Jackiw-Teitelboim Quantum Gravity On Finite Geometry

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Two dimensional quantum gravity theories come in three different versions:

1) Liouville gravity: consider all metrics on a two dimensional manifold (h,b). Can be formulated rigorously from combinatorics (counting of “maps”, i.e. polygonizations of surfaces) and from the “continuous” point of view (Liouville CFT). This field is so mature that it is now part of probability theory (Sheffield, Duplantier, Vargas, Rhodes, …).

2) Topological gravity: fix both bulk curvature and boundary extrinsic curvature (geodesic boundaries). Amounts to integrating over the moduli space of (h,b) surfaces with the Weil-Petersson measure (Witten, Kontsevitch, Mirzakhani; topological recursion, matrix model).

3) Jackiw-Teitelboim gravity: fix the bulk curvature but do not impose any condition on the boundary. Relevant for near-extremal black holes, holography, related to the duals of SYK and tensor/matrix models, etc. Very interesting on its own as well (see below). Three versions: negative curvature (much studied, holography), zero curvature (simplest set-up for understanding the UV properties) and positive curvature (most interesting? Cosmology).

Action

$$S_{\mathrm{dil}} = -\frac{1}{16\pi}\left[\int_{\mathscr{D}} \mathrm{d}^2x\sqrt{g}\,\Phi\left(R - \frac{2\eta}{L^2}\right) + 2\Phi_{\mathrm{b}}\oint_{\partial\mathscr{D}} \mathrm{d}s\,k\right]$$

$$\int_{\mathscr{D}} \mathrm{d}^2x\sqrt{g}\,R + 2\oint_{\partial\mathscr{D}} \mathrm{d}s\,k = 4\pi = -\frac{2\eta}{L^2}A[g] + 2\oint_{\partial\mathscr{D}} \mathrm{d}s\,k$$

$$S_{\Lambda} = \frac{\Lambda}{16\pi}A[g] \qquad \Lambda = -\frac{2\eta\Phi_{\mathrm{b}}}{L^2}$$

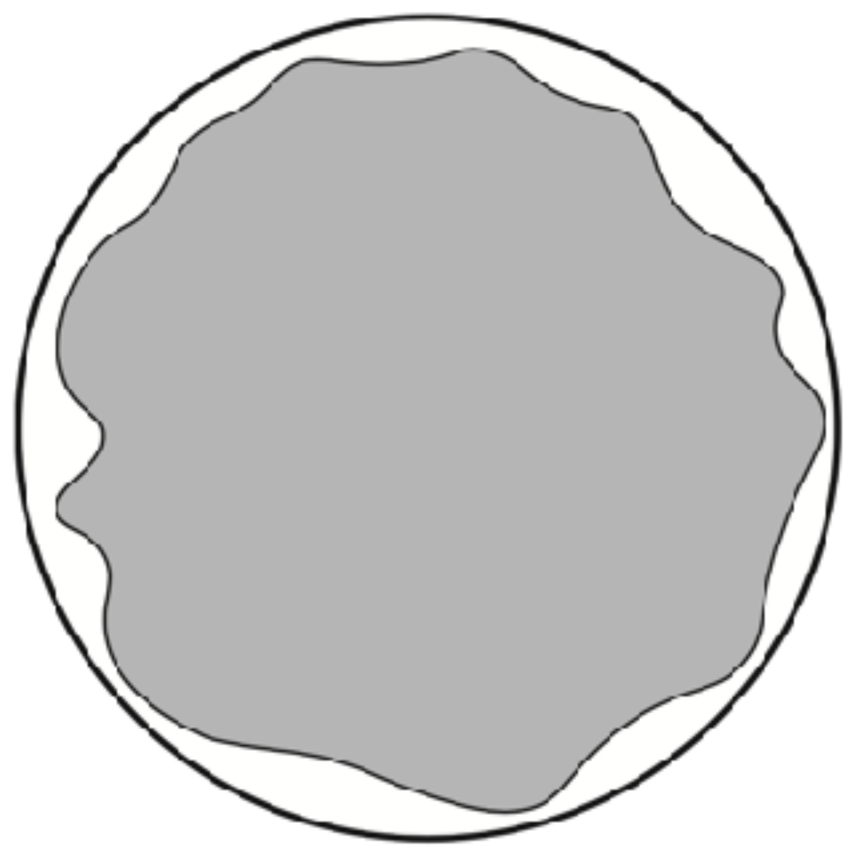
$$S_{\mathrm{dil}} = -\frac{1}{16\pi}\int_{\mathscr{D}} \mathrm{d}^2x\sqrt{g}\,\tilde{\Phi}\left(R - \frac{2\eta}{L^2}\right) + \frac{\Lambda}{16\pi}A[g]$$

$$\tilde{\Phi}_{\mathrm{b}} = 0$$

$$\eta = \pm 1 \text{ or } 0$$

Goals of this research

JT gravity has been studied almost exclusively in negative curvature in a particular limit where the relevant geometries are effectively infinite (Schwarzian theory).



Our main goal is to go beyond this approximation and study JT on finite geometries.

Note that this is essential for the positive curvature model, and probably for the zero curvature model as well.

Note also that on very short lengths scales (much shorter than the curvature length scale), the three models are equivalent, i.e. they have the same UV-completion.

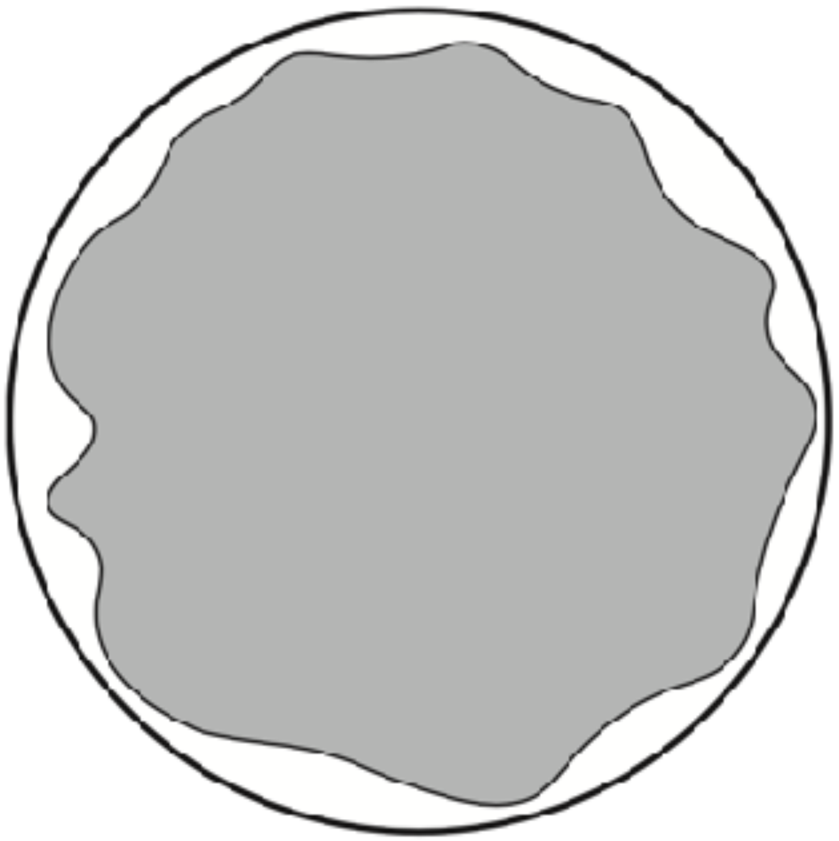
This observation is a strong indication that models like SYK, that have been considered in relation to JT gravity because they have a Schwarzian description in the IR, are not good candidates for a UV-complete formulation of JT. One needs to find a different description, which has a universal UV structure but yields very different IR physics depending on the choice of curvature.

