

$$H_{(N)} = \widehat{\Delta^{(N-1)}(\delta)}, \qquad \gamma_{(N)} = \widehat{\Delta^{(N-1)}(\gamma)},$$

For $N = 2, 3, \ldots$, we get:

$$\begin{aligned} H_{(2)} &= \mathbb{I}_2 \otimes \delta + \delta \otimes \mathbb{I}_2, \\ H_{(3)} &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \delta + \mathbb{I}_2 \otimes \delta \otimes \mathbb{I}_2 + \delta \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, \\ \gamma_{(2)} &= \mathbb{I}_2 \otimes \gamma + \gamma \otimes \mathbb{I}_2, \\ \gamma_{(3)} &= \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \gamma + \mathbb{I}_2 \otimes \gamma \otimes \mathbb{I}_2 + \gamma \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \end{aligned}$$

and so on.

The braided tensor product

The introduction of a non-trivial braiding requires specifying how Lie superalgebra generators are braided in a tensor product.
Let a, b, c, d be four generators of a Lie superalgebra represented by n-dimensional matrices. The braiding is expressed as

$$\boxed{(a \otimes b) \cdot (c \otimes d) = (a \otimes \mathbb{I}_n) \cdot \Psi(b, c) \cdot (\mathbb{I}_n \otimes d),}$$

$\Psi(b, c)$ is a $n^2 \times n^2$ matrix which encodes the braiding of b and c.
The dots in the right hand side denote ordinary matrix multiplication.
$\Psi(b, c)$ needs to satisfy certain braiding conditions.

For braided Majorana fermions we only need to specify the braidings of δ and γ.
The unique nontrivial braiding matrix is $\Psi(\gamma, \gamma)$ which encodes the braiding properties of the Majorana fermions (since γ is their creation operator):

$$\boxed{\Psi(\delta, \delta) = \delta \otimes \delta, \qquad \Psi(\delta, \gamma) = \gamma \otimes \delta, \qquad \Psi(\gamma, \delta) = \delta \otimes \gamma}$$

and

$$\boxed{\Psi(\gamma, \gamma) \equiv \Psi_t(\gamma, \gamma), \quad \text{where, for} \quad t \in \mathbb{C}^*, \quad \Psi_t(\gamma, \gamma) = B_t \cdot \gamma \otimes \gamma.}$$

B_t is a 4×4 constant matrix which depends on the parameter $t \neq 0$ and satisfies the braiding conditions; the dot in the r.h.s. denotes the standard matrix multiplication. $\mathbb{C}^* \equiv \mathbb{C} \backslash \{0\}$ is the punctured complex plane without the origin.

A consistent choice for B_t is

$$\boxed{B_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}.}$$

B_t is related to both the Burau representation of the braid group and the R-matrix of the quantum group $\mathcal{U}_q(\mathfrak{gl}(1|1))$.

The consistency is the braid relation satisfied by B_t:

$$\boxed{(B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) = (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t).}$$

Properties

I - B_t is dynamically compatible, since it commutes with the 2-particle Hamiltonian $H_{(2)}$:

$$[H_{(2)}, B_t] = 0.$$

II - for any integer N, the N-particle creation operator $\gamma_{(N)}$ creates one quantum of energy:

$$[H_{(N)}, \gamma_{(N)}] = \gamma_{(N)}.$$

III - B_t is bosonic. The even (odd) nonvanishing entries of the $\mathfrak{gl}(1|1)$ generators can be expressed as bullets (stars); in the tensor products we get

$$\begin{pmatrix} \bullet & * \\ * & \bullet \end{pmatrix} \otimes \begin{pmatrix} \bullet & * \\ * & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & * & * & \bullet \\ * & \bullet & \bullet & * \\ * & \bullet & \bullet & * \\ \bullet & * & * & \bullet \end{pmatrix}.$$

Truncations at roots of unity:

The t roots of unity which satisfy the polynomial equations produce truncations in multiparticle Hilbert spaces (and corresponding energy spectra) of the braided Majorana fermions.

The braided tensor product implies that

$$\boxed{(\mathbb{I}_4 \otimes \gamma) \cdot (\gamma \otimes \mathbb{I}_4) = \Psi_t(\gamma, \gamma) = -t\gamma \otimes \gamma.}$$

