Durham University

UK Research and Innovation

Diamonds of integrable models

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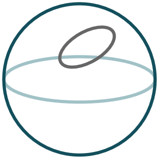
Based on **arXiv:2311.17551** and **arXiv:2407.09479**
with **Lewis Cole**, **Ryan Cullinan**, **Joaquin Liniado**
and **Dan Thompson**

Workshop on Noncommutative and Generalized Geometry in String theory, Gauge theory and Related Physical Models

September 2024

Introduction

- Integrable systems appear in many guises in mathematical and theoretical physics:
  - exact results in QFT,
  - quantum groups and algebras,
  - statistical mechanics,
  - string and gauge theory,
  - holographic duality.



Introduction

- The landscape of 2d integrable models continues to grow:
  - direct construction (challenging),
  - affine Gaudin models (integrable systems), [Feigin, Frenkel, Delduc, Lacroix, Magro, Vicedo, . . .]
  - higher-dimensional gauge theories, [Ward, Mason, Woodhouse, Costello, Yamazaki, Witten, . . .]
  - dualities and deformations (more general guiding principles). [. . .]
- These frameworks and approaches are closely connected.

Introduction

- Higher-dimensional gauge theories:

⋆ 4d Chern-Simons (4d CS) [Costello, Yamazaki, Witten, ...]

$$\frac{1}{2\pi i}\int_{\Sigma\times\mathbb{CP}^1}\omega\wedge\mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)$$

– localises on $\mathbb{CP}^1$ to give 2d IFTs.

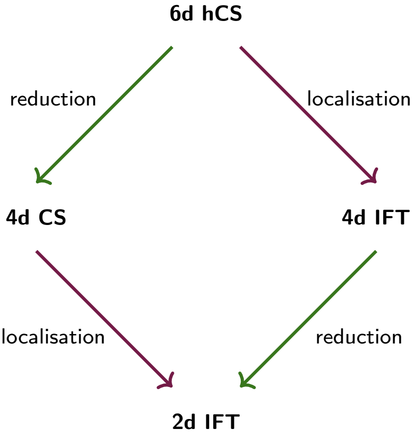
⋆ 4d anti-self-dual Yang-Mills (4d IFT) [Ward, Mason, Woodhouse, ...]

$$F_{AA'BB'}(A')=\epsilon_{AB}\Phi_{A'B'}+\epsilon_{A'B'}\tilde{\Phi}_{AB} \qquad \Phi_{A'B'}=0$$

– reduces along 2 dimensions to 2d IFTs.

Introduction

Conjecture (Costello):



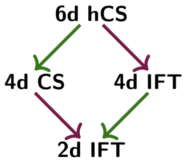
Introduction

- Higher-dimensional gauge theories:

⋆ 6d holomorphic Chern-Simons (6d hCS)

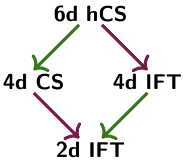
$$\frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\left(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{2}{3}\,\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right)$$

- 6d theory on twistor space,
- localises to give 4d IFTs,
- reduces along 2 dimensions to 4d CS.



Introduction

- What can we learn about:
  - the unification of classifications?
  - the landscape of integrable models?
  - the relation between integrable models?
  - properties of integrable models?



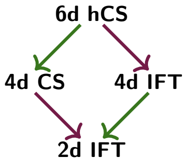
Introduction

What was known

- The diamond has been constructed for the PCM + WZ term: [Bittleston, Skinner]
  - Dirichlet (Cauchy) boundary conditions work as expected.
- Early attempts to generalise beyond this case where: [Chen, He, Tian]
  - unclear how to lift boundary conditions from 4d CS,
  - unclear what gauge symmetries to expect in the 2d and 4d IFTs.

What is known now [Cullinan, Cole, BH, Liniado, Thompson]

- The diamond has been successfully constructed for
  - the current-current deformation of the WZW CFT,
  - vectorial gaugings of the PCM plus WZ term.