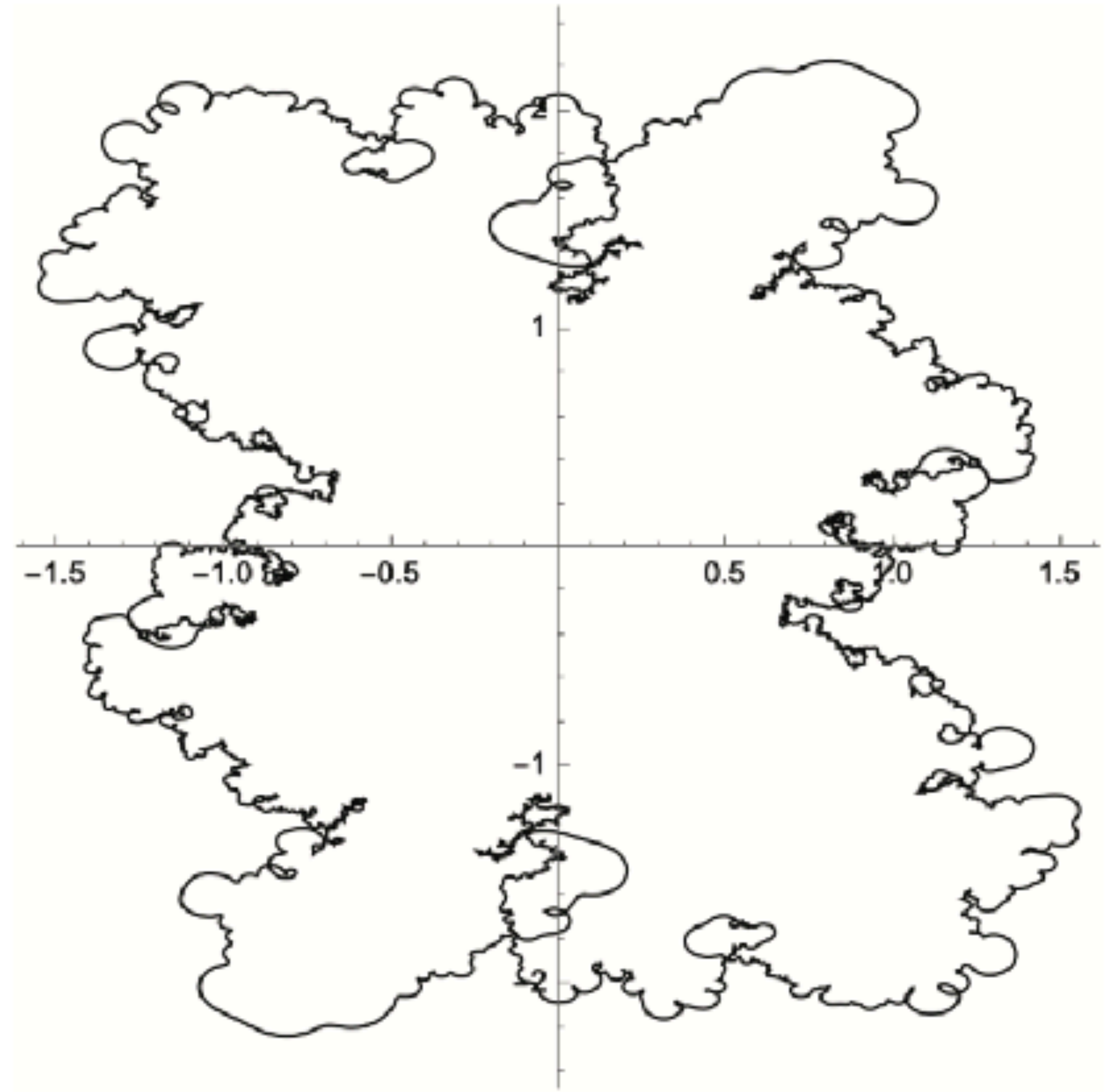
Two points of view

1) The models on finite geometries can be formulated in a combinatorial way, by discretizing the relevant spaces of metrics. This is similar to the matrix model approach to Liouville gravity. It reduces the problem to pure combinatorics. One finds a new model of random polygons (called self-overlapping) with remarkable properties.

2) One can also try to develop a direct continuum approach ``à la Liouville'' in finite geometries. In spirit, this is analogous to the FZZT Liouville branes, but now in the context of JT gravity. The result is surprisingly elegant. Ideas and techniques can be imported from Liouville into JT!

Upshot





Reparameterization ansatz/
smooth and gently wiggling boundary/
Schwarzian description

Typical configuration/
fractal boundary/
description in terms of a distribution-valued boundary conformal factor

References

F. F., Jackiw-Teitelboim Gravity, Random Disks of Constant Curvature, Self-Overlapping Curves and Liouville CFT_1, arXiv:2402.08052 (synthetic letter)

F. F., Random Disks of Constant Curvature: the Lattice Story, arXiv:2406.06875

F. F., Random Disks of Constant Curvature: the Conformal Gauge Story (to appear)

The semi-classical limit, using the framework developed above, has been studied in:

S. Chaudhuri and F. F., Finite cut-off JT and Liouville quantum gravities on the disk at one loop, arXiv:2404.03748

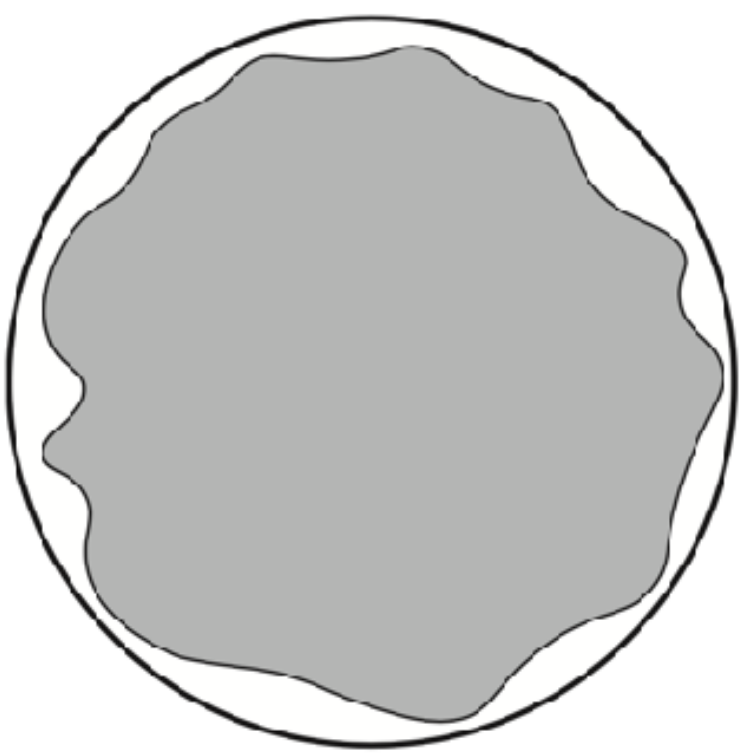
S. Chaudhuri and F. F., Dirichlet Scalar Determinants On Two-Dimensional Constant Curvature Disks, arXiv:2405.14958

More work in progress with Vincent Vargas, Baptiste Cerclé, Soumyadeep Chaudhuri…

The metric space for finite geometries

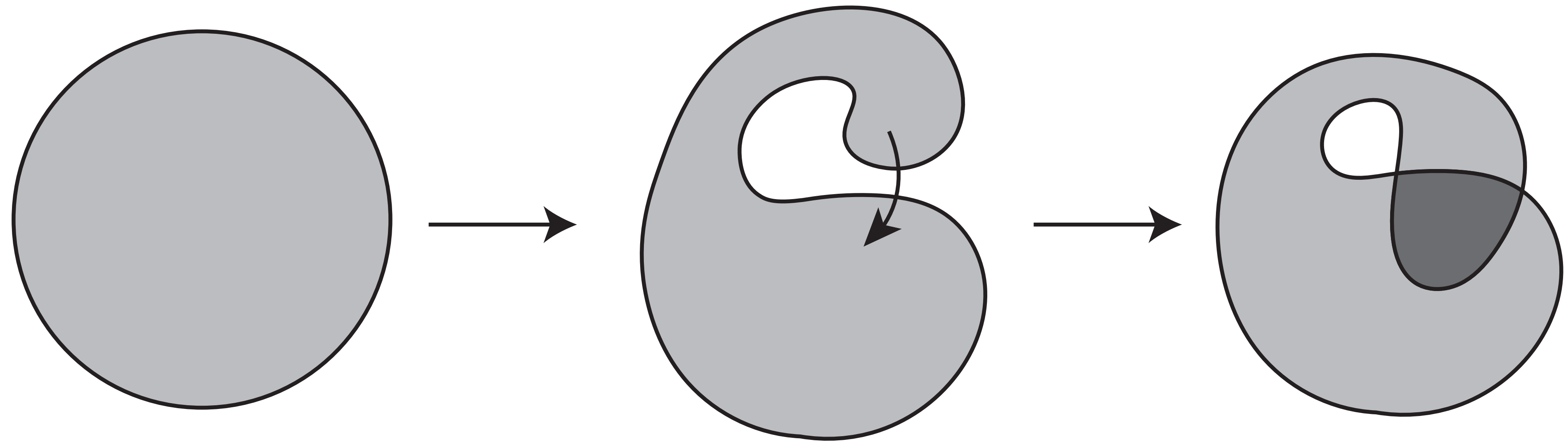
It can be described using immersions of the disk into a canonical space (hyperbolic space, Euclidean space or the two-sphere). This yields a ``distorted disk'' representing isometrically a metric on the original ``source'' disk.

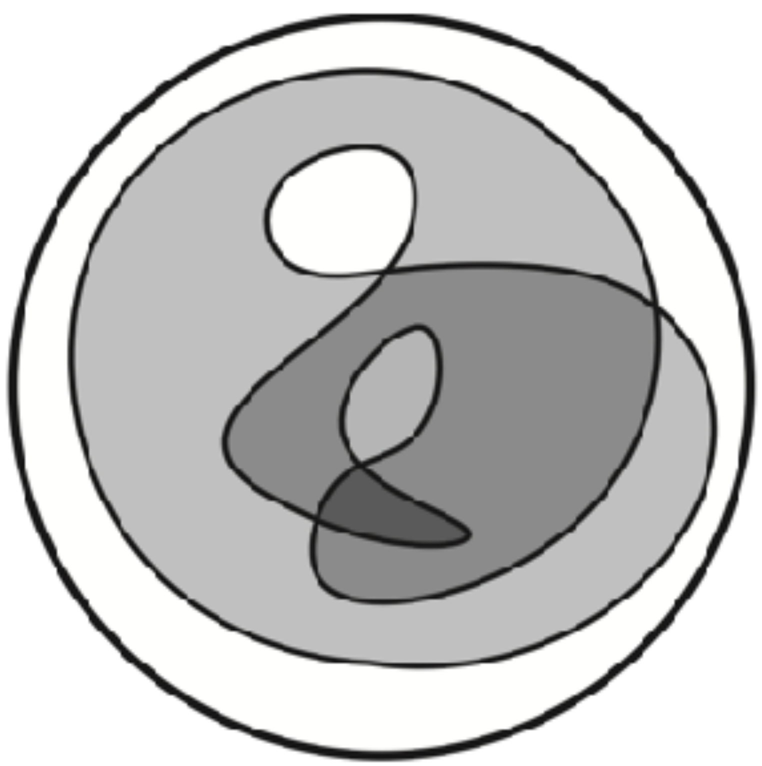
$$F : \mathscr{D} \to \mathbb{T} \qquad \mathbb{T} = \mathrm{E}^2\,,\ \mathrm{S}^2\,,\ \mathrm{H}^2 \qquad g = F^*\delta$$