By taking into account that $\gamma^2 = 0$ simple computations show that, for $N = 2, 3$, the only nonvanishing powers of $\gamma_{(N)}$ are

$$\begin{aligned}
\gamma_{(2)} &= 1 \cdot (\mathbb{I}_2 \otimes \gamma + \gamma \otimes \mathbb{I}_2), \\
\gamma_{(2)}^2 &= (1-t) \cdot (\gamma \otimes \gamma), \\
\gamma_{(3)} &= 1 \cdot (\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \gamma + \mathbb{I}_2 \otimes \gamma \otimes \mathbb{I}_2 + \gamma \otimes \mathbb{I}_2 \otimes \mathbb{I}_2), \\
\gamma_{(3)}^2 &= (1-t) \cdot (\mathbb{I}_2 \otimes \gamma \otimes \gamma + \gamma \otimes \mathbb{I}_2 \otimes \gamma + \gamma \otimes \gamma \otimes \mathbb{I}_2), \\
\gamma_{(3)}^3 &= (1-t)(1-t+t^2) \cdot (\gamma \otimes \gamma \otimes \gamma).
\end{aligned}$$

Important notion: “root of unity level”

A “level-k” root of unity, for $k = 2, 3, 4, \ldots$, a solution t_k of the $b_k(t_k) = 0$ equation such that, for any $k' < k$, $b_{k'}(t_k) \neq 0$.

Physical significance of a level-k root of unity: the corresponding braided multiparticle Hilbert space can accommodate at most $k-1$ Majorana spinors.

The special point $t = 1$, being the solution of the $b_2(t) \equiv 1 - t = 0$ equation, is a level-2 root of unity.
It givesthe ordinary total antisymmetrization of the fermionic wavefunctions.
The $t = 1$ level-2 root of unity encodes the Pauli exclusion principle of ordinary fermions.

With an abuse of language, the $t = -1$ root of unity, which does not solve any $b_k(t) = 0$ equation, can be called a root of unity of ∞ level.

Example: the 5 roots of $b_6(t) = 1 - t + t^2 - t^3 + t^4 - t^5$ are classified, for $t = exp(i\theta)$, into:

level-2 root, $\theta = 0$,
level-3 roots $\theta = \pi/3$ and $5\pi/3$,
level-6 roots $\theta = 2\pi/3$ and $4\pi/3$.

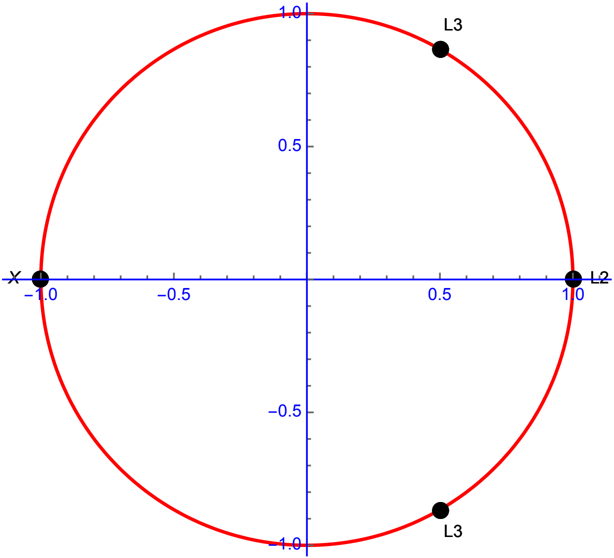
Physical significance of the level:

I - truncation of the energy spectrum: a level k root accommodates at most k inequivalent energy levels in the multiparticle states.

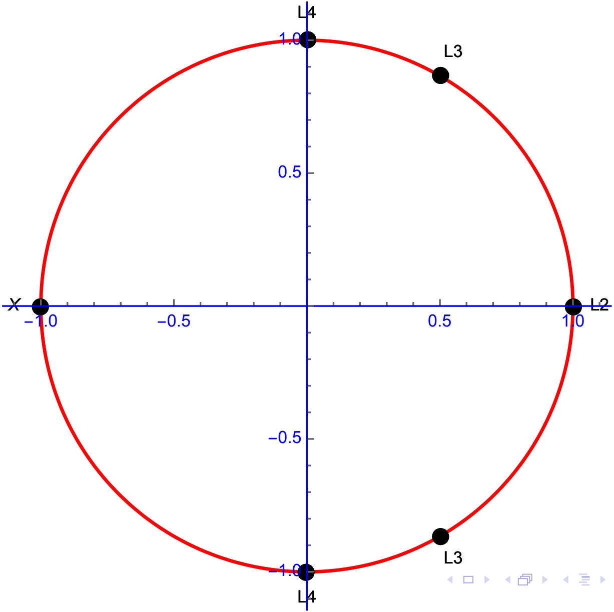
II - statistics' viewpoint: a level k root implies that at most $k-1$ Majorana parafermions can be created.

Comment: the lowest level $k = 2$ for $t = 1$ implies that the Majorana particles are ordinary fermions obeying the Pauli exclusion principle.