Outline

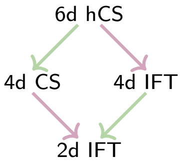
1. Introduction

2. The 2d IFTs

3. From 6d hCS to 4d CS to 2d IFT

4. From 6d hCS to 4d IFT to 2d IFT

5. Summary and future directions



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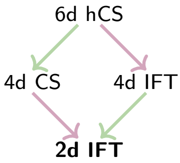

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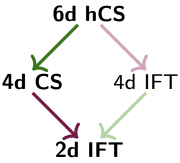

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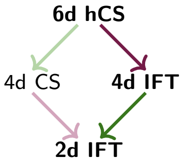

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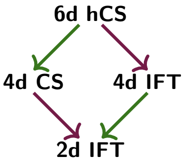


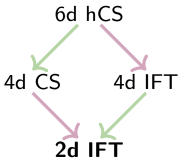
PCM plus WZ term

- The action of the PCM plus WZ term for the simple Lie group G is [Novikov, Witten]

$$S_{\mathrm{PCM}_2+\mathrm{WZ}_2} = \frac{k}{8\pi h}\int_{\Sigma} \mathrm{tr}\,(j\wedge\star j) + \frac{ik}{12\pi}\int_{\Sigma\times[0,1]} \mathrm{tr}\,(\tilde{j}\wedge\tilde{j}\wedge\tilde{j})$$

  - $j = g^{-1}dg \in \mathrm{Lie(G)} = \mathfrak{g}$ where $g \in C^{\infty}(\Sigma, \mathrm{G})$,
  - $\tilde{j} = \tilde{g}^{-1}d\tilde{g} \in \mathrm{Lie(G)} = \mathfrak{g}$ where $\tilde{g}$ is an extension of g to $\Sigma\times[0,1]$,
  - h is the sigma model coupling,
  - $k \in \mathbb{Z}$ is the level of the Wess-Zumino term,
  - tr is the normalised Killing form.

- The conformal points are $h = \pm 1$.



PCM plus WZ term – classical integrability

- The e.o.m. of the PCM plus WZ term is

$$d(\star - ih)dgg^{-1} = 0$$

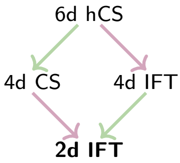
- There is a current that is both conserved and flat on-shell

$$K = -(1 + ih\star)dgg^{-1}$$

- Therefore, a Lax connection exists with spectral parameter $\zeta \in \mathbb{C}$

$$\mathcal{L} = \frac{1 + i\star}{1 - \zeta} K + \frac{1 - i\star}{1 + \zeta} K$$

- The Poisson bracket of the Lax matrix is of Maillet form and we have conserved charges in involution.

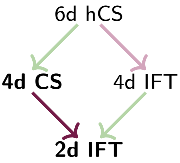


Origin from 4d CS

- The action of 4d CS is [Costello, Yamazaki, Witten, . . .]

$$\mathcal{S}_{\mathrm{CS}_4}(A) = \frac{1}{2\pi i}\int_{\Sigma\times\mathbb{CP}^1} \omega\wedge \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

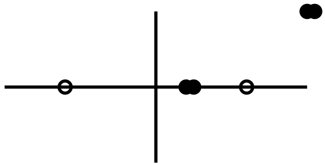
  - ζ is a complex coordinate on $\mathbb{CP}^1$,
  - ω is a specified meromorphic 1-form on $\mathbb{CP}^1$,
  - A is a $\mathfrak{g}$-valued 1-form on $\Sigma\times\mathbb{CP}^1$.
- To define the theory, we specify ω and boundary conditions ensuring that the boundary variation vanishes.
- The boundary conditions break the gauge symmetry.
- The 2d IFT is the edge mode theory that lives at the boundary.
- More mathematically rigorous treatments are available!



