What is the Large Charge Expansion?

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arXiv:2008.03308

WHO'S WHO

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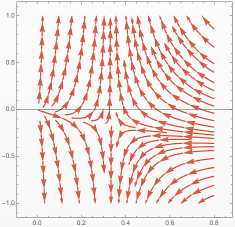
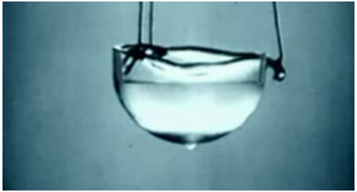
WHY ARE WE HERE? CONFORMAL FIELD THEORIES

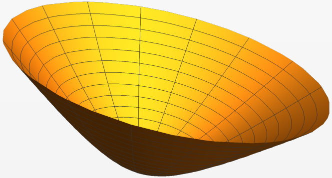
extrema of the RG flow



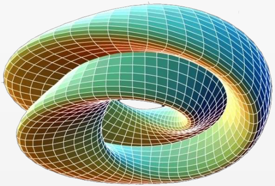
critical phenomena



quantum gravity



string theory



WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

Most conformal field theories (CFTs) lack nice limits where they become simple and solvable.

No parameter of the theory can be dialed to a simplifying limit.

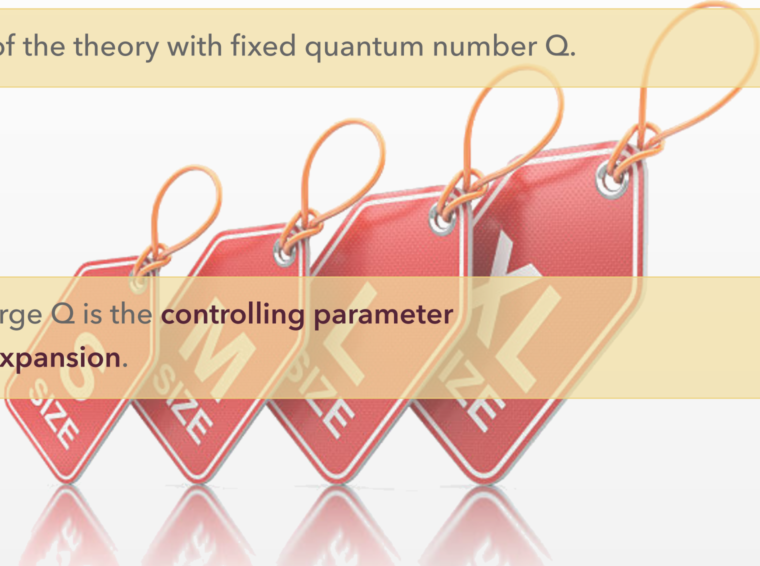
WHY ARE WE HERE? CONFORMAL FIELD THEORIES ARE HARD

In presence of a **symmetry** there can be **sectors of the theory** where anomalous dimension and OPE coefficients simplify.

THE IDEA

Study **subsectors** of the theory with fixed quantum number Q.

In each sector, a large Q is the **controlling parameter** in a **perturbative expansion**.



NOT AN ORIGINAL IDEA

NO BOOTSTRAP HERE!

This approach is orthogonal to bootstrap.
We will use an effective action.
We will access sectors that are difficult to reach with bootstrap.
(However, arXiv:1710.11161).

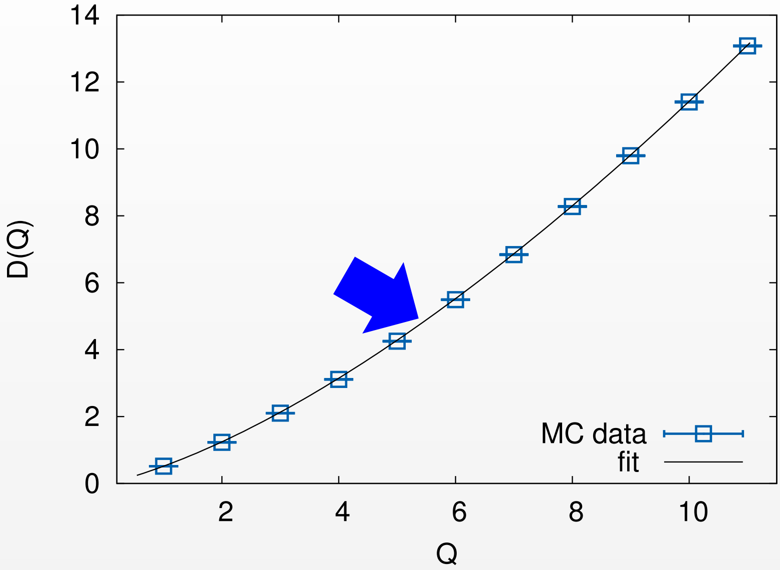
CONCLUSIONS

We consider the **$O(N)$ vector model in three dimensions**. In the IR it flows to a **conformal fixed point** [Wilson & Fisher].

