A toy model for supergravity

Fridrich Valach

University of Hertfordshire

w/ Julian Kupka and Charles Strickland-Constable

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- **Idea:** take $V = E$ and $E = \mathfrak{g}$ $\quad\rightsquigarrow\quad$ a finite-dimensional theory

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$$\text{generalised diffeomorphisms} \qquad \delta_\zeta \sigma = 0 \qquad \delta_\zeta \rho = \tfrac{1}{4}\zeta^a f_{abc}\gamma^{bc}\rho \qquad \zeta \in \mathfrak{g}$$

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$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\rho = \delta_\zeta\rho + \delta_\epsilon\rho - \tfrac{1}{2}\zeta_a\gamma^a \not{D}\rho, \qquad \zeta^a := 2\sigma^{-2}\bar\epsilon_2\gamma^a\epsilon_1, \quad \epsilon := -\tfrac{1}{2}\zeta_a\gamma^a\rho$$

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BV space:

$$T^*[-1](\mathbb{R}^{>0} \times \Pi S_+ \times \mathfrak{g}[1] \times \Pi S_-[1])$$

	σ	ρ	ξ	e	σ^*	ρ^*	ξ^*	e^*
$\mathbb{Z}_2$ degree	[0]	[1]	[0]	[1]	[0]	[1]	[0]	[1]
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$$S_{BV} = \mathcal{R}\sigma^2 - \bar{\rho}\not{D}\rho - \sigma^*\sigma^{-1}(\bar{\rho}e) + \frac{1}{4}\xi^a f_{abc}(\bar{\rho}^*\gamma^{bc}\rho) + \bar{\rho}^*\not{D}e$$
$$- \frac{1}{4}\xi^a f_{abc}(\bar{e}^*\gamma^{bc}e) + \frac{1}{2}\sigma^{-2}(\bar{e}\gamma^a e)(\bar{e}^*\gamma_a\rho) + \frac{1}{2}f^a{}_{bc}\xi^*_a\xi^b\xi^c$$
$$- \xi^*_a\sigma^{-2}(\bar{e}\gamma^a e) + \frac{1}{8}\sigma^{-2}(\bar{e}\gamma_a e)(\bar{\rho}^*\gamma^a\rho^*).$$

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$$\{S_{BV}, S_{BV}\} = 0 \qquad \Longrightarrow \qquad Q := \{S_{BV}, \cdot\} \text{ is a differential}$$

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Fierz identities:

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