4d CS origin of the PCM plus WZ term

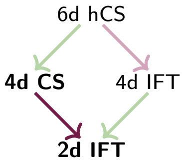
- For the PCM plus WZ term

$$\omega \sim \frac{\zeta^2 - 1}{(\zeta - h)^2}\, d\zeta$$



- The positions of the zeroes and the double pole at infinity are fixed by $SL(2, \mathbb{C})$.
- The remaining freedom is the position of the double pole on the finite plane.
- We take Dirichlet boundary conditions

$$A_\Sigma|_{\zeta=h} = 0 \qquad A_\Sigma|_{\zeta\to\infty} = 0$$



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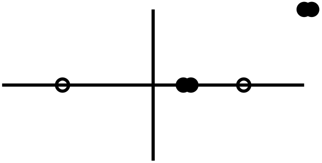
$$\omega \sim \frac{\zeta^2 - 1}{(\zeta - h)^2}\, d\zeta$$

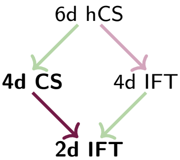


- We define

$$A = \hat{g}^{-1}\mathcal{L}\hat{g} + \hat{g}^{-1}d\hat{g} \qquad \mathcal{L}_{\bar{\zeta}} = 0$$

and solve the bulk e.o.m. subject to the boundary conditions.

- $\hat{g}$ contains the edge modes and $\mathcal{L}$ becomes the Lax connection.
- Substituting back into the 4d CS action we find the action of the PCM plus WZ term.



Origin from 4d CS

- For a general ω the boundary conditions can be understood systematically.

  [Delduc,Magro,Lacroix,Vicedo,...]

- From the gauge field and its derivatives evaluated at the poles we construct

$$\mathbb{A}_\pm \in \mathfrak{d}$$

  where $\mathfrak{d}$ is a Drinfel'd double, known as the defect algebra, with bilinear form $\langle\langle , \rangle\rangle$.

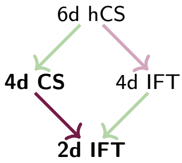
- Setting the boundary variation to vanish implies that

$$\langle\langle \mathbb{A}_+, \delta\mathbb{A}_- \rangle\rangle = \langle\langle \mathbb{A}_-, \delta\mathbb{A}_+ \rangle\rangle$$

- The choice for which the integrability is best understood is to take

$$\mathbb{A}_\pm \in \mathfrak{l}$$

  where $\mathfrak{l}$ is a **Lagrangian subalgebra** of $\mathfrak{d}$.

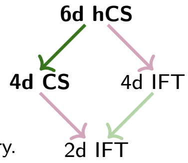


Reducing 6d hCS to 4d CS

- The action of 6d hCS is

$$\mathcal{S}_{\mathrm{hCS}_6}(\bar{\mathcal{A}}) = \frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \mathrm{tr}\left(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{2}{3}\,\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right)$$

  - Euclidean twistor space $\mathbb{PT}$ is diffeomorphic to $\mathbb{R}^4 \times \mathbb{CP}^1$,
  - $(x^{AA'}, \pi_{A'})$ are coordinates on $\mathbb{R}^4 \times \mathbb{CP}^1$.
  - A and A' are left- and right-handed spinor indices: $A \in \{1,2\}$, $A' \in \{1,2\}$,
  - Ω is a specified meromorphic 3-form,
  - $\bar{\mathcal{A}}$ is a $\mathfrak{g}$-valued anti-holomorphic 1-form on $\mathbb{PT}$.
- To define the theory, we specify Ω and boundary conditions ensuring that the boundary variation vanishes.
- The boundary conditions break the gauge symmetry.
- The 4d IFT is again the edge mode theory that lives at the boundary.