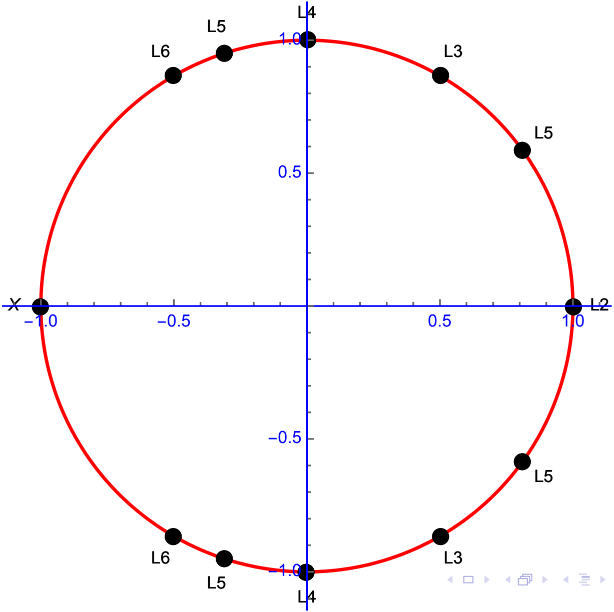
Roots of unity, levels up to 3:



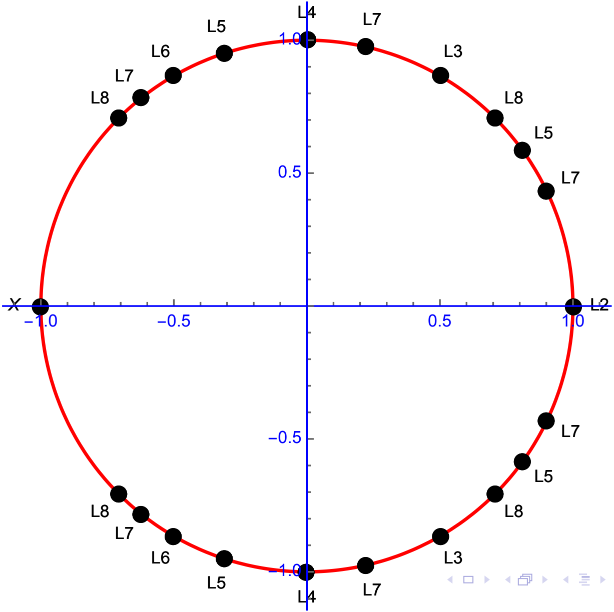
Roots of unity, levels up to 4:



Roots of unity, levels up to 6:



Roots of unity, levels up to 8:



Comment: the multiparticle energy spectra only depend on the "roots of unity levels".

Let's present some tables

Level $k=2$ root of unity: $t=1$; the N-particle energy levels are

$E\backslash N$	1	2	3	4	5	6	7
2							
1	X	X	X	X	X	X	X
0	X	X	X	X	X	X	X

Comment: this table corresponds to the ordinary, totally antisymmetrized, Majorana fermions, with only $E=0,1$ energy eigenvalues for any N.

Level $k=3$ roots of unity, given by $t=e^{i\vartheta}$ with $\vartheta=\frac{1}{3}\pi, \frac{5}{3}\pi$:

$E\backslash N$	1	2	3	4	5	6	7
3							
2		X	X	X	X	X	X
1	X	X	X	X	X	X	X
0	X	X	X	X	X	X	X

Comment: the energy eigenvalues are $E=0,1,2$ for any multiparticle sector with $N\geq 2$.

Level $k = 4$ roots of unity, given by $t = e^{i\vartheta}$ with $\vartheta = \frac{1}{2}\pi, \frac{3}{2}\pi$:

$E \backslash N$	1	2	3	4	5	6	7
4							
3			X	X	X	X	X
2		X	X	X	X	X	X
1	X	X	X	X	X	X	X
0	X	X	X	X	X	X	X

Comment: a "plateau" is reached; starting from $N \geq 3$ the energy eigenvalues are $E = 0, 1, 2, 3$.

Level $k = 5$ roots of unity given by $t = e^{i\vartheta}$, $\vartheta = \frac{1}{5}\pi, \frac{3}{5}\pi, \frac{7}{5}\pi, \frac{9}{5}\pi$:

$E \backslash N$	1	2	3	4	5	6	7
5							
4				X	X	X	X
3			X	X	X	X	X
2		X	X	X	X	X	X
1	X	X	X	X	X	X	X
0	X	X	X	X	X	X	X

Comment: the plateau is shifted at $N \geq 4$, with energy eigenvalues $E = 0, 1, 2, 3, 4$.

General formulas:

Truncated cases at level-k: N-particle energy eigenvalues E given by

$$
\begin{aligned}
E &= 0,1,\ldots,N \qquad \text{for} \quad N < k,\\
E &= 0,1,\ldots,k-1 \qquad \text{for} \quad N \geq k.
\end{aligned}
$$

Comment: the plateau is reached for the maximal energy level $k-1$; this is the maximal number of braided Majorana fermions that can be accommodated in a multiparticle Hilbert space.