We find an explicit formula for the **dimension of the lowest primary at fixed charge**:

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}\left(Q^{-1/2}\right)$$

CONCLUSIONS: O(2)



SCALES

We want to write a **Wilsonian effective action.**

Choose a cutoff Λ, separate the fields into high and low frequency φ_H, φ_L and do the path integral over the high-frequency part:

$$e^{iS_\Lambda(\varphi_L)} = \int \mathcal{D}\varphi_H \, e^{iS(\varphi_H, \varphi_L)}$$

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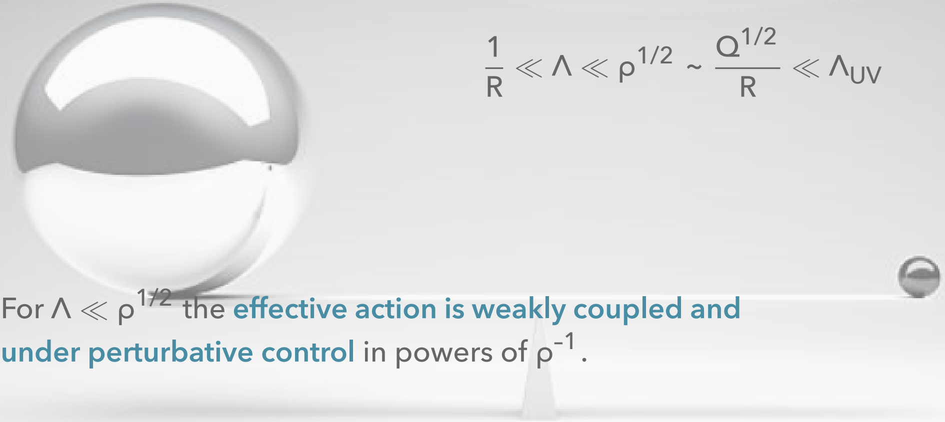
too hard

SCALES

- We look at a finite box of typical **length R**
- The U(1) charge Q fixes a **second scale $\rho^{1/2} \sim Q^{1/2}/R$**



$$\frac{1}{R} \ll \Lambda \ll \rho^{1/2} \sim \frac{Q^{1/2}}{R} \ll \Lambda_{UV}$$

For $\Lambda \ll \rho^{1/2}$ the **effective action is weakly coupled and under perturbative control** in powers of ρ^{-1}.

NON-LINEAR SIGMA MODEL

In a generic theory™, picking the lowest state of fixed charge induces a spontaneous symmetry breaking.
The low-energy physics is described by a **Goldstone field** χ.

Using conformal invariance, the most general action must take the form

$$L[\chi] = k_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$. All other terms are suppressed by powers of $1/Q$.

NON-LINEAR SIGMA MODEL

The **energy of the lowest state** for this action has the form

$$E = \frac{c_{3/2}}{\sqrt{V}} Q^{3/2} + R\sqrt{V} Q^{1/2} + \dots$$

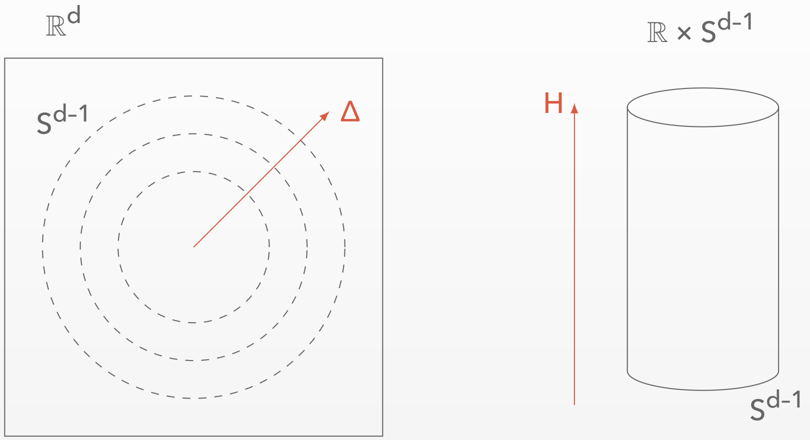
The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}} \zeta(-\tfrac{1}{2}|S^2) = -0.0937\dots$$

This is the unique contribution of order Q^0.