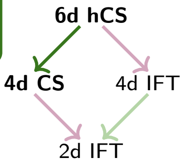
Reducing 6d hCS to 4d CS

- The meromorphic 3-form with the correct weight is

$$\Omega \sim \frac{1}{2}\,\frac{\langle \pi d\pi\rangle \wedge dx^{AA'} \wedge dx^{BB'} \epsilon_{AB}\pi_{A'}\pi_{B'}}{\langle \pi\alpha\rangle^2\langle \pi\beta\rangle^2}$$

- We have double poles at $\pi_{A'} = \alpha_{A'}$ and $\pi_{A'} = \beta_{A'}$.
- We introduce unit norm reference left- and right-handed spinors, μ^A and $\gamma^{A'}$ and define the coordinates on $\mathbb{R}^4$

$$dz = \mu_A\gamma_{A'}dx^{AA'} \qquad d\bar{z} = \hat{\mu}_A\hat{\gamma}_{A'}dx^{AA'}$$
$$dw = \hat{\mu}_A\gamma_{A'}dx^{AA'} \qquad d\bar{w} = -\mu_A\hat{\gamma}_{A'}dx^{AA'}$$



- The reduction along 2 directions imposes that $\partial_z\Phi = \partial_{\bar{z}}\Phi = 0$ for all fields Φ.

Reducing 6d hCS to 4d CS

- The reduction of 6d hCS gives 4d CS with the meromorphic 1-form

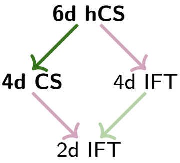
$$\omega \sim (\partial_z \wedge \partial_{\bar{z}}) \lrcorner \Omega \sim \frac{\langle \pi d\pi \rangle \langle \pi \gamma \rangle \langle \pi \hat{\gamma} \rangle}{\langle \pi \alpha \rangle^2 \langle \pi \beta \rangle^2} \sim \frac{(\zeta - \gamma_+)(\zeta - \gamma_-)}{(\zeta - \alpha)^2} \, d\zeta$$

where $\zeta = \pi_2/\pi_1$, $\gamma_+ = \gamma_2/\gamma_1$, $\gamma_- = -\bar{\gamma}_1/\bar{\gamma}_2$, $\alpha = (1, \alpha)$, $\beta = (0, 1)$.

- We can then use $SL(2, \mathbb{C})$ to fix $\gamma_\pm = \pm 1$ and identify

$$h = \frac{\gamma_- + \gamma_+ - 2\alpha}{\gamma_- - \gamma_+}$$

to recover the expected form.



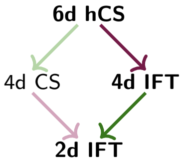
In the other direction

- First localising from 6d hCS to a 4d IFT, and then reducing to a 2d IFT works without any further issues.
- We again take Dirichlet boundary conditions

$$\mathcal{A}_{\mathbb{R}^4}|_{\pi=\alpha} = 0 \qquad \mathcal{A}_{\mathbb{R}^4}|_{\pi=\beta} = 0$$

- The e.o.m. of the 4d IFT can be recast, as a consequence of the construction, as the 4d anti-self-dual Yang-Mills equations.
- This is an example of Ward's conjecture:

> **"... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalisations) by reduction."**



The 4d IFT

- In this case the 4d IFT is the 4d WZW model

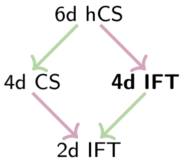
[Donaldson, Nair, Schiff, Losev, Moore, Nekrasov, Shatashvili]

$$\mathcal{S}_{\mathrm{WZW}_4} = \frac{k}{8\pi}\int_{\mathbb{R}^4} \mathrm{tr}\left(j \wedge \star j\right) + \frac{k}{12\pi}\int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathrm{tr}\left(\tilde{j}\wedge\tilde{j}\wedge\tilde{j}\right)$$

