On the deep string spectrum

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based on

CM and E. Skvortsov, JHEP 12 (2023) 055 [arXiv:2309.15988]

T. Basile and CM, JHEP 07 (2024) 184 [arXiv:2405.18467]

September 5, 2024, Corfu Summer Institute

Workshop on Quantum Gravity, Strings and the Swampland

Generalities of the spectrum

- two universal string parameters: α' and g_S
- infinitely many *physical* states

$$M^2 = \text{integer} \times \frac{1}{\alpha'}$$

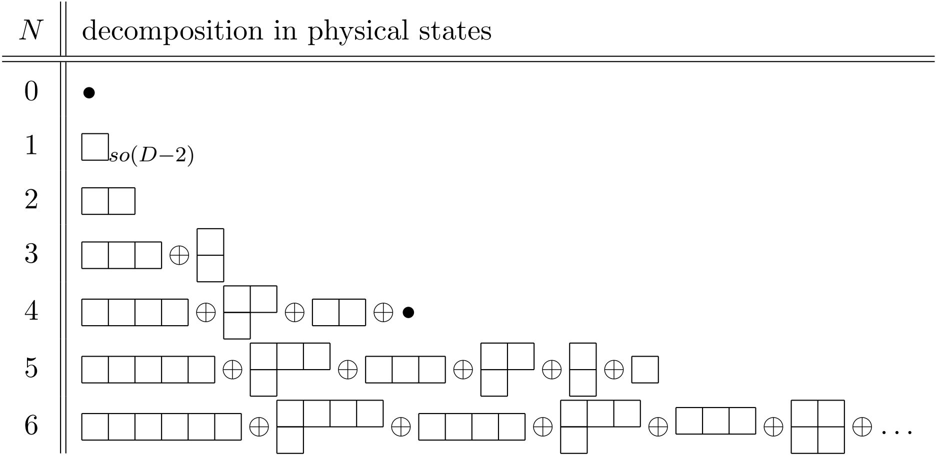
that are thought of as:

1. mass eigenstates $\Rightarrow$ on–shell mass
2. irreps of $SO(D-1)$ or $SO(D-2)$ $\Rightarrow$ TT

1–particle states à la Bargmann and Wigner ?

- among them $\exists$ ∞–many massive higher spins that are crucial to sting theory's *UV finiteness*
- What does the spectrum look like? Is there a bigger *symmetry*?

Let's start with the open bosonic string



see e.g. Weinberg 1985, Mañes, Vozmediano 1989

How can one construct the spectrum?

The space of physical states

- the string field X^μ can be expanded in *modes* α_n^μ with algebra $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \Rightarrow$ Fock space
  1. vacuum $|0; p^\mu\rangle$: momentum eigenstate, defined via $\alpha_{k>0}^\mu |0; p^\mu\rangle = 0$
  2. generic string states: functions of $\alpha_{k<0}^\mu$
  3. number operator: $N_m = \alpha_{-m} \cdot \alpha_m \, , m > 0$
- physical states: subset that satisfies the Virasoro constraints

$$(L_n - \delta_{n,0})|\text{phys}\rangle = 0 \, , \, \forall n \geq 0 \, , \quad L_n := \tfrac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{n-m} \cdot \alpha_m : \, , \; \alpha_0^\mu := \sqrt{2\alpha'} p^\mu$$

with the L_n's generating the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12} m(m^2 - 1)\delta_{m+n}$$

$\Rightarrow$ *three* constraints are sufficient

see e.g. Sasaki, Yamanaka 1985

$$(L_0 - 1)|\text{phys}\rangle = 0 \, , \quad L_1|\text{phys}\rangle = 0 \quad , \quad L_2|\text{phys}\rangle = 0$$

Traditional constructions of the spectrum

1. light–cone gauge (level–by–level, by construction non–covariant)

