Holography and first-order gravity formalism

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First order gravity

Standard general relativity uses metric $g_{\mu\nu}$ as a fundamental field , and connection is given by $\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho})$. However, it is usually stated that when one wish to include spinor fields, it is necessary to introduce local Lorentz frames. Therefore, we work with

$$e^{\mu}_{a} \to e^{a} = e^{a}_{\mu} \mathrm{d}x^{\mu}, \qquad \omega^{ab} = \omega^{ab}_{\mu} \mathrm{d}x^{\mu}.$$

Then we have

$$T^{a} = \mathrm{d}e^{a} + \omega^{a}{}_{b} e^{b}, \qquad R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c} \omega^{cb}.$$

If torsion vanishes, there is a relation between (spin) connection and vielbein field. However, introducing spinor fields is not the only motivation for working in the first-order formalism and introducing e^{a} and ω^{ab} as independent fields.

Motivation

We don't need to introduce spinors in order to

- Palatini formulation of GR (no need for LL indices).
- Poincare gauge theory
- Proper gauge theoretic formulation of gravity (useful to introduce NC gravity; torsion may be relevant here [Nikolic, Dimitrijevic Ciric, Ravocanovic '15], [Gocanin, DD '22])

$$\int \mathrm{Tr}(BF) \to \int \mathrm{d}^2x \, \sqrt{-g}\phi \, (R - \Lambda)$$

$$S_{CS^{(5)}} = -\frac{ik}{3} \int \mathrm{Tr}\Big(F \wedge F \wedge A - \frac{1}{2} F \wedge A \wedge A \wedge A + \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A \Big)$$

$$\to S^{(5)}_{\mathrm{CS}} = \frac{k}{8} \int \varepsilon_{abcde} \left(\frac{1}{l} R^{ab} R^{cd} e^e + \frac{2}{3l^3} R^{ab} e^c e^d e^e + \frac{1}{5l^5} e^a e^b e^c e^d e^e \right).$$

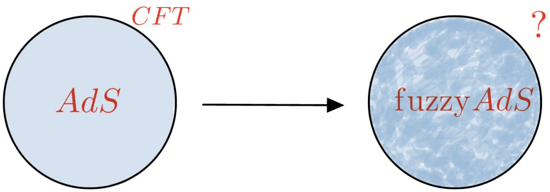
Example: $5D$ NC CS

The first order correction of $5D$ CS action, using SW map, is given by [Aschieri, Castellani, '14]

$$
\begin{aligned}
S^{(5)}_{CS,\theta} = \frac{k\theta^{IJ}}{12} \times \int \Bigg(& F^{ab}(F_I)_{bc}(\mathrm{D}F_J)^c{}_a + \frac{1}{l^2}F^{ab}(F_I)_{bc}(T_J)^c e_a \\
& + \frac{1}{l^2}F^{ab}(T_I)_b(\mathrm{D}T_J)_a + \frac{2}{l^2}F^{ab}(T_I)_b(F_J)_{ac}e^c \\
& + \frac{1}{l^2}T^a(T_I)^b(\mathrm{D}F_J)_{ba} + \frac{1}{l^2}T^a(\mathrm{D}T_I)^b(F_J)_{ba} \\
& + \frac{1}{l^2}T_a(F_I)^{ab}(F_J)_{bc}e^c + \frac{2}{l^4}T_a(T_I)_b(T_J)^{[b}e^{a]} \Bigg).
\end{aligned}
$$

We can easily see the importance of torsion in this theory.

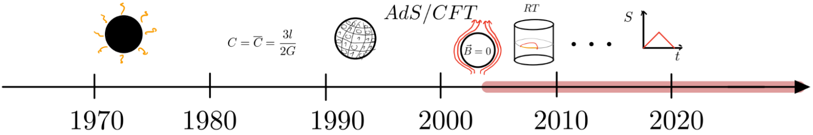
Motivation



So far, relation between noncommutativity an holography includes:

- NC field theory on a boundary (recent interest when studying deformations of string theory).
- NC gravity in the two dimensional bulk as a "quantum" bulk spacetimes (there are also some considerations done in a higher number of dimensions).
- Holography on quantum disc (DSSYK model?) [Almheiri, Popov '24].