  - $j = g^{-1}dg \in \mathrm{Lie(G)} = \mathfrak{g}$ where $g \in C^\infty(\mathbb{R}^4, \mathrm{G})$,
  - $\tilde{j} = \tilde{g}^{-1}d\tilde{g} \in \mathrm{Lie(G)} = \mathfrak{g}$ where $\tilde{g}$ is an extension of g to $\mathbb{R}^4 \times [0,1]$,
  - $\omega_{\alpha,\beta}$ is a 2-form depending on the spinors $\alpha_{A'}$ and $\beta_{B'}$ and is given by

$$\omega_{\alpha\beta} = \frac{1}{\langle\alpha\beta\rangle}\epsilon_{AB}\alpha_{A'}\beta_{B'}dx^{AA'}dx^{BB'}$$

and breaks the Euclidean space-time symmetry.



The 4d IFT

- The e.o.m. of the 4d WZW model is

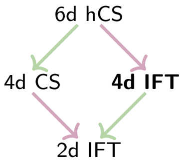
$$d(\star - \omega_{\alpha,\beta}\wedge) dgg^{-1} = 0$$

- The anti-self-dual Yang-Mills connection is

$$A'_{AA'} = -\frac{\beta_{A'}\alpha^{B'}}{\langle\alpha\beta\rangle}\partial_{AB'}gg^{-1}$$

which, as for the Lax connection in 4d CS, can be constructed from 6d hCS.

- This is known as Yang's parametrisation and g is known as Yang's matrix.



Gauged models

- The vector-gauged G_k/H_k WZW model with gauge field $B \in \mathfrak{h}$ and $\nabla = d + \mathrm{ad}_B$

[Witten]

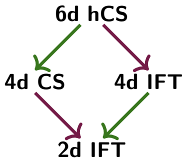
$$\mathcal{S}_{\mathrm{gWZW}_2}(g,B) = \frac{k}{8\pi h}\int_\Sigma \mathrm{tr}\left(\nabla g g^{-1} \wedge \star \nabla g g^{-1}\right) + \frac{ik}{12\pi}\int_{\Sigma\times[0,1]} \mathcal{L}_{\mathrm{gWZ}}(g,B)$$

can be written as the difference of two WZW models [Polyakov, Wiegmann]

$$\mathcal{S}_{\mathrm{gWZW}_2}(g,B) = S_{\mathrm{WZW}_2}(\tilde{g}) - S_{\mathrm{WZW}_2}(\tilde{h})$$

where $\tilde{g} = agb^{-1} \in G$, $\tilde{h} = ab^{-1} \in H$, $a \in H$, $b \in H$ and

$$B = \frac{1 - i\star}{2}\, a^{-1}da + \frac{1+i\star}{2}\, b^{-1}db$$



Gauged models

- This motivates us to consider the 6d action [see also Stedman]

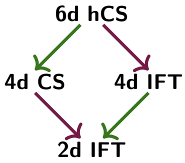
$$\mathcal{S}_{\mathrm{ghCS}_6}(\bar{\mathcal{A}},\bar{\mathcal{B}}) = \mathcal{S}_{\mathrm{hCS}_6}(\bar{\mathcal{A}}) - \mathcal{S}_{\mathrm{hCS}_6}(\bar{\mathcal{B}}) - \frac{1}{2\pi i}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{tr}\left(\bar{\mathcal{A}}\wedge\bar{\mathcal{B}}\right)$$

where $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are $\mathfrak{g}$-valued and $\mathfrak{h}$-valued anti-holomorphic 1-forms on $\mathbb{PT}$.

- Introducing the orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, we take the boundary conditions

$$\bar{\mathcal{A}}^{\mathfrak{k}}_{\mathbb{R}^4}|_{\pi=\alpha,\beta} = 0 \qquad \bar{\mathcal{A}}^{\mathfrak{h}}_{\mathbb{R}^4}|_{\pi=\alpha,\beta} = \bar{\mathcal{B}}_{\mathbb{R}^4}|_{\pi=\alpha,\beta}$$
$$\partial_0\bar{\mathcal{A}}^{\mathfrak{h}}_{\mathbb{R}^4}|_{\pi=\alpha,\beta} = \partial_0\bar{\mathcal{B}}_{\mathbb{R}^4}|_{\pi=\alpha,\beta}$$

where ∂_0 is the holomorphic derivative on $\mathbb{CP}^1$.