   Goddard, Thorn 1972
   Goddard, Goldstone, Rebbi, Thorn 1973

2. DDF (level–by–level, either ∃ reference momentum or need to solve constraints)

   Del Giudice, Di Vecchia, Fubini 1972, Brower 1972
   Skliros, Hindmarsh '11

3. obtain $SO(D-1)$ irreps from partition function (but no info on what they look like)

   Curtright, Thorn 1986
   Forcella, Hanany, J. Troost '10

4. old covariant way / building vertex operators (level–by–level):

$$|\mathrm{phys}\rangle = g_{\mathrm{o}}\, T^a\, F(\alpha^{\mu}_{-1}, \alpha^{\nu}_{-2}, \dots)\, |0; p\rangle$$

   1. *choose level*: $(L_0 - 1)|\mathrm{phys}\rangle = 0 \quad \Rightarrow \quad M^2 = (N-1)/\alpha'\,,$
      where $N := \sum_{m=1}^{\infty} N_m = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m$
   2. *write Ansatz* for F at chosen N and solve $L_1|\mathrm{phys}\rangle = 0 = L_2|\mathrm{phys}\rangle$

Examples

- first few levels:
  1. $N = 0$: $|\text{tachyon}\rangle = |0; p\rangle$
  2. $N = 1$: $|\text{vector}\rangle = \varepsilon \cdot \alpha_{-1}\, |0; p\rangle$ (T)
  3. $N = 2$: $F = \varepsilon_{\mu\nu}\, \alpha^{\mu}_{-1}\, \alpha^{\nu}_{-1} + \tilde{\varepsilon} \cdot \alpha_{-2}$
     use L_1 and L_2 to gauge away longitudinal d.o.f. and trace
     $\Rightarrow$ $|\text{phys}\rangle = \varepsilon_{\mu\nu}\, \alpha^{\mu}_{-1}\, \alpha^{\nu}_{-1}\, |0; p\rangle$ (TT) $\Rightarrow$ massive spin–2

- *leading* Regge trajectory: highest spins $\forall$ level

$$|\text{leading}\rangle = F_1\, |0; p\rangle = \varepsilon_{\mu_1 \dots \mu_s}\, \alpha^{\mu_1}_{-1} \dots \alpha^{\mu_s}_{-1}\, |0; p\rangle$$

  1. L_0 fixes p^2: $\alpha' p^2 = 1 - s$
  2. $L_1 \sim p \cdot \alpha_1 + \dots$ checks transversality: $p^{\nu} \varepsilon_{\nu \mu_2 \dots \mu_s} = 0$
  3. $L_2 \sim \alpha_1 \cdot \alpha_1 + \dots$ checks tracelessness: $\eta^{\nu\sigma} \varepsilon_{\nu\sigma\mu_3 \dots \mu_s} = 0$

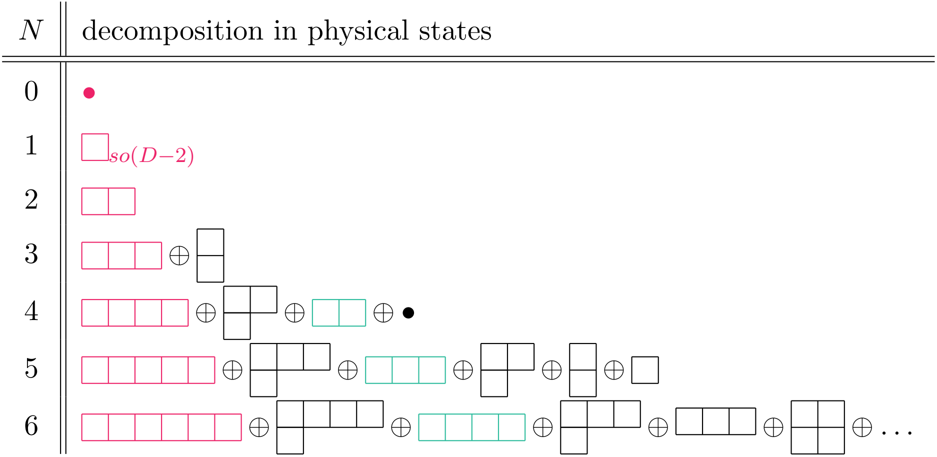
- state–operator correspondence:

$$\alpha^{\mu_1}_{-n_1} \dots \alpha^{\mu_k}_{-n_k} |0; p\rangle \quad \leftrightarrow \quad \partial^{n_1} X^{\mu_1} \dots \partial^{n_k} X^{\mu_k}\, e^{ip\cdot X}$$

$$\Rightarrow \text{TT become} \quad p \cdot \frac{\delta F_1}{\delta \partial X} = 0\,, \qquad \frac{\delta^2 F_1}{\delta\partial X \cdot \delta \partial X} = 0$$

see e.g. Sagnotti, Taronna '10

Visualisation: Regge *trajectories*



beyond the leading Regge, the spectrum seems *repetitive*

Main challenge: what do excited states look like?

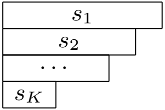
Is there a certain *pattern*?

Is the spectrum concealing a *bigger organizing symmetry*?

What does a state look like?

- *any* state has a polarization depicted by a Young diagram

$$\varepsilon^{\mu(s_1),\,\lambda(s_2),\ldots,\,\nu(s_K)}(p) \quad \Leftrightarrow$$



conventions: symmetric base (symmetric rows + additional relations, $s_1 \geq s_2 \geq \ldots \geq s_K$), $\quad \varepsilon^{\mu(s_1)} := \varepsilon^{\mu_1 \cdots \mu_{s_1}}$ for symmetric index groups

- for *physical* states: *dress* polarization by suitable polynomial
$\Rightarrow \quad \exists \quad \infty$–many physical polynomials $\forall$ diagram

- What is the simplest physical polynomial that *minimizes* the level?

$$F_K = \varepsilon_{\mu(s_1),\,\lambda(s_2),\ldots,\,\nu(s_K)}\, \alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s_1}} \ldots \alpha_{-K}^{\nu_1} \ldots \alpha_{-K}^{\nu_{s_K}} \quad \text{(TT)}$$

$$\Rightarrow \quad N = N_{\text{min}} = \sum_{i=1}^{K} s_i\, i$$

see e.g. Weinberg 1985

example: leading Regge: 1–row, spin–s at $N = s$

Where does a state appear?

- let's begin "parametrizing our ignorance":
  1. *any* diagram appears at $N_{\min}$ and at higher levels $N = N_{\min} + w$
     $\Rightarrow$ let's call w "depth"
  2. let's call a trajectory the set of states with a *fixed* number of rows at *fixed* w, e.g.: at $w = 0$, each value of K corresponds to a new trajectory ($K = 1$: leading Regge) $\Rightarrow$ *finite* K for given trajectory!

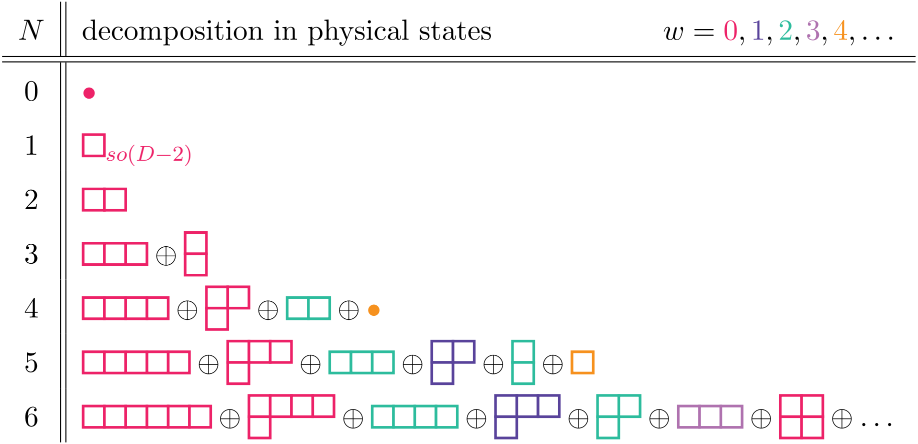
- idea: let's **use w to re–organize the spectrum**
  $\Rightarrow$ it consists of $w = 0$ trajectories and their ∞–many clones! e.g.:

trajectory	Young shape	spin	N	w	lightest member	polynomial
leading Regge	s	$s \geq 0$	s	0	•	$\varepsilon_{\mu(s)}\, \alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_s}$
first "clone"	s	$s \geq 2$	$s+2$	2	□□	**?**