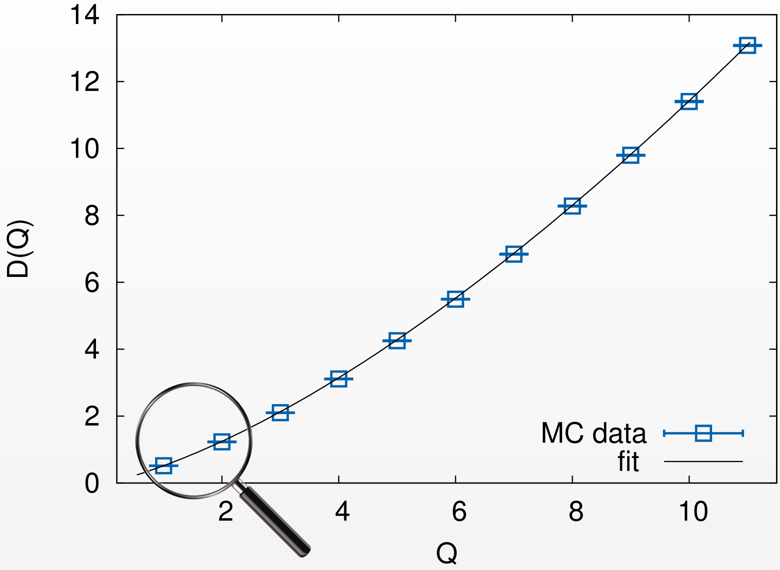
STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.



Protected by conformal invariance: a well-defined quantity.

TOO GOOD TO BE TRUE?



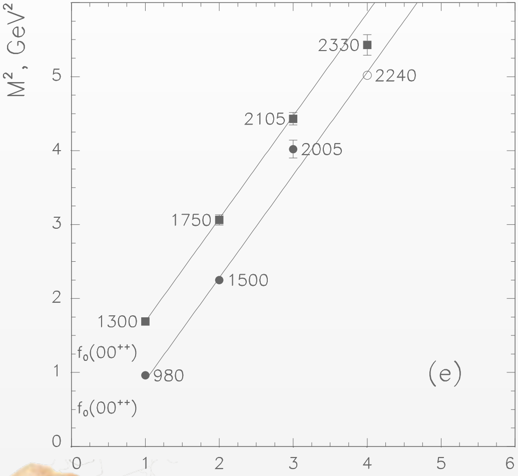
TOO GOOD TO BE TRUE?

Think of **Regge trajectories**.

The prediction of the theory is

$$m^2 \propto J\left(1 + \mathcal{O}\left(J^{-1}\right)\right)$$

but experimentally everything works so well at small J that String Theory was invented.



SELECTED TOPICS IN THE LARGE CHARGE EXPANSION

- O(2) model [Hellerman, DO, Reffert, Watanabe] [Monin, Pirtskhalava, Rattazzi, Seibold]
- fermions [Komargodski, Mezei, Pal, Raviv-Moshe] [Antipin, Bersini, Panopoulos] [Hellerman, Dondi, Kalogerakis, Moser, DO, Reffert]
- holography [Nakayama] [Loukas, DO, Reffert, Sarkar] [de la Fuente] [Guo, Liu, Lu, Pang] [Giombi, Komatsu, Offertaler]
- large N [Álvarez-Gaumé, DO, Reffert] [Giombi, Hyman]
- ε double-scaling [Badel, Cuomo, Monin, Rattazzi] [Arias-Tamargo, Rodriguez-Gomez, Russo] [Antipin, Bersini, Sannino, Wang, Zhang] [Jack, Jones]
- non-relativistic CFTs [Kravec, Pal] [Helleman, Swanson] [Favrod, DO, Reffert] [DO, Reffert, Pellizzani] [Hellerman, DO, Reffert, Pellizzani, Swanson]
- $\mathcal{N} = 2$ [Hellerman, Maeda] [Hellerman, Maeda, DO, Reffert, Watanabe] [Bourget, Rodriguez-Gomez, Russo] [Grassi, Komargodski, Tizzano] [Cremonesi, Lanza, Martucci]
- bootstrap [Jafferis, Zhiboedov]
- resurgence [Dondi, Kalogerakis, DO, Reffert] [Antipin, Bersini, Sannino, Torres] [Watanabe]

WHAT HAPPENED?

We started from a CFT.
There is no mass gap, there are **no particles**, there is **no Lagrangian**.

We picked a sector.
In this sector the physics is described by a **semiclassical configuration** plus massless fluctuations.