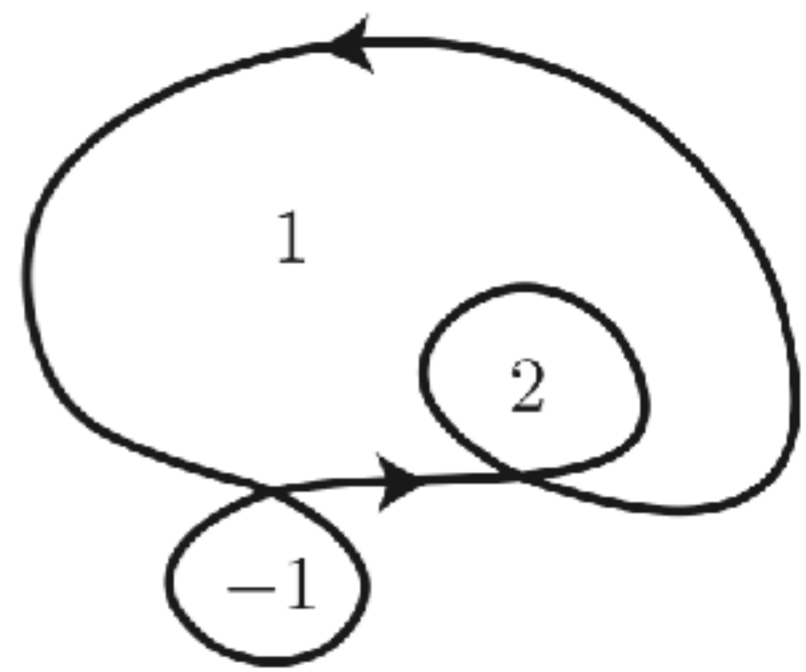


reparameterization ansatz

Self-avoiding loop

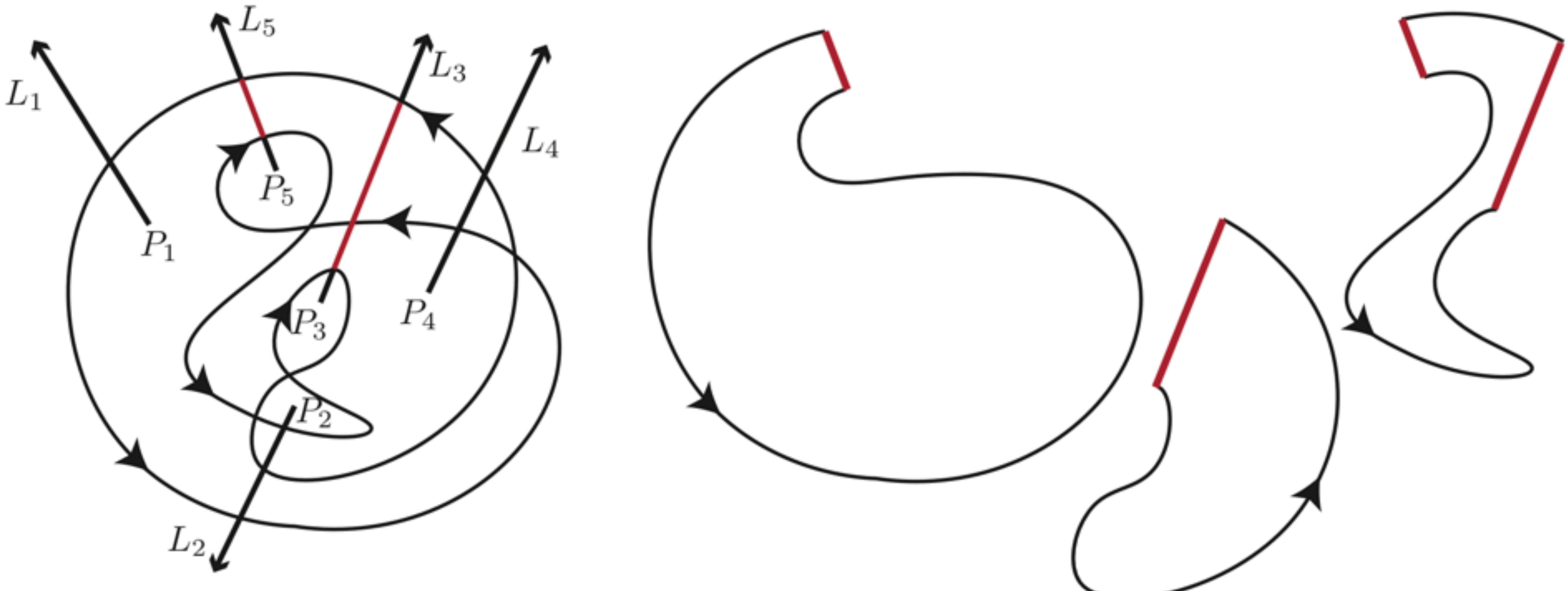


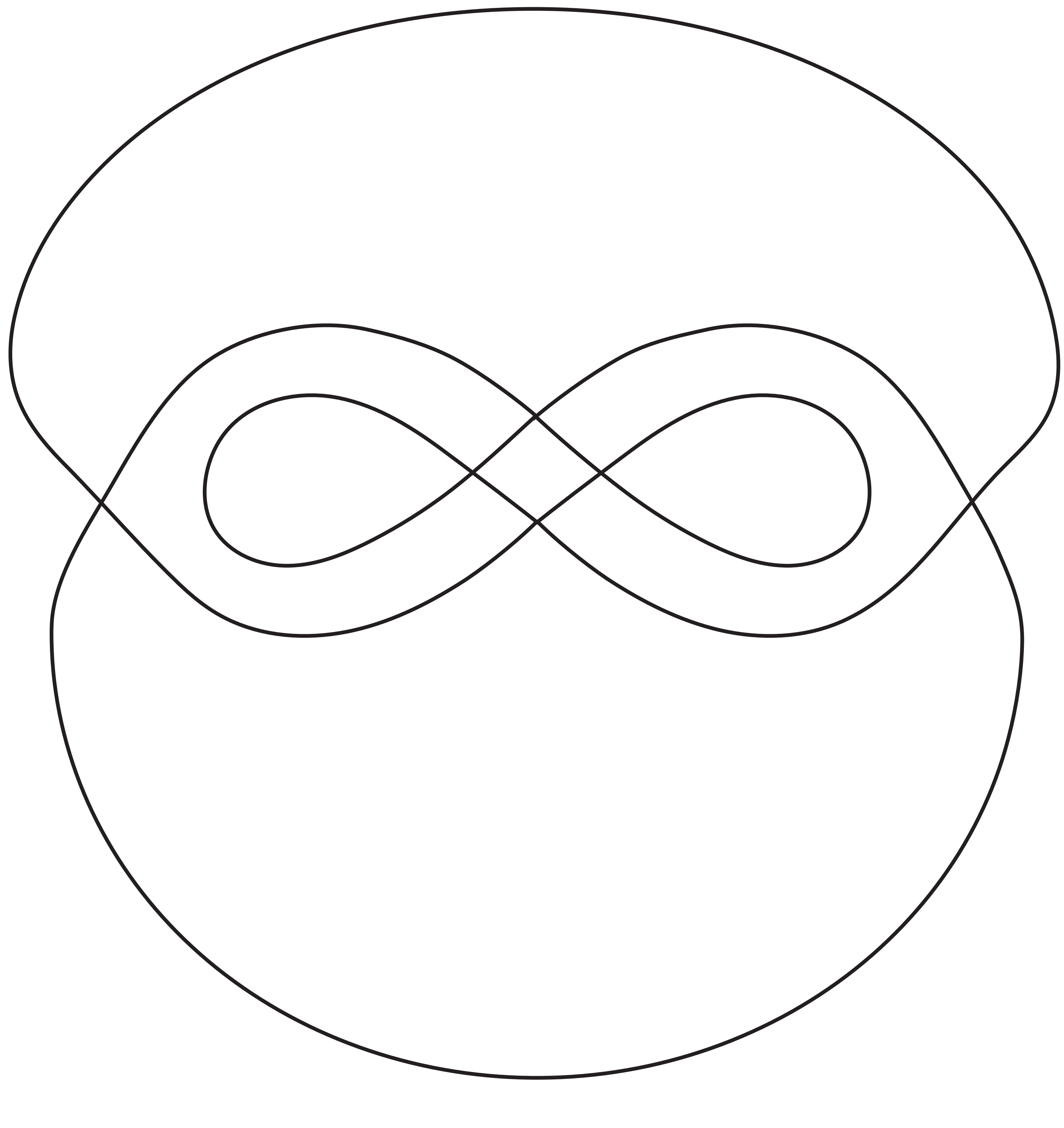


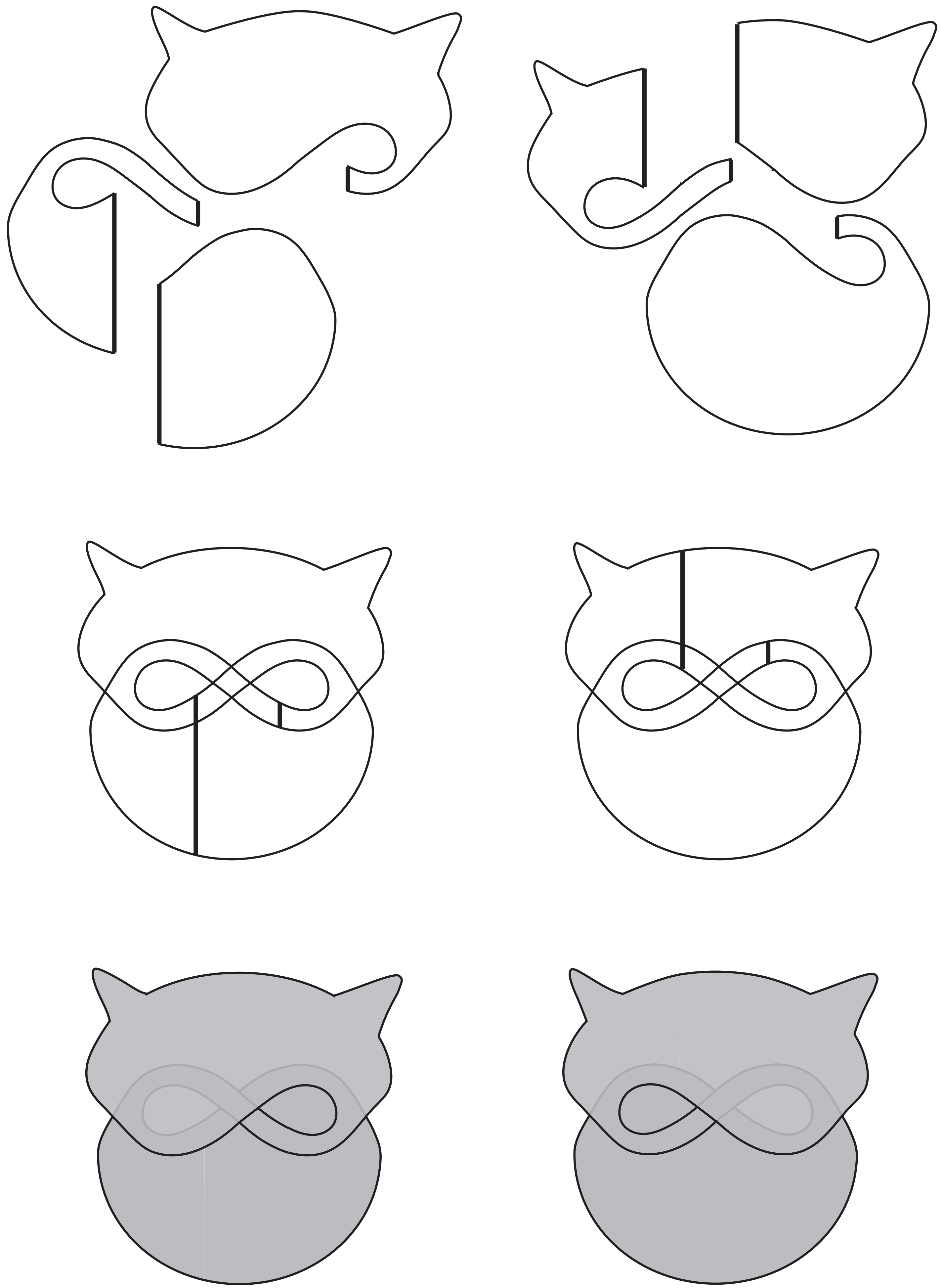

1
2
−1

Blank cuts

C. Titus ``The combinatorial topology of analytic functions of the boundary of a disk,'' Acta Mathematica 106(1) (1961) 45
S.J. Blank, ``Extending Immersions of the Circle,'' Ph.D.\ dissertation, Brandeis University (1967).
V. Poénaru, ``Extension des immersions en codimension 1,'' Astérisque 10 (1966--1968) 473 (Bourbaki seminar number 342).
P.W. Shor and C.J. Van Wyk, ``Detecting and decomposing self-overlapping curves,'' Computational Geometry 2 (1992) 31.