Untruncated case for $t=-1$ (level-∞):

$$E = 0,1,\ldots,N \qquad \text{for any} \quad N.$$

Comment: there is no plateau; the energy eigenvalues grow linearly with N.

II - solutions of open questions and new results

Convenient parametrization of roots of unity levels

Set for t belonging to the $|t| = 1$ unit circle:

$$t = -e^{2i\pi g}, \qquad \text{for real values} \quad g \in [0, 1[.$$

Level-s roots of unity L_s and the L_∞ untruncated case are given by

$$\begin{aligned} L_s \quad &: \quad g = \frac{r}{s} \qquad \text{with } r, s \text{ mutually prime integers}, \\ L_\infty \quad &: \quad g = 0. \end{aligned}$$

At the first orders the g values are

$$L_\infty = 0; \quad L_2 = \tfrac{1}{2}; \quad L_3 = \tfrac{1}{3},\ \tfrac{2}{3}; \quad L_4 = \tfrac{1}{4},\ \tfrac{3}{4}; \quad L_5 = \tfrac{1}{5},\ \tfrac{2}{5},\ \tfrac{3}{5},\ \tfrac{4}{5}.$$

The physics only depends on the s level and not on a given particular representative; without loss of generality one can set $r = 1$, so that

$$L_s \quad : \quad g_s = \frac{1}{s} \quad \text{and } t_s = e^{\pi i(\frac{2}{s} - 1)}.$$

The $g = 0$ case of the untruncated L_∞ level is recovered in the limit

$$g_\infty \quad = \quad \lim_{s \to \infty} g_s = 0.$$

A quantum group derivation of the truncations

Naively one could expect to directly work with the quantum superalgebra $\mathcal{U}_q(\mathfrak{gl}(1|1))$. This option is not viable: the creation operator γ entering is nilpotent and the same is true for its $\mathcal{U}_q(\mathfrak{gl}(1|1))$ quantum group counterpart. Due to the homomorphism of the coproduct, we get the nilpotent quantum group expression $\Delta_q(\gamma)^2 = \Delta_q(\gamma) \cdot \Delta_q(\gamma) = \Delta_q(\gamma^2) = 0$ for the $\mathcal{U}_q(\mathfrak{gl}(1|1))$ coproduct.

On the other hand a nonvanishing $(\Delta(\gamma))^2 \neq 0$ coproduct induced by the braiding is essential to produce the multi-particle spectra of the braided Majorana qubits. Clearly, some other construction has to be done.

The solution is found by working within the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$, inducing the multi-particle states by applying its coproduct to a specific representation and, furthermore, implementing a consistent superselection of the energy spectra. These steps allow to recover the multi-particle spectra of the braided Majorana qubits.

Another realization of the building blocks: braided tensors via intertwining operators

The braiding relation

$$(\mathbb{I}_2 \otimes_{br} \gamma)\cdot(\gamma \otimes_{br} \mathbb{I}_2) = -t(\gamma \otimes_{br} \mathbb{I}_2)\cdot(\mathbb{I}_2 \otimes_{br} \gamma) = -t(\gamma \otimes_{br} \gamma).$$

can be expressed in terms of an ordinary tensor product $\otimes$ by introducing a suitably defined intertwining operator W_t:

$$(\gamma \otimes_{br} \mathbb{I}_2) \mapsto \gamma \otimes \mathbb{I}_2, \qquad (\mathbb{I}_2 \otimes_{br} \gamma) \mapsto W_t \otimes \gamma.$$

The mappings

$$\begin{array}{rcl} (\mathbb{I}_2 \otimes_{br} \gamma)\cdot(\gamma \otimes_{br} \mathbb{I}_2) & \mapsto & (W_t \otimes \gamma)\cdot(\gamma \otimes \mathbb{I}_2) = (W_t\gamma) \otimes \gamma \\ (\gamma \otimes_{br} \mathbb{I}_2)\cdot(\mathbb{I}_2 \otimes_{br} \gamma) & \mapsto & (\gamma \otimes \mathbb{I}_2)\cdot(W_t \otimes \gamma) = (\gamma W_t) \otimes \gamma \end{array}$$

imply a consistency condition for the 2×2 interwining operator W_t given by

$$W_t\gamma \quad = \quad (-t)\gamma W_t.$$

A solution, expressed in terms of the $t = -e^{2i\pi g}$ position, is

$$W_t \quad = \quad \cos(-\pi g)\cdot\mathbb{I}_2 + i\sin(-\pi g)\cdot X, \qquad \text{where } X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