The full theory has no small parameters but we can study this sector with a **simple effective field theory (EFT)**.
We are in a **strongly coupled** regime but we can compute physical observables using **perturbation theory**.

TODAY'S TALK

The EFT for the $O(2)$ model in $2+1$ dimensions

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Double Scaling & Supersymmetry

TODAY'S TALK

The EFT for the $O(2)$ model in $2 + 1$ dimensions

- An EFT for a CFT.
- The physics at the saddle.
- State/operator correspondence for anomalous dimensions.

Double Scaling & Supersymmetry

TODAY'S TALK

The EFT for the $O(2)$ model in $2 + 1$ dimensions

Double Scaling & Supersymmetry

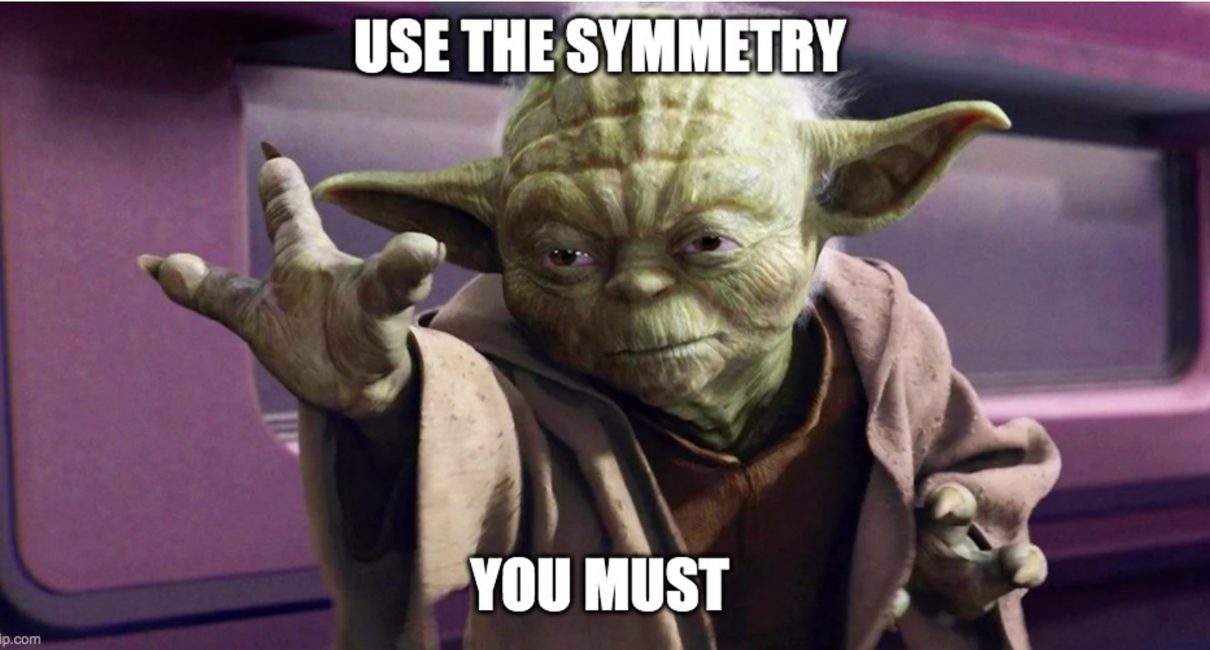
- Large N
- $\mathcal{N} = 2$ SQCD in four dimensions

PARENTAL

ADVISORY

EXPLICIT CONTENT

AN EFT FOR A CFT



THE O(2) MODEL

The simplest example is the Wilson-Fisher (WF) point of the O(2) model in three dimensions.

- Non-trivial fixed point of the φ^4 action

$$L_{UV} = \partial_\mu \varphi^* \partial_\mu \varphi - u(\varphi^* \varphi)^2$$

- Strongly coupled
- In nature: ^{4}He.
- Simplest example of spontaneous symmetry breaking.
- **Not accessible** in perturbation theory. **Not accessible** in $4 - \varepsilon$. **Not accessible** in large N.
- Lattice. Bootstrap.

CHARGE FIXING

We consider a **subsector of fixed charge Q**.

Generically, the classical solution at fixed charge **breaks spontaneously $U(1) \to \emptyset$**.

We have one **Goldstone boson** χ.

AN ACTION FOR χ

Start with two derivatives:

$$L[\chi] = \frac{f_\pi}{2} \partial_\mu \chi \partial_\mu \chi - C^3$$

(χ is a Goldstone so it is dimensionless.)