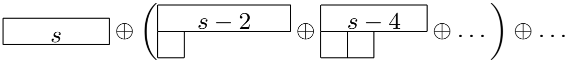
$\Rightarrow$ polynomial *complexity*: measured by w (and number of rows)

CM, Skvortsov '23

Re–organizing the spectrum



e.g. at $w = 0$:



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All trajectories at once

- let's consider the *most general* polynomial $F = F(\alpha^{\mu}_{-1}, \alpha^{\nu}_{-2}, \dots)$
- simplification: transverse subspace is sufficient
  $\Rightarrow$ F contains only transverse tensors

see e.g. Mañes, Vozmediano 1989

- idea: let's supply the Virasoro constraints with F $\Rightarrow$ obtain:

$$(L_0 - 1)F = \Big(\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \alpha' p^2 - 1\Big)F = 0$$

$$2L^{\perp}_{n>0}F = \Bigg[\sum_{m=1}^{n-1} \alpha_{n-m} \cdot \alpha_m + 2\sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_{n+m}\Bigg]F = 0$$

- our first proof: (in the full spacetime) using the v.o. language
  $\Rightarrow$ the Virasoro become differential constraints on F

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A bigger organizing symmetry

- let's define the operators $(k, l = 1, 2, \ldots, N)$

$$T^k{}_l := \frac{1}{k}\,\alpha_{-k}\cdot\alpha_l\,, \quad T_{kl} := \alpha_k\cdot\alpha_l\,, \quad T^{kl} = \frac{1}{kl}\,\alpha_{-k}\cdot\alpha_{-l}$$

- observations:
  1. use $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \;\Rightarrow\; \{T^k{}_l\,, T_{kl}, T^{kl}\}$ generate $\mathfrak{sp}(2N)$!
     can choose lowering & raising operators
     - ★ lowering: $T_{kl}, T^{k<l}{}_l$
     - ★ raising: $T^{kl}, T^{k>l}{}_l$

     $\Rightarrow$ lowest weight state: $T^k{}_k\,F = s_k\,F\,,\; T_{kl}F = 0\,,\; T^{k<l}{}_l\,F = 0$
  2. the lowering operators check Young symmetry & tracelessness!
     e.g.: 2–rows at $w = 0\,, \quad F = \varepsilon_{\mu(s_1),\,\nu(s_2)}\,\alpha^{\mu_1}_{-1}\ldots\alpha^{\mu_{s_1}}_{-1}\,\alpha^{\nu_1}_{-2}\ldots\alpha^{\nu_{s_2}}_{-2}$
     - ★ $T_{12}F_2 = 0 \quad\Leftrightarrow\quad \eta^{\rho\sigma}\varepsilon_{\mu(s_1-1)\rho,\nu(s_2-1)\sigma} = 0$ etc
     - ★ $T^1{}_2\,F_2 = 0 \quad\Leftrightarrow\quad \varepsilon_{\mu(s_1),\mu\nu(s_2-1)} = 0$
  3. $\mathfrak{sp}(2\bullet)$ commutes with $\mathfrak{so}(D-1,1)$! $\quad\Rightarrow\quad$ "*Howe dual*" pair
     they act on different index types: α^μ_n

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Building the spectrum: a new technology

- can now rewrite the constraints on a general F as:

$$(L_0 - 1)F = \left(\sum_{n=0}^{\infty} n\, T^n{}_n + \alpha' p^2 - 1 \right) F = 0$$

$$L^{\perp}_{n>0} F = \left[\sum_{m=1}^{n-1} T^{\perp}_{m,n-m} + 2 \sum_{m=1}^{\infty} m T^m{}_{n+m} \right] F = 0$$

$\Rightarrow$ observation: the lowest weight states of $\mathfrak{sp}(2\bullet)$ *solve* the Virasoro constraints and are the $w = 0$ trajectories!