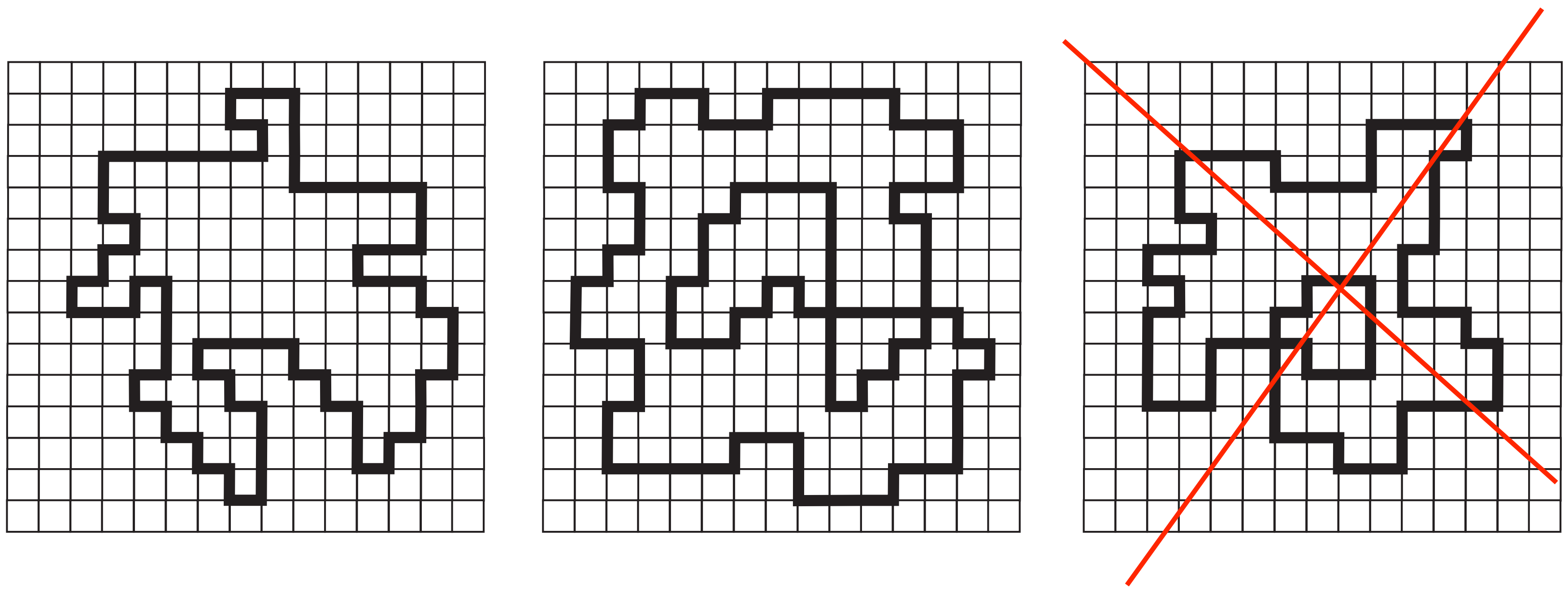


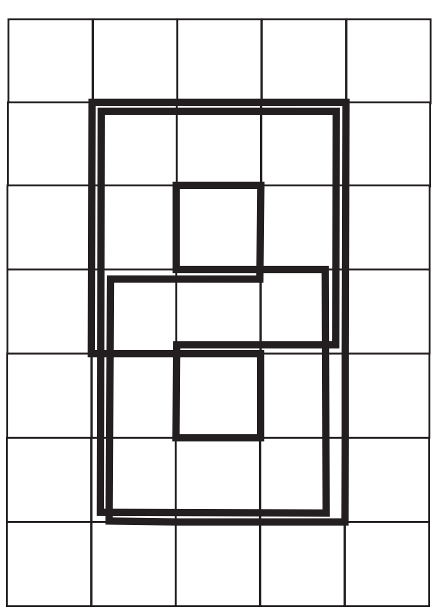


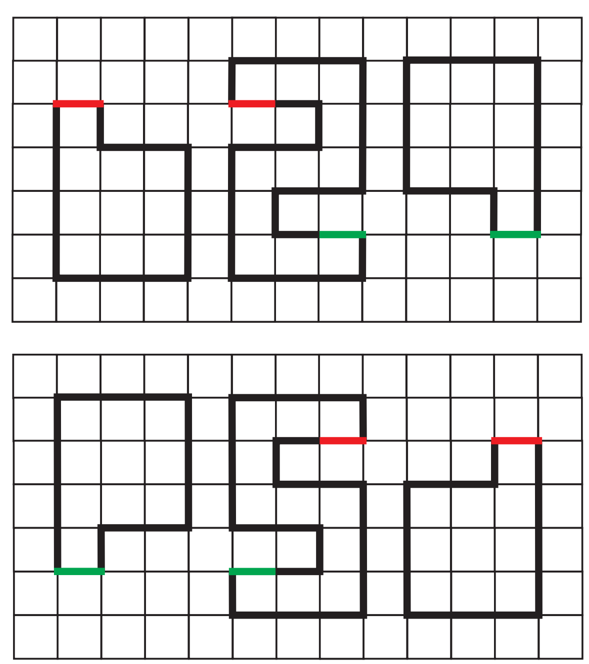


A given boundary curve may bound several distincts disks (i.e. be associated with several distinct metrics on the source disk!) -> multiplicity index

Upshot: JT gravity is a random curve model of a new type. We have to count self-overlapping curves, taking into account the multiplicity.







$$2n = 48\,, \quad p = 31$$

$$W(t,g) = \sum_{\gamma} \mu_\gamma t^{F(\gamma)} g^{v(\gamma)} = \sum_{n,p\geq 1} W_{2n,p} t^p g^{2n}$$

We land in the realm of random polygon models - a huge subject - and we can learn a lot from the general facts in random polygons.

There is a continuum limit for which the limiting curves will be fractals almost surely.

$$n \to \infty\,, \quad \ell_0 \to 0\,, \quad 2n\ell_0^{1/\nu} = \beta_{\mathrm{q}} \quad \text{fixed.}$$

$$\langle A^r \rangle \sim n^{2\nu r}$$

$$\frac{1}{\nu} = d_{\mathrm{H}} \text{ is the Hausdorff dimension.}$$

β_{q} is the quantum boundary length.

$$\langle A^r \rangle = \int_0^\infty A^r \rho_{\beta_{\mathrm{q}}}(A) \mathrm{d}A$$

One last remark on the discretized formulation: there is a matrix model formulation of the counting problem, based on a version of the so-called ``dually weighted graphs'' models first studied by Di Francesco and Itzykson.

Conformal gauge on the disk

$$g = e^{2\Sigma}\delta \qquad\qquad \sigma = \Sigma_{|\partial\mathscr{D}}$$

Extremely important message: the conformal factor (Liouville field) has free boundary conditions, that is to say, its bulk values and boundary values a priori fluctuate independently of each other.

$$\Sigma = \Sigma_{\mathrm{B}} + \Sigma_{\sigma}$$

$$\Sigma_{\mathrm{B}|\partial\mathscr{D}} = 0$$

$$\Sigma_{\sigma|\partial\mathscr{D}} = \sigma\,, \quad \Sigma_{\sigma}(x) \text{ depends only on the boundary } \sigma \text{ for all } x \in \mathscr{D}$$

This being understood, you can treat both ordinary Liouville and JT gravity along similar lines.