New building blocks:

$$\begin{array}{lcl} 2P & : & A_1^\dagger := \gamma \otimes \mathbb{I}_2, \qquad A_2^\dagger := W_t \otimes \gamma; \\ 3P & : & B_1^\dagger := \gamma \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, \quad B_2^\dagger := W_t \otimes \gamma \otimes \mathbb{I}_2, \quad B_3^\dagger := W_t \otimes W_t \otimes \gamma. \end{array}$$

The 2-particle creation and annihilation operators belong to a non-standard odd sector of a $\mathbb{Z}_2$-graded decomposition of 4×4 matrices: the nonvanishing entries (denoted with "$*$") of the even (odd) sector M_0 (M_1) are accommodated in

$$M_0 \equiv \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}, \qquad M_1 \equiv \begin{pmatrix} 0 & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & * \\ 0 & * & * & 0 \end{pmatrix}.$$

The $\mathbb{Z}_2$ grading is respected since

$$M_i \cdot M'_j = M''_{i+j} \qquad \text{for} \quad i,j = 0,1, \quad \text{with} \quad i+j = 0,1 \ \ mod\ 2.$$

The 2-particle creation building blocks $A_1^\dagger, A_2^\dagger$ and their conjugate are

$$A_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-i\pi g} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi g} & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & e^{i\pi g} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\pi g} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

An even 4×4 central charge c is defined as $c = diag(1,1,1,1)$.

Leites-Serganova introduced in 1990 the notions of: metamanifold, metaspace, metasymmetry, Volichenko algebra

Metasymmetries are transformations acting on "metaspaces": they do not respect even/odd gradings and generalize the $\mathbb{Z}_2$-grading preserving symmetries of ordinary superalgebras.

They lead to "mixed-brackets" which interpolate ordinary commutators/anticommutators.

They are impllemented in Volichenko algebras which are "metabelian": (metabelianess means that for any x, y, z triple of operators the ordinary $[[x, y], z] = 0$ commutators are vanishing.

The operators entering the mixed-brackets generalized Heisenberg-Lie algebras do not satisfy the metabelianess condition: they are not Volichenko (just the 2-particle subalgebras spanned by either the creation or the annihilation operators are).

Despite of that the notion of *metasymmetry* can be applied to the mixed-brackets Heisenberg-Lie algebras.

from Leites-Serganova, NATO ARW Proc., Kiev 2000:

1.3. An intriguing example: the general Volichenko algebra $\mathfrak{vgl}_\mu(p|q)$. Let the space $\mathfrak{h}$ of $\mathfrak{vgl}_\mu(p|q)$ be the space of $(p+q)\times(p+q)$-matrices divided into the two subspaces as follows:

$$\mathfrak{h}_{\hat{0}} = \left\{ \begin{matrix} A & 0 \\ 0 & D \end{matrix} \right\}; \quad \mathfrak{h}_{\hat{1}} = \left\{ \begin{matrix} 0 & B \\ C & 0 \end{matrix} \right\}. \tag{1.3.1}$$

Here $\mathfrak{h}_{\hat{1}}$ is a natural $\mathfrak{h}_{\hat{0}}$-module with respect to the bracket of matrices; fix $a, b \in \mathbb{C}$ such that $a : b = \mu \in \mathbb{C}P^1$ and define the multiplication $\mathfrak{h}_{\hat{1}} \times \mathfrak{h}_{\hat{1}} \longrightarrow \mathfrak{h}_{\hat{0}}$ by the formula

$$[X, Y] = a[X, Y]_- + b[X, Y]_+ \text{ for any } X, Y \in \mathfrak{h}_{\hat{1}}. \tag{1.3.2}$$

(The subscript $-$ or $+$ indicates the commutator and the anticommutator, respectively.) As we sill see, $\mathfrak{h}$ is a simple Volichenko algebra for any a, b except for $ab = 0$ when it becomes isomorphic to either the Lie algebra $\mathfrak{gl}(p+q)$ or the Lie superalgebra $\mathfrak{gl}(p|q)$. To show that $\mathfrak{vgl}_\mu(p|q)$ is indeed a Volichenko algebra, we have to realize it as a subalgebra of a Lie superalgebra. This is done in heading 2 of Theorem 2.7.

Introduction of the

“mixed-brackets”
generalized fermionic Heisenberg-Lie algebras

(satisfied by the creation/annihilation operators of the multiparticle braided Majorana qubits)

Here: 2-particle example

Definition

For two operators A, B, the mixed-bracket is defined as

$$(A,B)_{\theta_{AB}} \quad = \quad i \sin\theta_{AB}[A,B] + \cos\theta_{AB}\{A,B\}.$$

with the angle θ_{AB} to be determined.