Holography with torsion



Lovelock CS gravity:

$$S_{\rm CS}^{(5)} = \frac{k}{8} \int \varepsilon_{abcde} \left(\frac{1}{l} R^{ab} R^{cd} e^e + \frac{2}{3l^3} R^{ab} e^c e^d e^e + \frac{1}{5l^5} e^a e^b e^c e^d e^e \right).$$

FG gauge [Banados, Miskovic, Theisen '05], [Cvetkovic, Simic, Miskovic '18]:

$$e^1 = -\frac{\mathrm{d}\rho}{2\rho}, e^a = \frac{1}{\sqrt{\rho}} (\overline{e}^a + \rho \overline{k}^a), \omega^{a1} = \frac{1}{\sqrt{\rho}} (\overline{e}^a - \rho \overline{k}^a), \omega^{ab} = \overline{\omega}^{ab}.$$

More: 3D models, most based on CS description... Application to spin systems [Gallegos, Gursoy '20], [Hashimoto, Kimura '14]...

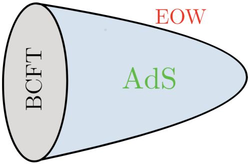
AdS/BCFT duality

We use the construction due to Takayanagi [Takayanagi '10]. Problem:

$$\partial \circ \partial \mathcal{M} = \emptyset$$

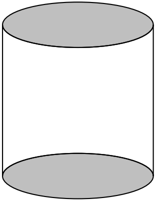
Solution: add EOW brane, such that appropriate (Neunmann) boundary conditions are satisfied.
Example:

$$\frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}x^3 \sqrt{-g}\,(R + 2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}x^2 \sqrt{|h|}K$$
$$- \frac{T}{8\pi G}\int_{\mathcal{Q}} \mathrm{d}x^2 \sqrt{|h|} \Rightarrow K_{ij} - (K - T)h_{ij} = 0$$



Motivation for studying AdS/BCFT is vast!

Holographic Entanglement Entropy



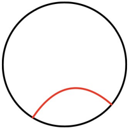
Topologically, AdS is similar to the cylinder.
We can use AdS/CFT to compute the entanglement entropy of a boundary subregion.

How to compute entanglement entropy in CFT: Replica trick!: conical singularities.

$$-\mathrm{Tr}(\rho \ln \rho) = -\lim_{n\to 1} \frac{1}{1-n} \mathrm{Tr}(\rho^n)$$

RT proposal [Ruy, Takayanagi '06]:

$$S_{EE} = \min_{\Xi \sim L} \frac{A(\Xi)}{4G\hbar}.$$



What can we do?

Along the lines of previously introduced ideas, we can [Gocanin, DD '23], [Gocanin, DD '24].

- Analyse Chamseddine's topological gravity in even dimensions (generalization of JT gravity)

  $$\int \mathrm{Tr}\,(\phi F F)\,.$$

  FG gauge:

  $$\hat{\phi}^1 = \frac{1}{\sqrt{\rho}}(\varphi - \rho\psi), \quad \hat{\phi}^a = \phi^a, \quad \hat{\varphi} = \frac{1}{\sqrt{\rho}}(\varphi + \rho\psi).$$

  Generalized conformal symmetry $e^a\langle\mathcal{T}_a\rangle + \varphi\langle\mathcal{O}\rangle = 0$, up to boundary terms.
- As for JT gravity, we can derive the profile of EoW brane, as well as equations for dilaton fields, using first order formalism.

The central question is the issue with the boundary terms [Erdmenger, Heß, Matthaiakakis, Meyer '23]. Focus therefore on a toy model of three dimensional gravity with negative cosmological constant.

$3D$ gravity

$3D$ Einstein-Hilbert gravity with negative cosmological constant can be writen as CS gauge theory for $SO(2,2) \approx SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ gauge group. In terms of $SL(2,\mathbb{R})$ gauge group, the action can be written as a difference of two CS actions for $A^{\pm} = (\omega^a \pm e^a)J_a^{\pm}$.

$$\kappa \int_{\mathcal{N}} \varepsilon_{abc} \left(R^{ab} e^c + \frac{1}{3} e^a e^b e^c \right) = \int CS(A^+) - \int CS(A^-) + \int_{\partial} (\dots)$$

Boundary central charges are given by $c = \bar{c} = \frac{3}{2G}$ [Brown, Hennaux '86]. We can

1. make two constants in front of CS terms different,
2. use $A^{\pm} = (\omega^a \pm q^{\pm} e^a)J_a^{\pm}$.

This results in a MB model of $3D$ gravity with torsion [Mielke, Baekler '91], [Vasilic, Blagojevic '03]

$$a \int \varepsilon_{abs} R^{ab} e^c + \Lambda \int \varepsilon_{abs} e^a e^b e^c + \alpha_3 \int \mathrm{CS}(\omega) + \alpha_4 \int T^a e_a .$$

For $\alpha_3 \neq 0$, two boundary central charges are different $\to$ no BCFT!