The fundamental difference between JT and Liouville is that, because the bulk curvature is fixed, the Liouville field must satisfy the Liouville equation in the bulk,

$$\Delta_\delta \Sigma = \frac{\eta}{L^2} e^{2\Sigma}$$

A fundamental mathematical theorem then ensures that, for a given boundary Liouville field, the bulk field is uniquely determined (in flat space, this is the usual statement about the uniqueness of harmonic functions on the disk, with prescribed value on the boundary).

If, in the decomposition $\Sigma = \Sigma_\mathrm{B} + \Sigma_\sigma$ we use Σ_σ satisfying the Liouville equation, then necessarily $\Sigma_\mathrm{B} = 0$.

That's the fundamental difference with Liouville. We shall have to integrate only over the boundary Liouville field in the path integral, whereas for Liouville one needs to integrate over both the bulk and the boundary fields.

We have tested these ideas (with S. Chaudhuri) by studying the semi-classical limits of both Liouville and JT in this formalism.

Exact result: UV structure

$$\sigma = \sum_{n\in\mathbb{Z}} \sigma_n e^{in\theta}$$

$$S = \frac{1}{4\pi}\int_0^{2\pi}\Big(\sigma\,\mathrm{Hilb}[\sigma'] + 2\sigma\Big)\mathrm{d}\theta = \sum_{n\geq 1} n|\sigma_n|^2 + \sigma_0$$

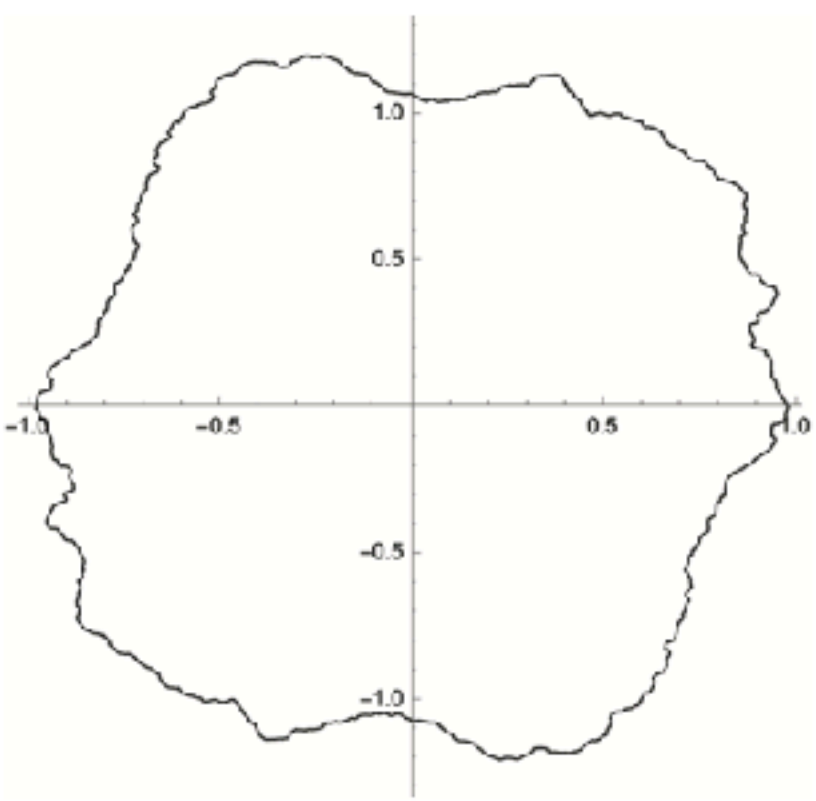
This is quite a remarkable action. It Is PSL(2,R)-invariant. It predicts that

$$|\sigma_n| \sim \frac{1}{\sqrt{n}}$$

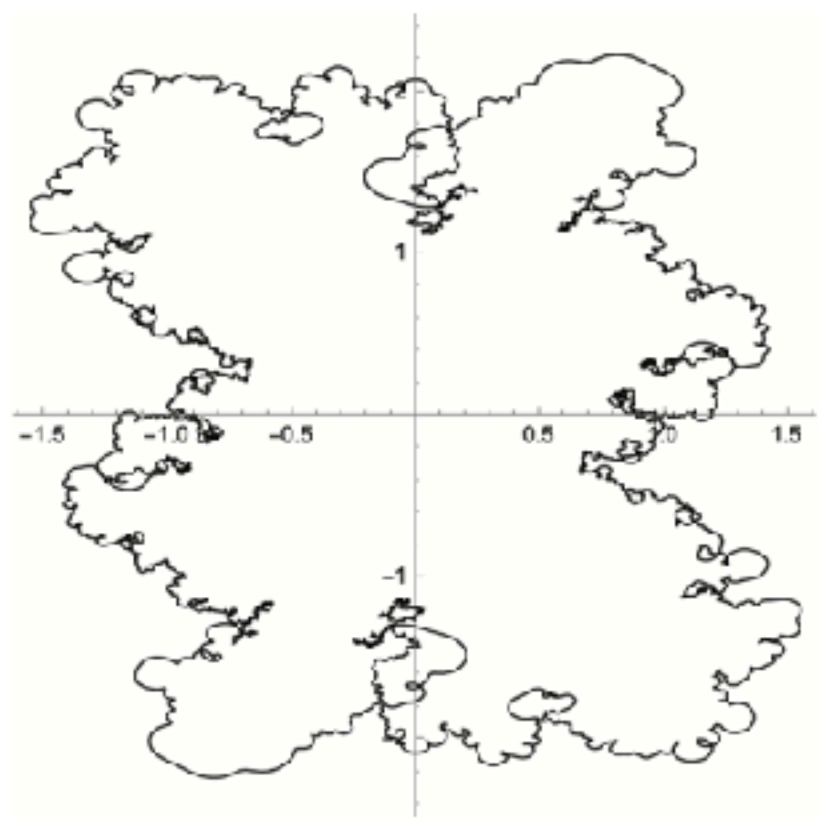
almost surely, implying that the boundary field is distribution-valued (very unlike a normal one-dimensional quantum variable q, which would have modes going like 1/n and would be continuous).

