Target space actions of spinning particles

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CaLIGOLA

Motivation: understand aspects of string field theory through the toy model example of a spinning particle.

With Hulík, Grassi and Sachs arXiv:2402.09868

The particle sigma model

Massless relativistic particle:

$$S[x,e] = \int_I \mathrm{d}t \; \det e \, e^{-2} \, \partial_t x^\mu \partial_t x_\mu \, .$$

Explicitly invariant under line diffeomorphisms.

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BRST: change parity, spin of v and introduce extra fields:

$$S[x,e,p,b,c,\bar{\pi}] = \int_I \mathrm{d}t \, p\dot{x} - e\frac{p^2}{2} + b\dot{c} - \bar{\pi}(e - e_{\text{fixed}})$$

$$\delta^{\text{BRST}} e = \mathrm{d}(c), \quad \delta^{\text{BRST}} b = \bar{\pi}, \quad \delta^{\text{BRST}} \dot{x} = c\,\dot{p}$$

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The action localises:

$$S[x,e,p,b,c,\bar{\pi}] = S[x,e] + \delta^{\text{BRST}}(b(e - e_{\text{fixed}}))$$

After fixing the einbein to the reference value, there remains a BRST symmetry generated by Poisson brackets with

$$Q = c\frac{p^2}{2}$$

which is nilpotent.

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$$[p, x] = 1, \quad [b, c]_+ = 1$$
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$$\mathbb{P}(s) = \sum (-1)^t s^t \dim V_t\,.$$

In our example we have

$$V = V_0 \oplus V_1$$

and it turns out that

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Dynamics: since $Q = c\Box$

$$Q\,|\Phi\rangle = 0 \Longleftrightarrow \Box\phi = 0, \quad \forall \phi^*$$
$$\mathrm{Im}Q|_{V_1} = \{\varphi(x)\,|0\rangle \;|\phi^* = \Box\varphi\}$$

which says that ϕ^* is pure gauge if it is not harmonic.

$$0 \xrightarrow{Q} \phi(x) \xrightarrow{Q} \phi^*(x) \xrightarrow{Q} 0$$

Off-shell formulation

Remaining ingredient: non-degenerate pairing

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Interactions by homotopy algebra:

$$M_k : \otimes^k V \mapsto V, \; k \geq 2, \quad Q =: M_1, \qquad \text{"} \sum_{k=n+1}^{\infty} (M_k)^2 = 0 \text{"}$$

For polynomial interactions: multiproducts are given e.g. by

$$M_2(\Phi_1, \Phi_2) = c\Phi_1\Phi_2$$

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that is associative. Maurer-Cartan form of the interacting action:

$$S_{BV}[\Psi] = (\Psi|Q\Psi) + \frac{\lambda_k}{(k+1)!}(\Psi|M_k(\Psi_1, \ldots, \Psi_k)) = \int \mathrm{d}x \, \phi\Box\phi + \frac{\lambda}{3!}\phi^3 + \ldots$$

Relativistic spinning particle

(1|1)-line:

$$x \mapsto X \equiv x(t) + \mathrm{i}\eta\psi(t) \text{ and } e \mapsto E \equiv e(t) + 2\mathrm{i}\eta\chi(t)$$

Superdiffeos:

$$W := v + \omega, \quad v = v(t)\partial_t, \quad \omega = \xi(t)\partial_\eta + \eta v(t)\partial_t.$$

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Invariant action (2nd order formulation) [Brink, Di Vecchia, Howe 1977]:

$$-\mathrm{i}\int \mathrm{d}t\,\mathrm{d}\eta\,\frac{E}{2}DX\partial_t X \qquad D := \partial_\eta + \eta\partial_t$$

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After Berezin integral:

$$\int_I \mathrm{d}t\, \eta_{\mu\nu}\, \frac{1}{e}\left(\dot{x}^\mu \dot{x}^\nu + \mathrm{i}\psi^\mu \dot{\psi}^\nu\right) - \frac{\mathrm{i}}{e^2}\chi\psi^\mu \dot{x}^\nu$$

For the path integral then :

$$S[x, \psi, e, \chi, p] + b\dot{c} + \beta\dot{\gamma} + \bar{\pi}(e - e_{\text{fixed}}) + \bar{\mu}(\chi - \chi_{\text{fixed}})$$