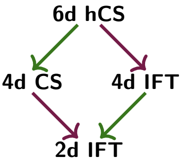
Gauged models

- The resulting gauged 4d IFT is

$$\mathcal{S}_{\mathrm{gWZW}_4}(g,B) = \frac{k}{8\pi}\int_{\mathbb{R}^4} \mathrm{tr}\left(\nabla g g^{-1} \wedge \star \nabla g g^{-1}\right) + \frac{k}{12\pi}\int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}(g,B)$$
$$- \int_{\mathbb{R}^4} \left(\mu_\alpha \wedge \mathrm{tr}(u\cdot F(B)) + \mu_\beta \wedge \mathrm{tr}(\tilde{u}\cdot F(B))\right)$$

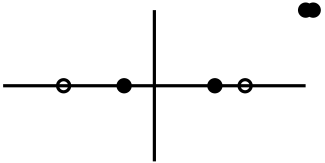
where $\mu_\alpha = \epsilon_{AB}\alpha_{A'}\alpha_{B'} dx^{AA'} dx^{BB'}$ and $\mu_\beta = \epsilon_{AB}\beta_{A'}\beta_{B'} dx^{AA'} dx^{BB'}$.

- The $\mathfrak{g}$-valued Lagrange multipliers u and $\tilde{u}$ impose constraints that mean this action can be written as the difference of two 4d WZW actions.
- Reducing to 2d, we find a much richer family of integrable models than simply 2d gauged WZW due to the additional adjoint scalars u, $\tilde{u}$, B_z and $B_{\bar{z}}$.
- In particular, we can recover the homogeneous sine-Gordon models.

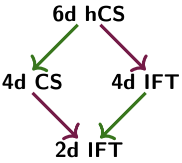


Dualities and deformations

- In the Chern-Simons constructions of integrable field theories
  - dualities are understood as different choices of boundary conditions for the same choice of ω or Ω,
  - deformations are understood as moving the positions of poles and zeroes, or splitting higher-order poles.

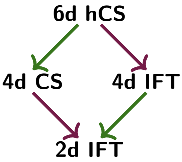


- For non Dirichlet boundary conditions the defect algebra description of boundary conditions in 4d CS does not lift to 6d hCS due to fibred structure of twistor space.
- We are led to consider wider classes of boundary conditions.



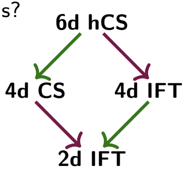
Summary

- The diamond has been constructed for: [Costello]
  - the PCM + WZ term, [Bittleston, Skinner]
  - the current-current deformation of the WZW CFT,
  - vectorial gaugings of the PCM plus WZ term. [Cole, Cullinan, BH, Liniado, Thompson]
- The study of deformed models starting from 6d hCS has led to new types of boundary conditions in 6d hCS and 4d CS.
- Different real forms of space-time can be considered if we replace twistor space by the correspondence space.
- The study of the gauged models has led to new 2d IFTs with adjoint scalars.
- Both setups lead to new 4d IFTs and new examples of the Ward conjecture.



Future directions

- Can we generalise further?
  - What happens if Ω has zeroes?
  - What is the Hamiltonian picture of 6d hCS?
  - What if $\mathbb{CP}^1$ is replaced by a more general Riemann surface? [Cole, Weck]
  - What is the relation to integrability in higher dimensions via categorification? [Vicedo, Schenkel, Chen, Liniado]
- What lessons can we learn about integrability in higher dimensions?
- Is it possible to quantize integrable field theories in higher dimensions?



Thank you!