AdS/BCFT with torsion

Total action is given by

$$\kappa \int_{\mathcal{N}} \varepsilon_{abc} \left(R^{ab} e^c + \frac{1}{3} e^a e^b e^c \right) + \alpha\kappa \int_{\mathcal{N}} T^a e_a$$
$$- 2\kappa \int_{\mathcal{Q}} \varepsilon_{abc} e^a n^b \mathrm{d} n^c + \kappa \int_{\mathcal{Q}} e^a (\delta^b_a - n_a n^b) \varepsilon_{bcd} \omega^{cd}$$
$$+ \kappa T \int_{\mathcal{Q}} \varepsilon_{abc} n^a e^b e^c.$$

Variation $\delta\omega$ yields the constraint

$$e_a n^a|_{\mathcal{Q}} = 0,$$

variation δe^a gives

$$P^b_a \varepsilon_{bcd} \omega^{cd} - 2\varepsilon_{abc} n^b \mathrm{d} n^c + 2T \varepsilon_{abc} n^b e^c + \alpha e_a = 0$$

If we have AdS with torsion in the bulk $ds^2 = e^{2\rho}(-\mathrm{d}t^2 + \mathrm{d}\phi^2) + l^2 \mathrm{d}\rho^2$, we can compute the profile of the brane $\rightarrow$ same results as in Riemannian case (expected as torsion here plays a role of cosmological constant).

UIrreps of $SL(2,\mathbb{R})$

We can take generators of $\mathfrak{sl}(2,\mathbb{R}) \cong$ to be

$$[H, E_+] = E_+\,, \quad [H, E_-] = -E_-\,, \quad [E_+, E_-] = 2H.$$

As this group is noncompact, all unitary irreps are infinitely dimensional. They are classified by the value of a quadratic Casimir operator as belonging to

- Principal continuous series
- Complementary continuous series
- Discrete series $\rightarrow$ useful for us (AdS)!

Not all unitary irreps are highest/lowest weight representations!

What are they useful for?

For CS formulation of $3D$ gravity, RT surface can be modeled with a gravitational Wilson line in a highest weight representation [Ammon, Castro, Iqbal '13], as their backreaction induces a conical defect.

$$\mathrm{Tr}_{\mathcal{R}}\left(\mathcal{P}e^{\int_\gamma A}\right)$$

What can we do with there representations?

By relying on their boundary conditions, we can derive the RT formula for our gravity model. Again, the formula suggest that we have the standard minimal surface prescription, which indeed results with a famous CFT entanglement entropy formula, for central charge $c = \frac{3l}{2G}$, with l satisfying $\frac{1}{l^2} = 1 - \frac{3\alpha^2}{4}$.

Second, we can explore the holography on Fuzzy AdS! We quantize our spacetime using frame formalism [Madore '00],

$$e^{\mu}_{\alpha} = e_{\alpha}(x^{\mu}) \text{ (classical)} = [p_{\alpha}, x^{\mu}] \text{ (quantum)} \quad .$$

Momenta p_a are from $\mathfrak{so}(2,1)$ Lie algebra in two dimensions, or $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ in three dimensions. For a scalar field on this spacetime satisfying NC Klein-Gordon equation, we have as in the commutative case

$$\langle \phi \rangle \sim \langle z \rangle^{\Delta}, \qquad \text{for small } \langle z \rangle.$$

Away from the boundary, we have (in 2D)

$$\langle z \rangle^{\frac{1}{2}} \left(\left(\sqrt{2-\beta^2} - i\frac{\langle t \rangle}{l}\beta \right)^2 + \beta^2\lambda^2 \right)^{-\frac{\mu}{2}} P^{-\nu}_{\mu-1}\left(\frac{\sqrt{2-\beta^2} + i\frac{\langle t \rangle}{l}\beta}{\sqrt{(\sqrt{2-\beta^2} + i\frac{\langle t \rangle}{l}\beta)^2 + \beta^2\frac{\langle z \rangle^2}{l^2}}} \right).$$

Conslusions

- It is useful to consider first order formulation of gravity.
- We have a partial understanding of the holographic properties of those theories.
- First order formulation of gravity may be important for NC gravity - both because of SW type constructions and frame formalism.
- The latter presents an interesting playground to study holographic properties of quantum spacetimes.
- We are still lacking a more systematic approach to the holographic description of first-order gravity, including torsion.
- Work to be done includes dealing with more general gravity theories, understanding the nature of boundary theories, generalization to different numbers of spacetime dimensions...

Thank you for your attention!