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We want to describe a CFT: we can **dress with a dilaton**

$$L[\sigma, \chi] = \frac{f_\pi e^{-2f\sigma}}{2} \partial_\mu \chi \partial_\mu \chi - e^{-6f\sigma} C^3 + \frac{e^{-2f\sigma}}{2} \left(\partial_\mu \sigma \partial_\mu \sigma - \frac{\xi R}{f^2} \right)$$

The fluctuations of χ give the Goldstone for the broken $U(1)$, the fluctuations of σ give the (massive) Goldstone for the broken conformal invariance.

LINEAR SIGMA MODEL

We can put together the two fields as

$$\Sigma = \sigma + i f_\pi \chi$$

and rewrite the action in terms of a complex scalar

$$\varphi = \frac{1}{\sqrt{2f}} e^{-f\Sigma}$$

We get

$$L[\varphi] = \partial_\mu \varphi^* \partial^\mu \varphi - \xi R \varphi^* \varphi - u(\varphi^* \varphi)^3$$

Only depends on dimensionless quantities $b = f^2 f_\pi$ and $u = 3(Cf^2)^3$.

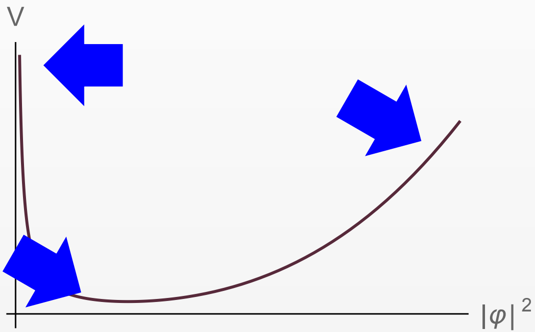
Scale invariance is manifest.

The field φ is some complicated function of the original φ.

CENTRIFUGAL BARRIER

The $O(2)$ symmetry acts as a shift on χ.

Fixing the charge is the same as adding a **centrifugal term** $\propto \dfrac{1}{|\varphi|^2}$.



GROUND STATE

We can find a fixed-charge solution of the type

$$\chi(t,x) = \mu t \qquad\qquad \sigma(t,x) = \frac{1}{f}\log(v) = \text{const.},$$

where

$$\mu \propto Q^{1/2} + \ldots \qquad\qquad v \propto \frac{1}{Q^{1/2}}$$

The classical energy is

$$E = c_{3/2}/\sqrt{V}Q^{3/2} + c_{1/2}R\sqrt{V}Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right)$$

FLUCTUATIONS

The fluctuations over this ground state are described by two modes.

- A universal "**conformal Goldstone**". It comes from the breaking of the U(1).

$$\omega = \frac{1}{\sqrt{2}} p$$

- The **massive dilaton**. It controls the magnitude of the quantum fluctuations. **All quantum effects are controled by 1/Q.**

$$\omega = 2\mu + \frac{p^2}{2\mu}$$

(This is a heavy fluctuation around the semiclassical state. It has nothing to do with a light dilaton in the full theory)

NON-LINEAR SIGMA MODEL

Since σ is heavy we can integrate it out and write a non-linear sigma model (NLSM) for χ alone.

$$L[\chi] = k_{3/2}(\partial_\mu \chi \partial^\mu \chi)^{3/2} + k_{1/2} R (\partial_\mu \chi \partial^\mu \chi)^{1/2} + \dots$$

These are the leading terms in the expansion around the classical solution $\chi = \mu t$.
All other terms are suppressed by powers of $1/Q$.

In 3 + 1 NRCFT the analogous story in a background potential A_0 leads to

$$L[\chi] = c_0 U^{5/2} + c_1 U^{-1/2} \partial_i U \partial_i U + c_2 U^{1/2} ((\partial_i \partial_i \chi)^2 - 9 \partial_i \partial_i A_0) + \dots \tag{1}$$

where $U = \partial_t \chi - A_0 \chi - \partial_i \chi \partial_i \chi / 2$.