- let's distinguish 2 kinds of embeddings:
  "*principal*" $(w = 0)$ and "*non–principal*" $(w > 0)$

- idea: employ Howe duality between $\mathfrak{sp}(2\bullet)$ and $\mathfrak{so}(D-1,1)$

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irreps of $\mathfrak{sp}(2\bullet)$ $\overset{\text{"1–1"}}{\longleftrightarrow}$ irreps of $\mathfrak{so}(D-1,1)$

for the duality of commuting algebras see Howe 1989, ...

Building the spectrum: a new technology

- implementation: use raising operators of $\mathfrak{sp}(2\bullet)$ to reach $w > 0$

$$F^{f}_{w>0} = f_{w>0}(T_{\perp}^{mn}, T^{k>l}{}_{l}\,)\, F_{w=0}$$

- example: reach the first clone, at $w = 2$, of the leading Regge

$$F^{f}_{w=2} = \Big[\beta_1 T_{\perp}^{11} + \beta_2 T^{3}{}_{1} + \beta_3 (T^{2}{}_{1}\,)^2\Big] F_{w=0}$$

$\Rightarrow$ solve the Virasoro constraints for $F^{f}_{w=2}$:

$$\Rightarrow \quad \beta_2 = -\beta_1 \frac{D+2s-1}{3s}\,, \quad \beta_3 = \beta_1 \frac{D+2s-1}{2(s-1)}$$

spin is not fixed $\Rightarrow$ full trajectory is known!

e.g.: the lightest member–state 

$$\begin{aligned} F_B &= B_{\mu\nu}\, i\partial X^{\mu} i\partial X^{\nu} \\ F_{\widetilde{B}} &= \widetilde{B}_{\mu\nu}\Big[-\tfrac{3}{\alpha'}\, i\partial X^{\mu} i\partial X^{\nu} i\partial X^{\kappa} i\partial X_{\kappa} + 29\, i\partial X^{\mu} i\partial^3 X^{\nu} - 87\, i\partial^2 X^{\mu} i\partial^2 X^{\nu}\Big] \end{aligned}$$

- amplitudes available! we have several examples

CM, Skvortsov '23

A new superstring technology, super quickly

- add *fermionic* modes b_r^μ, with $\{b_m^\mu, b_n^\nu\} = \delta_{m+n}\,\eta^{\mu\nu}$
  $\Rightarrow$ favor columns
- $\exists$ two sectors, NS : $\mathfrak{osp}(2\mathsf{M}|2\mathsf{N},\mathbb{R})$, R : $\mathfrak{osp}(2\mathsf{M}+1|2\mathsf{N},\mathbb{R})$
  but how do we *glue rows to columns*?
- look at super–Virasoro constraints $\Rightarrow$ principal embedding:
  1. NS :

$b_{-\frac{1}{2}}$	α_{-1}	$\cdots$		α_{-1}
$b_{-\frac{1}{2}}$	$b_{-\frac{3}{2}}$	α_{-2}	$\cdots$	α_{-2}

  2. R :

2^3 configurations :

	α_{-1}	$\cdots$			α_{-1}
b_{-1}		α_{-2}	$\cdots$	α_{-2}	
b_{-1}	b_{-2}		α_{-3}	$\cdots$	α_{-3}

Basile, CM, '24

A new superstring technology, super quickly

- R : there exists a zero weight operator, ${Q^n}_n := \frac{1}{n}\,\alpha_{-n}\cdot b_n$!