Property:

$$(B,A)_{-\theta_{AB}} \quad = \quad (A,B)_{\theta_{AB}}.$$

Level-s 2-particle operators:

$$A_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-i\pi/s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/s} & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & e^{i\pi/s} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\pi/s} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$c = diag(1, 1, 1, 1).$

Rename them:

$$G_0 = c, \ G_1 = A_1^\dagger, \ G_2 = A_2^\dagger, \ G_3 = A_3, \ G_4 = A_4.$$

Mixed-brackets:

$$(G_I, G_J)_{\theta_{IJ}} \qquad \text{for} \quad I, J = 0, 1, 2, 3, 4.$$

Level-s 2-particle generalized fermionic Heisenberg-Lie algebra:

$$(G_1, G_3)_{\theta_{13}} = (G_3, G_1)_{\theta_{31}} = (G_2, G_4)_{\theta_{24}} = (G_4, G_2)_{\theta_{42}} = G_0.$$

All other $(G_I, G_J)_{\theta_{IJ}}$ brackets are vanishing.

Determination of the θ_{IJ} angles:

$$\theta_{IJ} = \frac{s+2}{4s}\pi \cdot \mu_I \mu_J \cdot (\nu_I - \nu_J).$$

where $\mu_I, \mu_J, \nu_I, \nu_J$ are determined as follows

Let

$$N_L = -\frac{1}{2} \cdot diad(1, 1, -1, -1), \qquad N_R = -\frac{1}{2} diag(1, -1, 1, -1)$$

and define for a given operator G:

$$[N_L, G] = \lambda_L G, \qquad [N_R, G] = \lambda_R G.$$

We can set

$$\mu = \lambda_L + \lambda_3, \qquad \nu = \lambda_L^2 - \lambda_R^2.$$

The corresponding μ_I, ν_I values for G_I are read from the table

	μ	ν
G_0	0	0
G_1	1	1
G_2	1	-1
G_3	-1	1
G_4	-1	-1

Comments:

The generalized mixed-brackets level-s multiparticle Heisenberg-Lie algebras have formally the same presentation of the ordinary Heisenberg-Lie algebras.

The 2-particle construction is immediately generalized to the N-particle case.

We get

$$(A_i, A_j^\dagger) = \delta_{ij} \cdot c = \delta_{ij} \cdot \mathbb{I} \quad \text{(all other brackets are vanishing)}$$

for the suitable angles entering $(\cdot, \cdot)$.

The $s \to \infty$ limit which reproduces a bosonic spectrum for the graded Majorana qubits

In that limit the 2-particle operators are:

$$A_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c = diag(1,1,1,1).$$

They close a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded extension of the 2-particle fermionic Heisenberg algebra:

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra with grading assignment:

$$
\begin{aligned}
c &\in 00 && (\textit{boson}),\\
A_1, A_1^\dagger &\in 10 && (\textit{parafermions}),\\
A_2, A_2^\dagger &\in 01 && (\textit{parafermions}),\\
&\in 11 && (\textit{empty exotic boson sector}).
\end{aligned}
$$

$$
\begin{gathered}
\{A_1, A_1^\dagger\} = \{A_2, A_2^\dagger\} = c,\\
\{A_1, A_1\} = \{A_2, A_2\} = \{A_1^\dagger, A_1^\dagger\} = \{A_2^\dagger, A_2^\dagger\} = 0,\\
[A_1, A_2] = [A_1, A_2^\dagger] = [A_1^\dagger, A_2] = [A_1^\dagger, A_2^\dagger] = 0,\\
[c, *] = 0.
\end{gathered}
$$

Dynamical “metasymmetry” of the mixed-brackets Heisenberg-Lie algebras

The mixed-brackets generalizations of the fermionic Heisenberg-Lie algebras appear as dynamical symmetry of Ordinary Differential Equations given by Matrix Schrödinger equations in $0+1$ dimensions.

2-particle example: Matrix Schrödinger equation

$$\left(i\partial_t \cdot \mathbb{I}_4 - H_2\right)\Psi(t) \quad = \quad 0,$$

where $H_2 = diag(0,1,1,2)$ and $\Psi(t)$ is a 4-component vector.
The $\Psi_{ij}(t)$ solutions can be expressed in terms of the creation $(A_1^\dagger, A_2^\dagger)$ and annihilation (A_1, A_2) operators defined for the given angle πg:

$$\begin{array}{lcll}
\Psi_{00}(t) & = & v_{00}, & \text{where } v_{00}^T = (1,0,0,0),\\
\Psi_{10}(t) & = & e^{-it}A_1^\dagger v_{00} = e^{-it}v_{10}, & \text{where } v_{10}^T = (0,0,1,0),\\
\Psi_{01}(t) & = & e^{-it}A_2^\dagger v_{00} = e^{-it}v_{01}, & \text{where } v_{01}^T = (0,e^{i\pi g},0,0),\\
\Psi_{11}(t) & = & e^{-2it}A_1^\dagger A_2^\dagger v_{00} = e^{-2it}v_{11}, & \text{where } v_{11}^T = (0,0,0,e^{i\pi g}).
\end{array}$$