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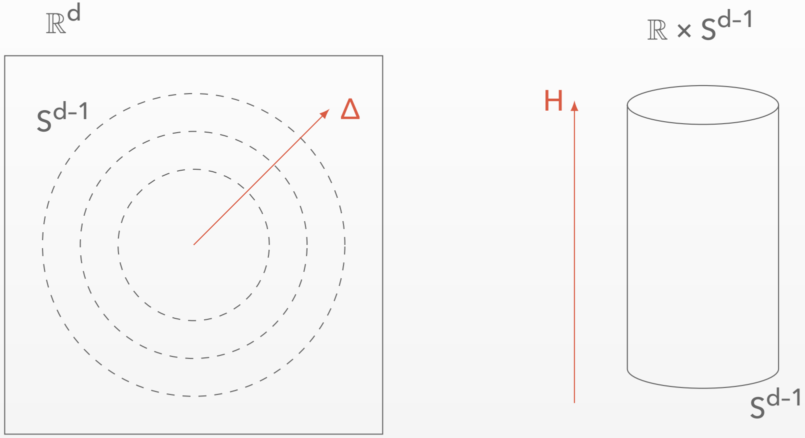
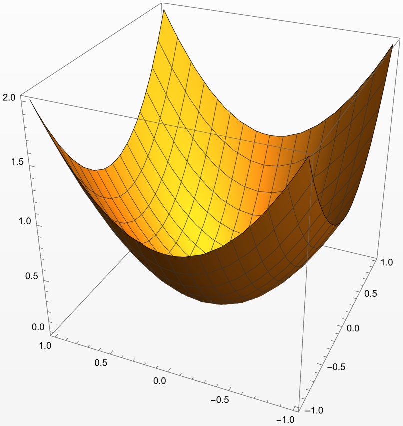
The anomalous dimension on $\mathbb{R}^d$ is the energy in the cylinder frame.

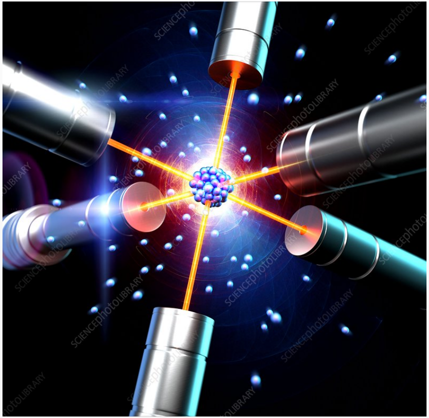


Protected by conformal invariance: a well-defined quantity.

NRCFT STATE-OPERATOR CORRESPONDENCE

The anomalous dimension on $\mathbb{R}^d$ is the energy in a harmonic trap.





Protected by conformal invariance: a well-defined quantity.

CONFORMAL DIMENSIONS

We know the energy of the ground state.

The leading quantum effect is the **Casimir energy of the conformal Goldstone**.

$$E_G = \frac{1}{2\sqrt{2}}\zeta(-\tfrac{1}{2}|S^2) = -0.0937\ldots$$

This is the unique contribution of order Q^0.

Final result: the **conformal dimension of the lowest operator of charge Q** in the $O(2)$ model has the form

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}}Q^{3/2} + 2\sqrt{\pi}c_{1/2}Q^{1/2} - 0.094\ldots + \mathcal{O}\left(Q^{-1/2}\right)$$

In 3 + 1 NRCFT we find

$$\Delta_Q = c_{4/3}Q^{4/3} + c_{2/3}Q^{2/3} + b_{5/9}Q^{5/9} + b_{1/3}Q^{1/3} + b_{1/9}Q^{1/9} - \frac{1}{3\sqrt{3}}\log(Q) + c_0$$

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DOUBLE SCALING

DOUBLE SCALING

The large-charge expansion works for strongly-coupled systems that are not otherwise (analytically) accessible.

What if there is another parameter? e.g. large N, small ε?

In a double-scaling limit (**Q/N fixed, εQ fixed**) the fixed-charge is a **refinement** of the theory.
Equivalent to the resummation of all-order Feynman diagrams.

O(N) AT LARGE N

- at small double-scaling parameter, infinite number of Feynman diagrams

$$\Delta(Q) = \frac{Q}{2} + \frac{2}{\pi^2}\frac{Q^2}{N} + \frac{16\left(\pi^2 - 12\right)Q^3}{9\pi^4 N^2} + \ldots$$

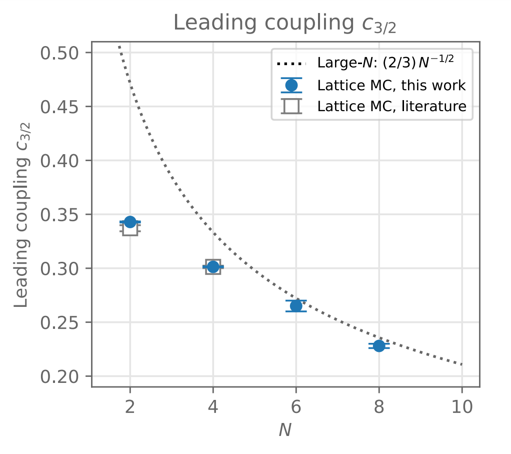
- at large double-scaling parameter, large-charge behavior