$$\nu = \frac{1}{2}\left[1 + \sqrt{\frac{c}{c-24}}\right]$$

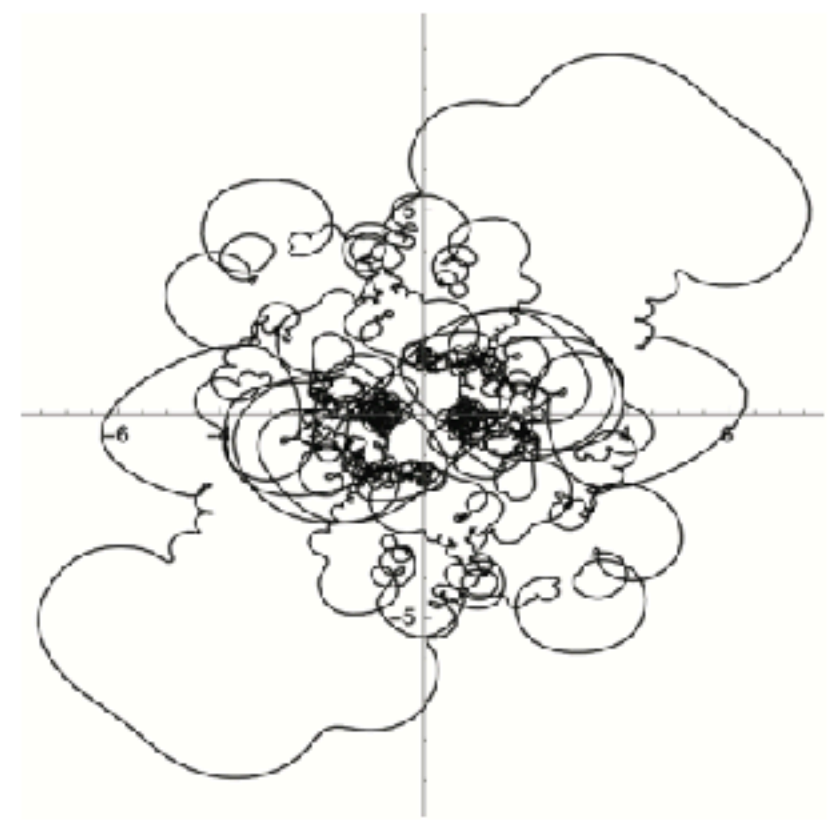
There is a c=0 barrier (analogous to the c=1 barrier in Liouville)



$c = -125$



$c = 0$



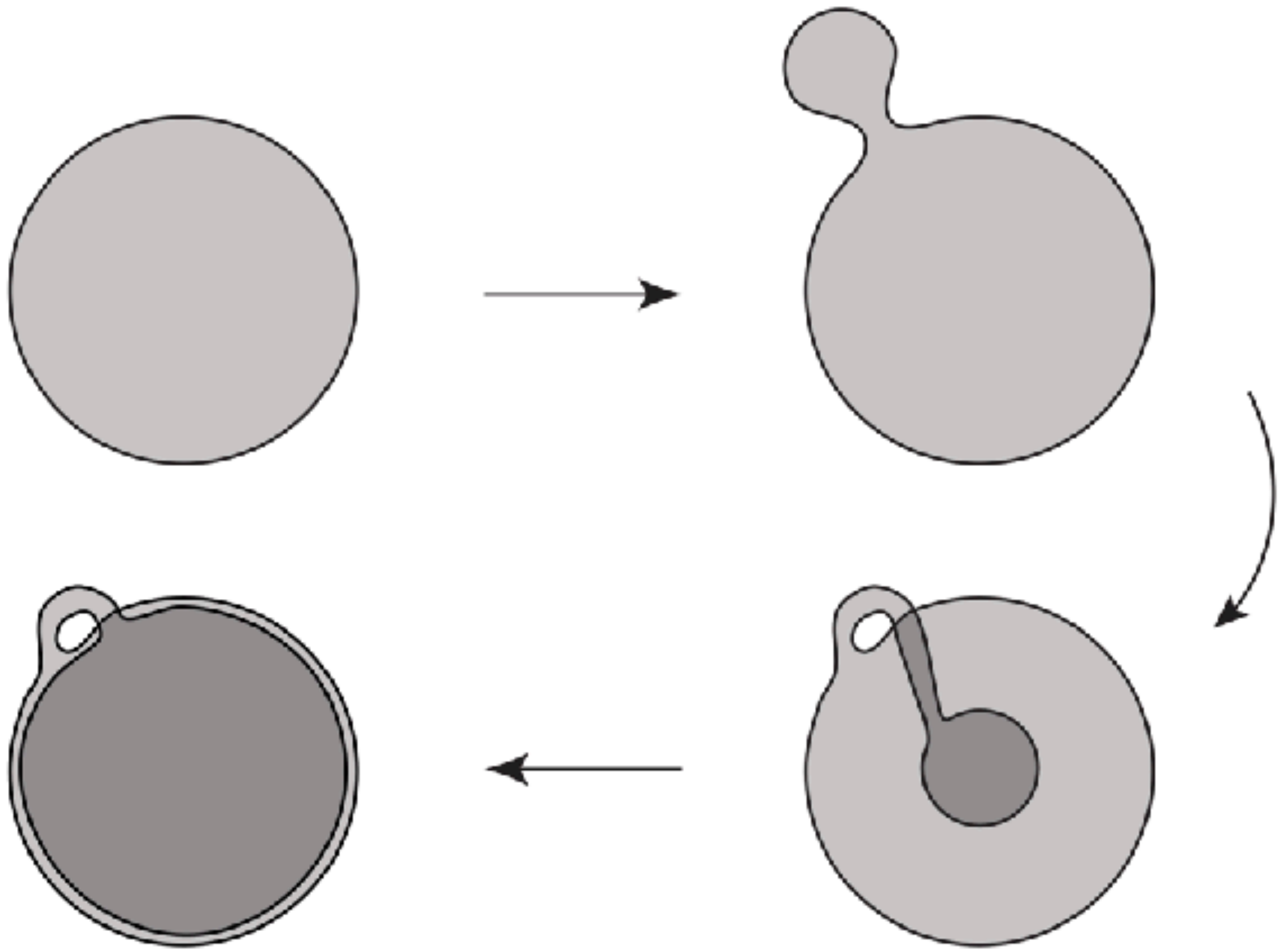
$c = 12$

Long distance properties, quantum version of the isoperimetric inequalities

$$A \leq 2\pi L^2 \left[-1 + \sqrt{1 + \frac{\ell^2}{4\pi^2 L^2}} \right] \sim \ell L \quad \text{when } R = -2/L^2.$$

$$A \leq \frac{\ell^2}{4\pi} \quad \text{when } R = 0$$

No such inequality in the case of positive curvature (see below).



One step in the construction of a metric of constant positive curvature on the disk having arbitrarily large area and arbitrarily small boundary length.

$$W(\Lambda) = W(0) \int_0^\infty \rho(A) e^{-\frac{\Lambda}{16\pi} A} \, \mathrm{d}A$$

1) in the positive curvature theory, the density probability for the area decreases like a power law. The theory exists only for positive cosmological constants.

2) in the zero curvature theory, the density probability for the area decreases like an exponential. The theory exists only for cosmological constants greater than a critical, strictly negative value.

3) in the negative curvature theory, the density probability for the area decreases like exp(-cst A^2). The theory exists for all cosmological constants, in particular for very large and negative cosmological constant (which is the Schwarzian limit).

Thank you for your attention !