By setting

$$S_1^\dagger = e^{-it}A_1^\dagger, \qquad S_2^\dagger = e^{-it}A_2^\dagger, \qquad S_1 = e^{it}A_1, \qquad S_2 = e^{it}A_2,$$

we end up with four plus one symmetry operators (the extra operator being the 4×4 identity operator $c := \mathbb{I}_4$) satisfying

$$[S_\sharp, i\partial_t \cdot \mathbb{I}_4 - H_2] \quad = \quad 0 \qquad \text{for } S_\sharp = S_1^\dagger,\ S_2^\dagger,\ S_1,\ S_2,\ c.$$

These operators close the 2-particle mixed-bracket generalization of the fermionic Heisenberg-Lie algebra.

Nonminimal realization of the intertwining operators: Connection with ternary algebras

In the minimal matrix representation, the N-particle sector of the braided Majorana qubits is realized by $2^N \times 2^N$ matrices. Equivalent descriptions which produce isomorphic Hilbert spaces can be obtained from nonminimal representations.

Let's consider the third root of unity; an example of a set of nonminimal representations is given by $2 \cdot 6^{N-1} \times 2 \cdot 6^{N-1}$ matrices. Unlike the minimal representations with the special third root of unity case this nonminimal set encodes a $\mathbb{Z}_3$ ternary grading.

The ternary construction of the braided Majorana qubits employs tensor products of the three 3×3 matrices Q_i (defined for $j = e^{i\frac{2}{3}\pi}$ with $j^3 = 1$) and their $Q_i^\dagger$ hermitian conjugates:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \qquad Q_2^\dagger = \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

A consistent Z_3 grading can be assigned by setting, *mod* 3,

$$deg(Q_i) = 1, \qquad deg(Q_i^\dagger) = 2, \qquad \text{for } i = 1, 2, 3.$$

The non-minimal building blocks of the braided 2-particle are

$$\widetilde{A}_1^\dagger = \gamma \otimes \mathbb{I}_2 \otimes Q_1, \quad \widetilde{A}_2^\dagger = \gamma \otimes \mathbb{I}_2 \otimes Q_2, \quad \widetilde{A}_1 = \gamma^\dagger \otimes \mathbb{I}_2 \otimes Q_1^\dagger, \quad \widetilde{A}_2 = \gamma^\dagger \otimes \mathbb{I}_2 \otimes Q_2^\dagger.$$

They satisfy the same relations as their minimal counterparts.

Note about quons:

The "mixed brackets" which interpolate commutators and anticommutators can be defined for other types of parastatistics. The most notable example is the algebra of quons introduced by Greenberg and Mohapatra. Quons are q-deformed oscillators, defined for $-1 \leq q \leq 1$ which interpolate between fermions ($q = -1$) and bosons ($q = 1$). The "q-mutators" of n creation/annihilation $a_i^\dagger, a_i$ quons, with $i = 1, 2, \ldots, n$ are defined to satisfy

$$a_i a_j^\dagger - q a_i^\dagger a_j \quad = \quad \delta_{ij}.$$

It is a trivial exercise to express the q-mutator of one ($n = 1$) quon as a mixed bracket, interpolating commutator and anticommutator. One has to set

$$aa^\dagger - qa^\dagger a = 1 \qquad \Leftrightarrow \qquad \cos^2(\theta_q) \cdot [a, a^\dagger] + \sin^2(\theta_q) \cdot \{a, a^\dagger\} = 1,$$

where the angle θ_q, comprised in the range $\theta_q \in [0, \pi/2]$, is given by

$$\theta_q \quad = \quad \arcsin\left(\sqrt{\frac{1-q}{2}}\right), \qquad (q = 1 - 2\sin^2\theta_q).$$

The Volichenko-type mixed brackets which define the generalized fermionic Heisenberg-Lie algebras and give the multi-particle parastatistics of the braided Majorana qubits is not reproduced by the the quonic "mixed brackets" formulas.

Braid statistics (anyons) have been experimentally observed in two-dimensional material. They can also have relevant applications.

What about parastatistics? (beyond bosons/fermions in any D)

(Permutation group, not braid group, with $S^2 = \mathbb{I}$)

Along the years some arguments have been put forward to explain why fundamental paraparticles have not been observed in Nature.

Main idea: paraparticles are not observable because they can be reproduced by ordinary particles

(Conventionality of parastatistics' argument)

A nice and nuanced discussion is found in the following paper

The Conventionality of Parastatistics

David John Baker
Hans Halvorson
Noel Swanson*

March 6, 2014

Abstract

Nature seems to be such that we can describe it accurately with quantum theories of bosons and fermions alone, without resort to parastatistics. This has been seen as a deep mystery: paraparticles make perfect physical sense, so why don't we see them in nature? We consider one potential answer: every paraparticle theory is physically equivalent to some theory of bosons or fermions, making the absence of paraparticles in our theories a matter of convention rather than a mysterious empirical discovery. We argue that this equivalence thesis holds in all physically admissible quantum field theories falling under the domain of the rigorous Doplicher-Haag-Roberts approach to superselection rules. Inadmissible parastatistical theories are ruled out by a locality-inspired principle we call Charge Recombination.