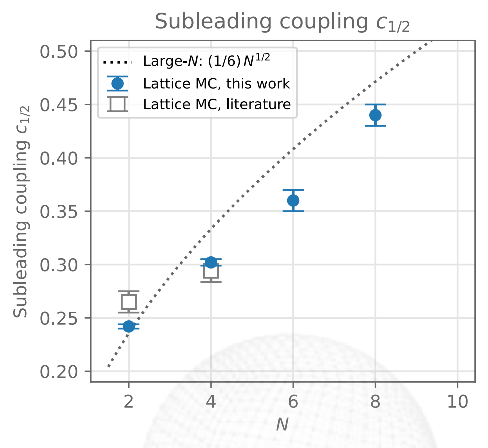
$$\Delta(Q) = \frac{4N}{3}\left(\frac{Q}{2N}\right)^{3/2} + \frac{N}{3}\left(\frac{Q}{2N}\right)^{1/2} - \frac{7N}{360}\left(\frac{Q}{2N}\right)^{-1/2} - \frac{71N}{90720}\left(\frac{Q}{2N}\right)^{-3/2} + \mathcal{O}\left(e^{-\sqrt{Q/(2N)}}\right)$$

- exact result for all values of Q/N (from resurgence)

$$\operatorname{Leg}\left[\Delta(Q)\right] = \frac{2rm^2}{\pi}\int_0^\infty dy\, \frac{K_2(2mry)}{y\sin(y)} = \frac{r^2}{3}m^3 - \frac{m}{24} + \cdots - \frac{2ir^{1/2}m^{3/2}}{(4\pi)^{3/2}}e^{-2\pi rm} + \ldots$$

HOW GOOD IS LARGE N?





[Singh, arXiv:2203.00059]

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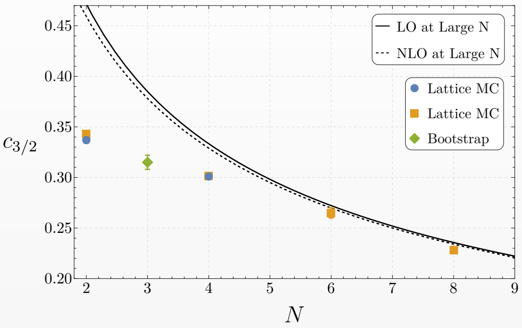
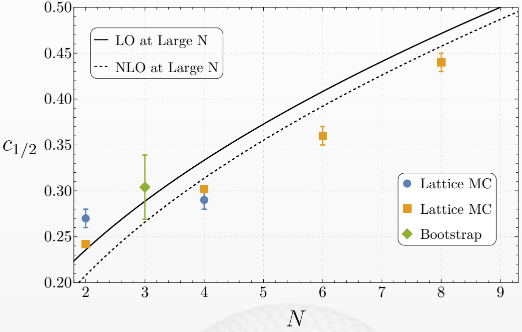




[Dondi and Sberveglieri, arXiv:2409.xxx]

SUPERSYMMETRIC MODELS

Systems with **eight supercharges** (or more) behave differently.
(Ask me about four for a surprise).
The lowest state is BPS, and has dimension proportional to the charge.

What happens when there is a flat direction?

Many known examples of (non-Lagrangian) $\mathcal{N} \geq 2$ SCFT in four dimensions.
Coulomb branch with a dimension-one moduli space: all the physics is encoded in a single operator $\mathcal{O}$ and every chiral operator is just $\mathcal{O}^n$.

SUPERSYMMETRIC MODELS

$$\left\langle \mathcal{O}^{n_1}(x_1)\mathcal{O}^{n_2}(x_2)\bar{\mathcal{O}}^{n_1+n_2}(x_3)\right\rangle = \frac{C^{n_1,n_2,n_1+n_2}}{|x_1 - x_3|^{2n_1\Delta}|x_2 - x_3|^{2n_2\Delta}}$$

The OPE of $\mathcal{O}$ with itself is regular, so we can set $x_2 = x_1$ and the three-point function is actually a two-point function (C^{n_1,n_2,n_1+n_2} is meaningful).
$Q = n\Delta$ is the controlling parameter (it's the R-charge).