  $$\Rightarrow \text{write } |\mathsf{R}\,\rangle_{w=0} = \Big(\mathsf{b}_0 + \sum_{n=1}^{K} \tfrac{1}{n!}\,\mathsf{b}_{i_1\ldots i_n}\,{Q^{i_1}}_{i_1}\ldots {Q^{i_n}}_{i_n}\Big)\,|\mathsf{NS}\text{–like}\rangle_{w=0}$$

  and impose super–Virasoro ⇒ find $\mathsf{b}'s$

  ⇒ non–trivial multiplicity in the R sector even at $w = 0$

- new technology: reach deeper trajectories via

  $$F^{f}_{w>0} \equiv f\big(T^{mn}_{\perp}, {T^k}_l\,, M^{mn}_{\perp}, {M^k}_l\,, Q^{mn}_{\perp}, {Q^k}_l\,, {Q_l}^k, M^m, Q^m\big)\,F_{w=0}$$

  with $k > l$ (include ${Q^n}_n$ in the R sector)

- whole trajectories available

Basile, CM, '24

restriction: the dictionary for the state–operator correspondence is *trajectory–dependent*

Conclusion

we have developed a **new technology** that excavates entire string trajectories deeper in the string spectrum

it is **covariant** and **efficient**

- key observation: the Virasoro constraints entail $\mathfrak{osp}$ generators
- idea: employ *Howe duality* between $\mathfrak{osp}$ and $\mathfrak{so}(D-1,1)$
- gearwheel: $\mathfrak{osp}$ raising operators
- framework: open string, critical dimension

1. bosonic string — CM, Skvortsov '23
2. superstring — Basile, CM '24

Is the *complete* spectrum within reach?

Black–hole amplitudes from *string* amplitudes?
Chaos in string scattering? ...

What can the complete spectrum teach us about fields and strings and, ultimately, about quantum gravity?

Backup

- super–Virasoro constraints: (L_0 and G_0 fix m^2)

$$G^{\perp}_{1/2} F_{\mathsf{NS}} = \sum_{n\geq 1} \left[n\, {Q^n}_{n+1} + {Q_n}^n \right] F_{\mathsf{NS}} = 0$$

$$G^{\perp}_{3/2} F_{\mathsf{NS}} = \left[Q^{\perp}_{11} + \sum_{k\geq 1} \left(k\, {Q^k}_{k+2} + {Q_{k+1}}^k \right) \right] F_{\mathsf{NS}} = 0$$

$$G_1 F^{\perp}_{\mathsf{R}} = \left[Q_1 + \textstyle\sum_{k\geq 1} \left(k\, {Q^k}_{k+1} + {Q_{k+1}}^k \right) \right] F_{\mathsf{R}} = 0$$

- orthosymplectic generators: $Q_m := \frac{1}{\sqrt{2}}\, \gamma \cdot \alpha_m$ and

$${Q^n}_r := \frac{1}{n}\, \alpha_{-n} \cdot b_{r-\frac{1}{2}}\,, \quad {Q_n}^r := b_{-r+\frac{1}{2}} \cdot \alpha_n\,, \quad Q_{nr} := \alpha_n \cdot b_{r-\frac{1}{2}}$$

$$\Rightarrow \quad \mathsf{NS} : \mathfrak{osp}(2\mathsf{M}|2\mathsf{N}, \mathbb{R})\,, \quad \mathsf{R} : \mathfrak{osp}(2\mathsf{M}+1|2\mathsf{N}, \mathbb{R})$$

lowest weight vector: ${Q_m}^r F = 0\,,\ r \leq m$ and ${Q^m}_r F = 0\,, m < r$

Basile, CM, '24

- R dictionary:

$$b^{\mu}_{-r}\, |p; A; 0\rangle_{\mathsf{R}} = \lim_{w\to 0} \oint \frac{\mathrm{d}z}{2\pi i} \frac{1}{z^{r+1/2}}\, \psi^{\mu}(z) S_A(w)\, |p; 0\rangle_{\mathsf{NS}}$$