Article *in* The British Journal for the Philosophy of Science · February 2013

DOI: 10.1093/bjps/axu018

Recent advances challenge the
"Conventionality of parastatistics" argument

Theoretical advances

(based on the "n-bit parastatistics")

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras

	00	**10**	**01**	**11**
00	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$
10	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$
01	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$
11	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$

Comment. In $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra physics the particles are accommodated in 2 bits of information:

- **ordinary bosons (00),**
- **exotic bosons (11),**
- **parafermions of (10) type,**
- **parafermions of (01) type.**

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **54** (2021) 115203 (35pp) https://doi.org/10.1088/1751-8121/abe2f2

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parastatistics in multiparticle quantum Hamiltonians

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Received 6 September 2020, revised 18 January 2021
Accepted for publication 3 February 2021
Published 25 February 2021

CrossMark

IOP Publishing

Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **54** (2021) 355202 (21pp)

https://doi.org/10.1088/1751-8121/ac17a5

Inequivalent quantizations from gradings and $\mathbb{Z}_2 \times \mathbb{Z}_2$ parabosons

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Received 29 April 2021, revised 7 July 2021
Accepted for publication 26 July 2021
Published 10 August 2021

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Further theoretical advances

1. arXiv:2308.05203 [pdf, other] quant-ph cond-mat.stat-mech hep-th math-ph

Free particles beyond fermions and bosons

Authors: Zhiyuan Wang, Kaden R. A. Hazzard

Abstract: It is commonly believed that there are only two types of particle exchange statistics in quantum mechanics, fermions and bosons, with the exception of anyons in two dimension. In principle, a second exception known as parastatistics, which extends outside of two dimensions, has been considered but was believed to be physically equivalent to fermions and bosons. In this paper we show that nontrivia... ▽ More

Submitted 9 August, 2023; **originally announced** August 2023.

Comments: 16 pages, 7 figures

1. arXiv:2309.00965 [pdf, other] hep-th cond-mat.stat-mech math-ph quant-ph

Inequivalent Z_2^n-graded brackets, n-bit parastatistics and statistical transmutations of supersymmetric quantum mechanics

Authors: M. M. Balbino, I. P. de Freitas, R. G. Rana, F. Toppan

Abstract: Given an associative ring of Z_2^n-graded operators, the number of inequivalent brackets of Lie-type which are compatible with the grading and satisfy graded Jacobi identities is $b_n = n + \lfloor n/2 \rfloor + 1$. This follows from the Rittenberg-Wyler and Scheunert analysis of "color" Lie (super)algebras which is revisited here in terms of Boolean logic gates. The inequivalent brackets, recovered f... ▽ More

Submitted 2 September, 2023; **originally announced** September 2023.

Comments: 57 pages, 16 figures

Report number: CBPF-NF-002/23

Experimental advances

1. arXiv:2108.05471 [pdf, other] quant-ph

Experimental realization of para-particle oscillators

Authors: C. Huerta Alderete, Alaina M. Green, Nhung H. Nguyen, Yingyue Zhu, B. M. Rodríguez-Lara, Norbert M. Linke

Abstract: Para-particles are fascinating because they are neither bosons nor fermions. While unlikely to be found in nature, they might represent accurate descriptions of physical phenomena like topological phases of matter. We report the quantum simulation of para-particle oscillators by tailoring the native couplings of two orthogonal motional modes of a trapped ion. Our system reproduces the dynamics of... ▽ More

Submitted 11 August, 2021; **originally announced** August 2021.

Comments: 11 pages, 7 figures, 1 table

1. arXiv:2207.02430 [pdf, other] quant-ph

Para-particle oscillator simulations on a trapped ion quantum computer

Authors: C. Huerta Alderete, Alaina M. Green, Nhung H. Nguyen, Yingyue Zhu, Norbert M. Linke, B. M. Rodríguez-Lara

Abstract: Deformed oscillators allow for a generalization of the standard fermions and bosons, namely, for the description of para-particles. Such particles, while indiscernible in nature, can represent good candidates for descriptions of physical phenomena like topological phases of matter. Here, we report the digital quantum simulation of para-particle oscillators by mapping para-particle states to the st… ▽ More

Submitted 6 July, 2022; **originally announced** July 2022.

Comments: 7 pages, 5 figures, 1 table

Report number: LA-UR-22-26358

Thanks for the attention!