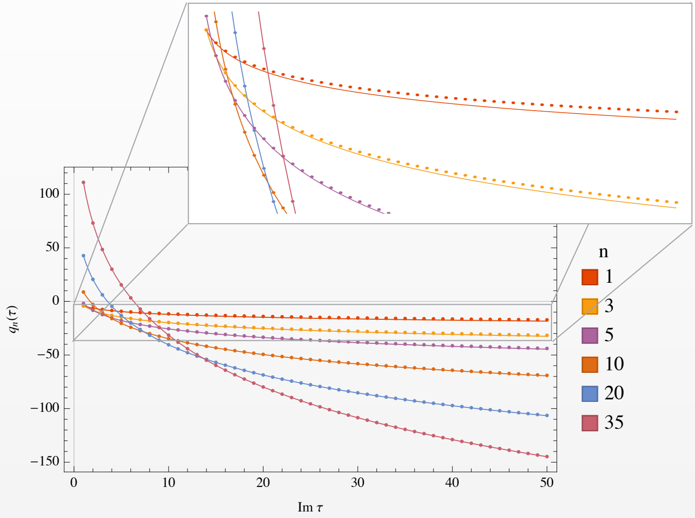
Final result:

$$\left\langle \mathcal{O}^{n}(x_1)\bar{\mathcal{O}}^{n}(x_2)\right\rangle = C_n \frac{\Gamma(n\Delta + \alpha + 1)}{|x_1 - x_2|^{2n\Delta}}$$

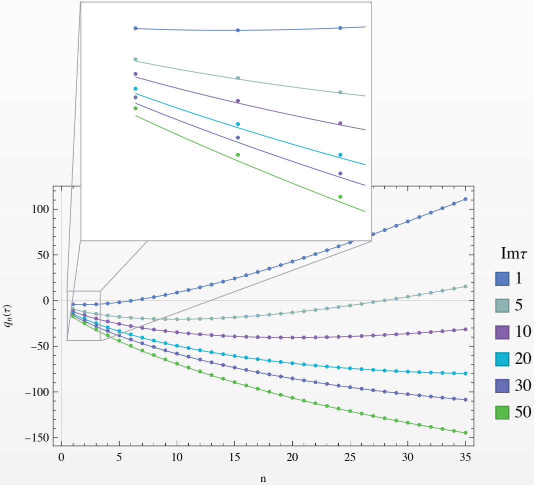
For SU(2) SQCD (τ is UV coupling, σ is the IR coupling)

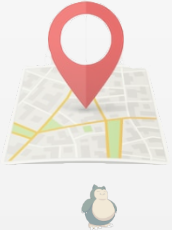
$$C_n = \frac{Y_G^{12}}{2^{9/2} e \pi^{3/2}} \frac{|\lambda(\sigma)|^{2/3}|1 - \lambda(\sigma)|^{8/3}}{|\eta(\sigma)|^8 \,\mathrm{Im}(\sigma)^2} \frac{1}{\left(4\,\mathrm{Im}(\tau) + 8/\pi \log(2)\right)^{2n}}$$

COMPARISON WITH LOCALIZATION



COMPARISON WITH LOCALIZATION



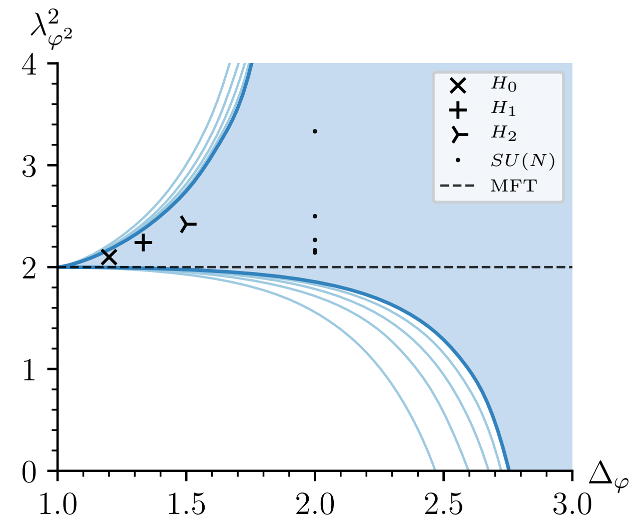


COMPARISON WITH BOOTSTRAP

For strongly coupled theories one can use bootstrap to place bounds on the three-point coefficients with n = 1.

This is the worst possible situation for us. And still…

COMPARISON WITH BOOSTRAP



Taken from arXiv:2006.01847

CONCLUSIONS

- With the large-charge approach we can study **strongly-coupled systems perturbatively**.
- Select a sector and we write a **controllable effective theory**.
- The strongly-coupled physics is (for the most part) subsumed in a **semiclassical state**.
- Precise and **testable predictions**.
- Qual(nt)itative control of the **non-pertubative** effects.
- **CFT constraints**: perturbative/non-perturbative **interplay**.
- Remarkable agreement with **lattice**.