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Now the BRST operator is:

$$Q = c\frac{p^2}{2} + \gamma\psi \cdot p + \gamma^2 b$$

and acts on the fields as:

$$\{Q, x\} = cp + \gamma\psi, \quad \{Q, \psi\} = \gamma p, \quad \{Q, b\} = \frac{p^2}{2}, \quad \{Q, \beta\} = \psi \cdot p + \gamma b, \quad \{Q, c\} = \gamma^2$$

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The graded vector space (with function coefficients) has an extra $U(1)$ grading (generator is $\psi \cdot \bar{\psi} - \gamma\bar{\beta} + \beta\bar{\gamma}$):

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First order in r gives:

$$r(D-\frac{1}{s}-s)(1-s)$$

which is possible to arrange as:

$$0\xrightarrow{Q}\underbrace{V_{1,-1}}_{C(x)}\xrightarrow{Q}\underbrace{V_{1,0}}_{A_\mu(x)\psi^\mu,\varphi(x)}\xrightarrow{Q}\underbrace{V_{1,1}}_{A^*_\mu(x)\psi^\mu,\varphi^*(x)}\xrightarrow{Q}\underbrace{V_{1,2}}_{C^*(x)}\xrightarrow{Q}0$$

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of respective dimensions: 1, D+1, D+1, 1. Interpretation: Maxwell theory in BV form:

$$|\Psi\rangle:=\left(C(x)\beta+A_\mu(x)\psi^\mu+c\beta\varphi(x)+cA^*_\mu(x)\psi^\mu+\gamma\varphi^*(x)+c\gamma C^*(x)\right)|0\rangle$$

$$Q\,|\Psi\rangle=\mathrm{d}C+\gamma(\mathrm{d}^\dagger A+\varphi)-c(\Box A+\mathrm{d}\varphi)+c\beta\Box C+c\gamma(\Box\varphi^*+\mathrm{d}^\dagger A^*)$$

Off-shell formulation

For the pairing:

Hodge star

A Hodge star isomorphism can be found (non-unique)

$$\star : \ \Omega^p \xrightarrow{\sim} \Omega^{p_{\max}-p}.$$

The target space BV symplectic pairing is produced by means of $\star$.

Interactions

Would it be possible to build the interacting BV theory?

- higher products are not easy to cook up;
- another solution based on imposing some *operator-state correspondence*:

Operator-state correspondence

Looking for a surjection $Q(\cdot) : \mathcal{F} \mapsto (V \mapsto V)$ s.t.

$$Q(\omega)\beta\,|0\rangle = |\Psi\rangle\ .$$

$$\begin{aligned} Q(\omega) = &- c\left(p^2 + p\cdot B + B\cdot p - G_{\mu\nu}\psi^\mu\bar{\psi}^\nu - \phi - [p, B]\right) \\ &+ \gamma\Pi\cdot\bar{\psi} + \bar{\gamma}\Pi\cdot\psi + C - c\bar{\gamma}\psi\cdot A^* + c\gamma\bar{\psi}\cdot A^* \\ &+ \gamma\bar{\gamma}\phi^* + c\gamma\bar{\gamma}C^* + \gamma\bar{\gamma}b\,. \end{aligned}$$

$[Q(\omega), Q(\omega)] \equiv 0 \iff$ **BV eoms of YM with** $B_\mu = A_\mu$ **and** $G_{\mu\nu} = -2[\Pi_\mu, \Pi_\nu]$**.**

Then the interacting action functional of BV YM is:

$$\int \mathrm{d}x\mathrm{d}\psi\mathrm{d}\gamma\mathrm{d}\beta\mathrm{d}c \ \mathrm{Tr}\left(\star(\beta Q(\omega)\left|0\right\rangle)\left(\frac{1}{2}Q(0)+\frac{1}{3}Q(\omega)\right)Q(\omega)\beta\left|0\right\rangle\right)$$

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BV-BRST invariance: from

$$\begin{aligned} m_1(\omega) &:= [Q(0), Q(\omega)]\beta \\ m_2(\omega_1,\omega_2) &:= [Q(\omega_1), Q(\omega_2)]\beta \end{aligned}$$

$m_1 \circ m_2 = 0$ and $m_2 \circ m_2 = 0$, moreover m_2 is cyclic w.r.t. pairing.

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- index/partition function helps navigate what happens with different pictures;
- canonical quantization of the BRST theory yields on-shell free BV theory in target space;
- the off-shell, interacting theory can also be produced.

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Thanks for the attention!